



One dimensional Saint-Venant system

Ngoc Tuoi Vo Thi

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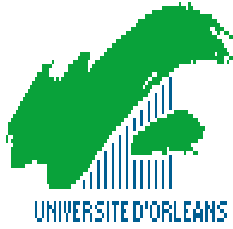
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UNIVERSITY OF ORLEANS

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MASTER 2 PUF HCMC
MATHEMATICS

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Subject

ONE DIMENSIONAL SAINT - VENANT SYSTEM

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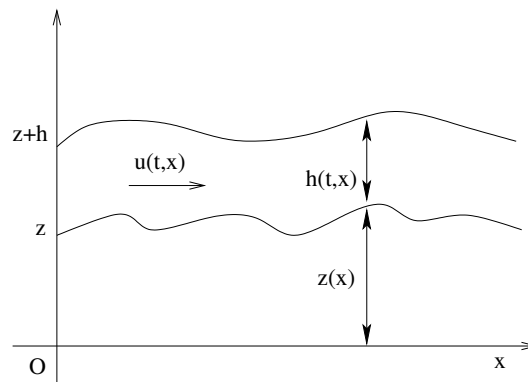
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Introduction



The study of free-surface water flow in channels has many important applications, one of the most significant being in the area of river modelling. Using numerical methods to compute the water surface profile and discharge, for both unsteady and steady open channel flows, is now very common in civil engineering hydraulics. The Saint-Venant equations, first formulated by De St. Venant (1871), are almost always used to model the flow. Very often in nature a flow will approach a steady state, that is where the flow is essentially unchanging in time. The case of steady flow is considered in this report. Under steady conditions the Saint-Venant equations reduce in complexity yielding a single nonlinear ordinary differential equation which describes the variation of the free surface. Finding analytical solutions are very useful to understand model and structure of the equation. It is also useful to construct test problems for numerical schemes. Using some measure of the difference between the numerical solution and the exact solution, the performance of a particular numerical scheme can be evaluated.

A simple method for constructing test problems with known analytical solutions to the steady Saint-Venant equation in 2D is presented in MacDonald [5]. The method is an "inverse method" in that some hypothetical depth profile is chosen and the bed slope that makes this profile an actual solution of the steady equation is then found. This method can be used to construct test problems with almost any desired features, including hydraulic jumps. Hence these test problems can be used to compare the numerical results, for any algorithm, with an exact solution. In this report, we will consider the "inverse method" for the steady Saint-Venant equation in 1D, including friction term and hydraulic jumps. The results in 1D are compared with the results in MacDonald [3] and [5]. Especially, we will pay attention to another friction law (Darcy-Weisbach friction) and the case rainfall.

Notation

| | |
|----------------------|---|
| x | Distance along channel(m) |
| t | Time (s) |
| L | Length of channel (m) |
| g | Acceleration due to gravity (m/s^2) |
| $h(x, t)$ | Height of water or Depth (m) |
| $u(x, t)$ | Velocity of water (m/s) |
| $z(x)$ | Bed level or Topography (m) |
| $S_f(x, h, Q)$ | Friction slope |
| $S_0(x) = -dz/dx$ | Bed slope |
| $Q(x, t)$ | Discharge (m^3/s) |
| $A(x, h)$ | Wetted area (m^2) |
| $T(x, h)$ | Free surface width (m) |
| $q(x, t) = hu = Q/T$ | Discharge (m^2/s) |

1 The Saint - Venant equation

1.1 The Saint - Venant equation in 2D

From the principles of mass and momentum balance, we obtain the Saint - Venant equations (MacDonald [5])

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = R, \quad (1.1)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) + gA \frac{\partial h}{\partial x} - gA(S_0 - S_f) = 0. \quad (1.2)$$

where x is the distance along the channel, t is the time, $h(x, t)$ is the height of water, $Q(x, t)$ is the discharge, $T(x, h)$ is the free surface width, $A(x, h)$ is the wetted area, $S_0(x)$ is the bed slope, $S_f(x, h, Q)$ is the friction slope, R is the lateral inflow per unit length and g is the acceleration due to gravity.

The bed slope, S_0 , is given by

$$S_0 = -\frac{dz}{dx},$$

where $z(x)$ is the bed level, the elevation of the bed above some horizontal datum.

The friction slope, S_f , is given by

$$S_f = \frac{Q |Q| n^2 P^{4/3}}{A^{10/3}},$$

where $P(x, h)$ is the wetted perimeter and n is the Manning friction coefficient.

We only consider the steady flow problem, so it is assumed that $h = h(x)$ and $Q = Q(x)$. Under these steady conditions equations (1.1) and (1.2) reduce to

$$\frac{dQ}{dx} = R, \quad (1.3)$$

$$\frac{d}{dx} \left(\frac{Q^2}{A} \right) + gA \frac{dh}{dx} - gA(S_0 - S_f) = 0. \quad (1.4)$$

To simplify, the lateral inflow R is assumed to be zero. So, equation (1.3) becomes trivial with solution $Q \equiv \text{constant}$. Since the discharge can have no jumps it must be constant throughout the entire channel reach. From now on x will be measured in the direction of this constant discharge and hence $Q > 0$.

Differentiating the momentum term in equation (1.4) and dividing by gA then yields the equation

$$\frac{dh}{dx} - \frac{Q^2}{gA^3} \frac{\partial A}{\partial x} - S_0 + S_f = 0. \quad (1.5)$$

Suppose that for some reach $0 \leq x \leq L$ the functions T and P , representing channel width and wetted perimeter respectively, are arbitrarily defined for $0 < h \leq h_{max}$.

We will consider a rectangular channel with $A = Th$ and $P = T + 2h$. The equation (1.5) become

$$\left(1 - \frac{Q^2 T}{g A^3}\right) \frac{dh}{dx} - \frac{Q^2 h}{g A^3} \frac{\partial T}{\partial x} - S_0 + S_f = 0. \quad (1.6)$$

It is convenient to re-write this equation as

$$S_0(x) = f_1(x, h(x))h'(x) + f_2(x, h(x)), \quad (1.7)$$

where

$$f_1 = 1 - \frac{Q^2 T}{g A^3} = 1 - Fr^2, \quad (1.8)$$

Fr is the Froude number and

$$f_2 = \frac{Q^2 n^2 P^{4/3}}{A^{10/3}} - \frac{Q^2 h}{g A^3} \frac{\partial T}{\partial x}. \quad (1.9)$$

To have the smooth solution, it will be required that $T, P, \partial T/\partial x, \partial T/\partial h$ are continuous. These requirements are sufficient to ensure that differential equation (1.6) is valid and can be obtained from the integral form of the steady Saint Venant equation. If the discharge Q and the Manning friction coefficient n are chosen, then the function f_1 and f_2 have been defined by equations (1.8) and (1.9).

We will choose some function \hat{h} for $0 \leq x \leq L$, with $0 < \hat{h}(x) \leq h_{max}$ and having a continuous first derivative. This function will be referred to as the hypothetical depth profile. Finally, if the bed slope of the channel is given by

$$S_0(x) = f_1(x, \hat{h}(x))\hat{h}'(x) + f_2(x, \hat{h}(x)), \quad (1.10)$$

then it is easy that \hat{h} satisfies the differential equation (1.7) for the entire reach $0 \leq x \leq L$.

From the above, a complete test problem can be specified by the length L of the reach, the functions T, P (and hence A) which define the cross-sectional shape throughout the reach, values for the discharge Q and manning coefficient n , and the bed slope of the channel given by equation (1.10). The analytic solution to the steady problem is now given by $h \equiv \hat{h}$.

For many computational models, the bed level z is required rather than the bed slope. This cannot normally be found analytically from S_0 , so we can use numerical method to compute z and a starting value such as $z(L) = 0$ is required.

```
bedslope2D := proc(T, Q, n, g, h)
S0 =  $\left(1 - \frac{Q^2 T}{g A^3}\right) \frac{d\hat{h}}{dx} - \frac{Q^2 \hat{h}}{g A^3} \frac{\partial T}{\partial x} + S_f$ 
end proc.
```

Remark: The Froude number, Fr , is a dimensionless value that describes different flow regimes of open channel flow

$$Fr = \sqrt{\frac{Q^2 T}{g A^3}},$$

or

$$Fr = \frac{u}{\sqrt{gh}}.$$

When

$Fr = 1$: the type of flow is *critical flow* and occurs when inertial forces and gravitational forces exactly balance.

$Fr < 1$: the type of flow is *subcritical flow* (slow or tranquil flow) and occurs when gravitational forces dominate over inertial forces.

$Fr > 1$: the type of flow is *supercritical flow* (fast rapid flow) and occurs when inertial forces dominate over gravitational forces.

Critical flow is unstable and often sets up standing waves between supercritical and subcritical flow. When the actual water depth is below critical depth, it is called supercritical because it is in a higher energy state. Likewise actual depth above critical depth is called subcritical because it is in a lower energy state.

1.2 The Saint - Venant equation in 1D

We only consider the rectangular channel with $A = Th$, $P = T + 2h$ and $Q = Tq$. from equations (1.1) and (1.2), we obtain

$$\frac{\partial(Th)}{\partial t} + \frac{\partial(Tq)}{\partial x} = R, \quad (1.11)$$

$$\frac{\partial(Tq)}{\partial t} + \frac{\partial}{\partial x} \left(\frac{T^2 q^2}{Th} \right) + gTh \frac{\partial h}{\partial x} - gTh(S_0 - S_f) = 0. \quad (1.12)$$

Dividing equation (1.11) by T

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = \frac{R}{T} = R_1. \quad (1.13)$$

Changing equation (1.12)

$$\begin{aligned} (1.12) &\Leftrightarrow T \left[\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q^2}{h} \right) + gh \frac{\partial h}{\partial x} - gh(S_0 - S_f) \right] = 0 \\ &\Leftrightarrow \frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q^2}{h} + \frac{gh^2}{2} \right) - gh(S_0 - S_f) = 0. \end{aligned} \quad (1.14)$$

Finally, the Saint - Venant equation in 1D is obtained from equations (1.13) and (1.14)

$$\begin{cases} \partial_t h + \partial_x(uh) = R & (a) \\ \partial_t(uh) + \partial_x(u^2h + \frac{gh^2}{2}) = gh(S_0 - S_f) & (b) \end{cases} \quad (1.15)$$

where $q = uh$, $h(x, t) \in \mathbb{R}^+$ is the height of water, $u(x, t) \in \mathbb{R}$ is the velocity of water, $S_0(x) = -\partial_x z$ is the bed slope, $S_f(x, h, Q)$ is the friction slope, R is the rainfall intensity and g is the acceleration due to gravity.

The Manning friction slope in 2D is given by

$$\begin{aligned} S_f &= \frac{Q^2 n^2 P^{4/3}}{A^{10/3}} = \frac{Q^2 n^2 (T + 2h)^{4/3}}{(Th)^{10/3}} \\ &= \frac{Q^2 n^2 T^{4/3} \left(1 + \frac{2h}{T}\right)^{4/3}}{T^{10/3} h^{10/3}} = \frac{q^2 n^2 \left(1 + \frac{2h}{T}\right)^{4/3}}{h^{10/3}}. \end{aligned}$$

Because $T \gg h$, we can approximate

$$\left(1 + \frac{2h}{T}\right)^{4/3} \approx 1 + \frac{4}{3} \frac{2h}{T} + \dots$$

Let

$$n_1^2 = n^2 \left(1 + \frac{2h}{T}\right)^{4/3} = n^2 \left(1 + \frac{4}{3} \frac{2h}{T} + \dots\right). \quad (1.16)$$

It follows the Manning friction slope in 1D

$$S_f = \frac{n_1^2 q |q|}{h^{10/3}} = \frac{n_1^2 u |u|}{h^{4/3}}.$$

Besides, the Darcy - Weisbach friction is also used

$$S_f = \frac{f q |q|}{8gh^3} = \frac{f u |u|}{8gh},$$

where f is Darcy - Weisbach coefficient.

The value of f can be determined by

$$f \approx \frac{8gn_1^2}{h^{1/3}}.$$

1.3 Steady state

In this report, we only consider the steady flow problem, so we have

$$\partial_t h = 0 \quad \text{and} \quad \partial_t(uh) = 0.$$

Under the steady condition, equation (a) and (b) reduce to

$$\begin{aligned} (a) &\Rightarrow \partial_x(uh) = R \quad \text{where } R \text{ is constant,} \\ (b) &\Rightarrow \partial_x(u^2h + \frac{gh^2}{2}) = -gh\partial_x z - ghS_f. \end{aligned} \quad (1.17)$$

1.3.1 The case no rain $R = 0$

We have

$$\begin{aligned} \partial_x(uh) = 0 &\Rightarrow q = uh = \text{constant}, \\ (1.17) \Leftrightarrow u\partial_x(uh) + uh\partial_x u + gh\partial_x h &= -gh\partial_x z - ghS_f \\ \Leftrightarrow gh\partial_x z &= -ghS_f - u\partial_x(uh) - uh\partial_x u - gh\partial_x h \\ \Leftrightarrow gh\partial_x z &= -ghS_f - uh\partial_x u - gh\partial_x h \\ \Leftrightarrow \partial_x z &= -S_f - \frac{u}{g}\partial_x u - \partial_x h \\ \Leftrightarrow \partial_x z &= -S_f - \frac{q}{gh}\partial_x(\frac{q}{h}) - \partial_x h \quad (u = q/h) \\ \Leftrightarrow \partial_x z &= (\frac{q^2}{gh^3} - 1)\partial_x h - S_f. \end{aligned}$$

We obtain

$$S_0(x) = f_1(x, h(x))\partial_x h + f_2(x, h(x)), \quad (1.18)$$

where $f_1(x, h(x)) = 1 - \frac{q^2}{gh^3}$ and $f_2(x, h(x)) = S_f$.

For each formula of the friction slope S_f , equation (1.18) become

- Manning friction : $S_f = \frac{n^2 u |u|}{h^{4/3}}$

$$S_0 = (1 - \frac{q^2}{gh^3})\partial_x h + \frac{n^2 q^2}{h^{10/3}}, \quad (1.19)$$

where n is Manning friction coefficient.

- Darcy - Weisbach friction: $S_f = \frac{f u |u|}{8gh}$

$$S_0 = (1 - \frac{q^2}{gh^3})\partial_x h + \frac{f q^2}{8gh^3}, \quad (1.20)$$

where f is Darcy - Weisbach friction coefficient.

1.3.2 Rain $R \neq 0$

We have

$$\partial_x(uh) = R \Rightarrow q = uh = Rx + q_0,$$

q_0 is discharge at $x = 0$.

$$\begin{aligned} (1.17) \quad &\Leftrightarrow \partial_x\left(\frac{q^2}{h} + \frac{gh^2}{2}\right) = -gh\partial_x z - ghS_f \\ &\Leftrightarrow gh\partial_x z = -ghS_f - \frac{2q}{h}\partial_x q + \frac{q^2}{h^2}\partial_x h - gh\partial_x h \\ &\Leftrightarrow \partial_x z = -S_f - \frac{2qR}{gh^2} + \frac{q^2}{gh^3}\partial_x h - \partial_x h \\ &\Leftrightarrow \partial_x z = \left(\frac{q^2}{gh^3} - 1\right)\partial_x h - \frac{2qR}{gh^2} - S_f. \end{aligned}$$

We obtain

$$S_0(x) = f_1(x, h(x))\partial_x h + f_2(x, h(x)), \quad (1.21)$$

where $f_1(x, h(x)) = 1 - \frac{q^2}{gh^3}$ and $f_2(x, h(x)) = \frac{2qR}{gh^2} + S_f$.

We consider the friction slope S_f

- Manning friction: $S_f = \frac{n^2 u |u|}{h^{4/3}}$

$$S_0 = \left(1 - \frac{q^2}{gh^3}\right)\partial_x h + \frac{2qR}{gh^2} + \frac{n^2 q^2}{h^{10/3}}, \quad (1.22)$$

- Darcy - Weisbach friction: $S_f = \frac{f u |u|}{8gh}$

$$\partial_x z = \left(1 - \frac{q^2}{gh^3}\right)\partial_x h + \frac{2qR}{gh^2} + \frac{f q^2}{8gh^3}. \quad (1.23)$$

1.3.3 Energy and specific charge

For the case $R = 0$, $S_f = 0$ and steady state ($\partial_t h = \partial_t(uh) = 0$).

The equation (1.18) become

$$\begin{aligned} \partial_x z &= \left(\frac{q^2}{gh^3} - 1\right)\partial_x h \\ \Leftrightarrow \partial_x h - \frac{q^2}{gh^3}\partial_x h + \partial_x z &= 0. \end{aligned} \quad (1.24)$$

Integrate equation (1.24) from $x = a$ to $x = b$, we obtain

$$h(a) + \frac{q^2(a)}{2gh^2(a)} + z(a) = h(b) + \frac{q^2(b)}{2gh^2(b)} + z(b) = \text{constant}.$$

Define the **energy**

$$H(x) = h(x) + \frac{u^2(x)}{2g} + z(x).$$

Then we have the conservation of energy.

The **specific charge** is defined by

$$\begin{aligned} H_S(x) &= h(x) + \frac{u^2(x)}{2g} = h(x) + \frac{q^2(x)}{2gh^2(x)} \\ \Rightarrow q &= \sqrt{2gh^2(H_S - h)} = h\sqrt{2g}\sqrt{H_S - h}. \end{aligned}$$

Assume that H_S is constant,

$$\begin{aligned} \frac{\partial q}{\partial h} &= \sqrt{2g}\sqrt{H_S - h} - \frac{\sqrt{2gh}}{2\sqrt{H_S - h}} = \frac{2\sqrt{2g}(H_S - h) - \sqrt{2gh}}{2\sqrt{H_S - h}} = \frac{\sqrt{2g}(2H_S - 3h)}{2\sqrt{H_S - h}} \\ \Rightarrow \frac{\partial q}{\partial h} &= 0 \Leftrightarrow h = \frac{2}{3}H_S. \end{aligned}$$

The value of q can vary between 0 and q_{max} , where

$$q_{max} = \frac{2}{3}H_S\sqrt{2g}\sqrt{H_S - \frac{2}{3}H_S} = \frac{2}{3}H_S\sqrt{2g}\sqrt{\frac{1}{3}H_S} = \sqrt{\frac{8gH_S^3}{27}} = \sqrt{gh^3}.$$

We will consider the case $q = \text{constant}$ and $0 < q < q_{max}$.

Take again

$$H_S(x) = h(x) + \frac{q^2(x)}{2gh^2(x)}.$$

- $H_S \rightarrow \infty$ as $h \rightarrow 0$.
- $H_S \rightarrow \infty$ as $h \rightarrow \infty$.
- $\frac{\partial H_S}{\partial h} = 0$ at $h = \left(\frac{q^2}{g}\right)^{1/3} = h_c$.
- $Fr = 1$ or $u = \sqrt{gh}$ at $h = \left(\frac{q^2}{g}\right)^{1/3} = h_c$.

H_S obtain a minimum value as $h = h_c$ and h_c is called *critical depth*.

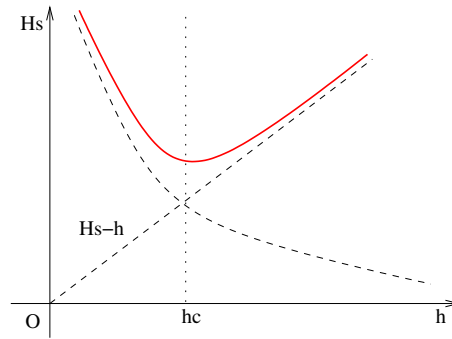


Figure 1: Specific charge

For a **subcritical flow**: $Fr = \frac{u}{\sqrt{gh}} < 1$ with $z_a < z_b$.

We have $H(x) = \text{constant}$,

$$H(a) = H(b) \Rightarrow H_S(a) + z(a) = H_S(b) + z(b)$$

$$\Rightarrow H_S(b) = H_S(a) + z(a) - z(b),$$

so $H_S(a) > H_S(b)$: the specific charge is decreasing,
and $h_a > h_b$.

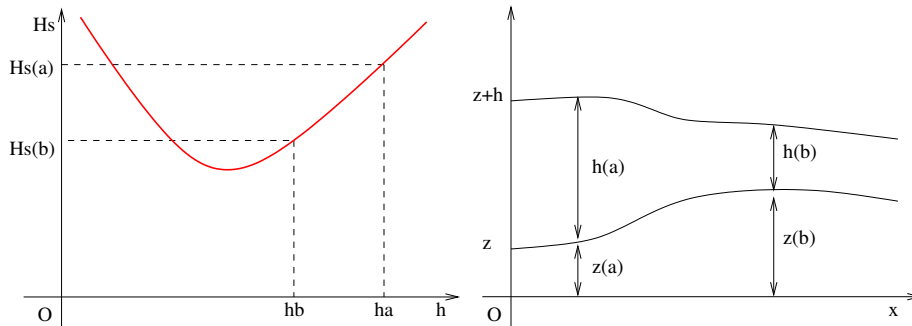


Figure 2: Subcritical flow

For a **supercritical flow**: $Fr = \frac{u}{\sqrt{gh}} > 1$ with $z_a < z_b$.

We have $H(x) = \text{constant}$,

$$H(a) = H(b) \Rightarrow H_S(a) + z(a) = H_S(b) + z(b)$$

$$\Rightarrow H_S(b) = H_S(a) + z(a) - z(b),$$

so $H_S(a) > H_S(b)$: the specific charge is decreasing,
and $h_a < h_b$.

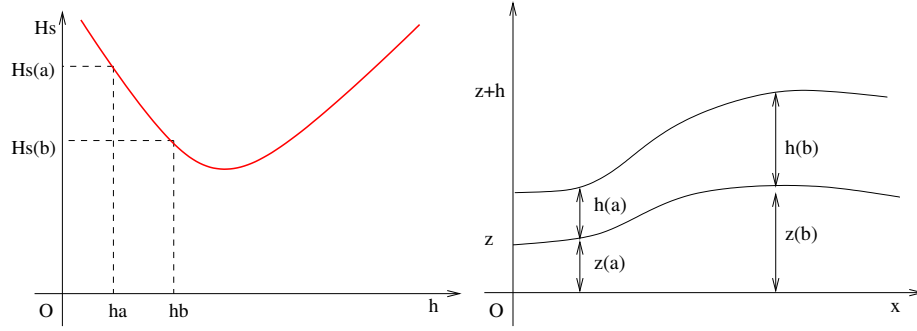


Figure 3: Supercritical flow

1.3.4 Hydraulic jump

For the case $\partial_x z = 0$, steady state and without friction.

Let $q = uh = C_1$ and $u^2h + \frac{gh^2}{2} = C_2$, where C_1, C_2 is constant.

Assume that there is a discontinuity with h_1, u_1 at upstream and h_2, u_2 at downstream.

We should have

$$u_1 h_1 = u_2 h_2 = q, \quad (1.25)$$

$$u_1^2 h_1 + \frac{gh_1^2}{2} = u_2^2 h_2 + \frac{gh_2^2}{2}. \quad (1.26)$$

A obvious solution is $u_1 = u_2$ and $h_1 = h_2$.

The other one isnot obvious, from condition (1.25) we have

$$u_1 = \frac{q}{h_1} \quad \text{and} \quad u_2 = \frac{q}{h_2}.$$

Then condition (1.26) become

$$\begin{aligned} \frac{q^2}{h_1^2} h_1 + g \frac{h_1^2}{2} &= \frac{q^2}{h_2^2} h_2 + g \frac{h_2^2}{2} \\ \Leftrightarrow (h_2 - h_1) \left(\frac{q^2}{h_1 h_2} - \frac{g}{2} (h_1 + h_2) \right) &= 0 \\ \Rightarrow h_1 h_2 (h_1 + h_2) &= \frac{2q^2}{g} \quad \text{as } h_1 \neq h_2. \end{aligned}$$

This is the hydraulic jump relation. It depend on discharge.

$$h_1 h_2 (h_1 + h_2) = 2h_c^3 \quad \text{with} \quad h_c^3 = \frac{q^2}{g}.$$

We also have

$$\frac{u_1^2 u_2^2}{u_1 + u_2} = \frac{gq}{2} = \frac{u_c^3}{2} \quad \text{with} \quad u_c = \sqrt{gh_c}.$$

1.3.5 Test problems with smooth solutions

The form of bed slope is given by

$$S_0(x) = f_1(x, h(x))h'(x) + f_2(x, h(x)).$$

If values for the discharge q and the Manning friction coefficient n (or Darcy - Weisbach coefficient f) are chosen, then the functions f_1 and f_2 have been completely defined. The important problem is to choose a function \hat{h} for $0 \leq x \leq L$, with $0 < \hat{h} \leq h_{max}$ and having a continuous first derivative. Finally, if the bed slope of the channel is given by

$$S_0(x) = f_1(x, \hat{h}(x))\hat{h}'(x) + f_2(x, \hat{h}(x)),$$

then \hat{h} will satisfy the differential equation (1.18) for the entire reach $0 \leq x \leq L$. The analytic solution to the steady problem is now given by $h \equiv \hat{h}$.

bedslope1D := proc(q, n, g, \hat{h})

$$S_0 = \left(1 - \frac{q^2}{g\hat{h}^3}\right) \frac{d\hat{h}}{dx} + S_f$$

end proc.

bedslope1DRain := proc(q, R, n, g, \hat{h})

$$S_0 = \left(1 - \frac{q^2}{g\hat{h}^3}\right) \frac{d\hat{h}}{dx} + \frac{2qR}{g\hat{h}^2} + S_f$$

end proc.

When the bed slope S_0 is found, we can compute the bed level z . However this cannot normally be found analytically from $S_0 = -\partial_x z$. We can approximate z by numerical method with a starting value such as $z(L) = 0$. For example, we use a uniform grid, $x_i = i\Delta x$, $i = 0, 1, \dots, N$, of spacing $\Delta x = \frac{L}{N}$, where $N \in \mathbb{Z}^+$. We have

$$z_{i+1} = z_i + \Delta x \partial_x z,$$

$i = 0..N$ with $z_0 = z(L)$.

1.3.6 Test problems with hydraulic jumps

It is useful if test problems could be constructed where the known solution has a hydraulic jump. Let the hypothetical depth profile $0 < \hat{h}(x) \leq h_{max}$, with a hydraulic jump at some point $x = x^*$, be given by

$$\hat{h}(x) = \begin{cases} \hat{h}_L(x) & 0 \leq x \leq x^* \\ \hat{h}_R(x) & x^* < x \leq L \end{cases}$$

the functions \hat{h}_L and \hat{h}_R having continuous derivatives respectively on the intervals $0 \leq x \leq x^*$ and $x^* \leq x \leq L$, with the derivatives being one-sided at the end points. The hydraulic jump must satisfy a jump condition and there cannot be a gain in energy across the jump. We should have

$$H_S(x^*, \hat{h}_L(x^*)) \geq H_S(x^*, \hat{h}_R(x^*)).$$

The bed slope of the channel is also defined in a piecewise manner by

$$S_0(x) = \begin{cases} S_{0L}(x) & 0 \leq x \leq x^* \\ S_{0R}(x) & x^* < x \leq L \end{cases} \quad (1.27)$$

with

$$\begin{aligned} S_{0L}(x) &= f_1(x, \hat{h}_L(x))\hat{h}'_L(x) + f_2(x, \hat{h}_L(x)), \\ S_{0R}(x) &= f_1(x, \hat{h}_R(x))\hat{h}'_R(x) + f_2(x, \hat{h}_R(x)). \end{aligned}$$

We see that \hat{h} satisfies the equation (1.18) everywhere except at the jump. So the bed slope is discontinuous at $x = x^*$,

$$S_{0L}(x^*) \neq S_{0R}(x^*).$$

At first sight this discontinuity may seem perfectly acceptable, since many valid test problems have such a feature. However, the hydraulic jump does not often occur at the same position as the bed slope discontinuity. So we would like to construct problems where the jump does not coincide with a bed slope discontinuity. For instance, we have chosen values for $\hat{h}_L(x^*)$ and $\hat{h}_R(x^*)$, choose values for $\hat{h}'_L(x^*)$ and $\hat{h}'_R(x^*)$ satisfying the linear relationship

$$\begin{aligned} S_{0L}(x^*) &= f_1(x^*, \hat{h}_L(x^*))\hat{h}'_L(x^*) + f_2(x^*, \hat{h}_L(x^*)) \\ &= f_1(x^*, \hat{h}_R(x^*))\hat{h}'_R(x^*) + f_2(x^*, \hat{h}_R(x^*)) = S_{0R}(x^*). \end{aligned} \quad (1.28)$$

If the functions are smooth enough, equation (1.27) can be differentiated to find a linear relationship between $\hat{h}''_L(x^*)$ and $\hat{h}''_R(x^*)$ in order to make the bed slope differentiable at the jump

$$S'_{0L}(x^*) = S'_{0R}(x^*).$$

When the function \hat{h}_L is chosen arbitrarily, the bed slope is required to have a continuous first derivative. We can determine the necessary values $\hat{h}_R, \hat{h}'_R, \hat{h}''_R$ at $x = x^*$ from hydraulic jump relation and equations (1.27), (1.28). In particular \hat{h}_R have to always remain positive. The obvious choice for \hat{h}_R is a cubic or higher order polynomial in $(x - x^*)$. The examples in MacDonald (1994) use this form. However, the difficulty with polynomials is that they can be highly oscillatory and hence difficult to control and keep positive. There are many order possible forms for \hat{h}_R . The examples in MacDonald (1995) [5] use sums of exponential functions.

2 Examples

Example 1: A 1km long rectangular channel of width $T = 10\text{m}$ has a discharge of $Q = 20\text{m}^3/\text{s}$. The flow is subcritical at inflow and is subcritical at outflow with depth 0.748409m . The Manning roughness coefficient for the channel is 0.03 .

We have the depth

$$\hat{h}(x) = \left(\frac{4}{g}\right)^{1/3} \left(1 + \frac{1}{2} \exp\left(-16 \left(\frac{x}{1000} - \frac{1}{2}\right)^2\right)\right),$$

and

$$\hat{h}'(x) = -\left(\frac{4}{g}\right)^{1/3} \frac{2}{125} \left(\frac{x}{1000} - \frac{1}{2}\right) \exp\left(-16 \left(\frac{x}{1000} - \frac{1}{2}\right)^2\right).$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4n^2}{\hat{h}(x)^{10/3}},$$

or the bed slope with Darcy - Weisbach friction

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4f}{8g\hat{h}(x)^3},$$

where $n = 0.033$ and $f = 0.093$.

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figures 4 and 5.

Example 2: A 1km long rectangular channel of width $T = 10\text{m}$ has a discharge of $Q = 20\text{m}^3/\text{s}$. The flow is supercritical at inflow with depth 0.741599m and is supercritical at outflow. The Manning roughness coefficient for the channel is 0.02 .

We have the depth

$$\hat{h}(x) = \left(\frac{4}{g}\right)^{1/3} \left(1 - \frac{1}{5} \exp\left(-36 \left(\frac{x}{1000} - \frac{1}{2}\right)^2\right)\right),$$

and

$$\hat{h}'(x) = \left(\frac{4}{g}\right)^{1/3} \frac{9}{625} \left(\frac{x}{1000} - \frac{1}{2}\right) \exp\left(-36 \left(\frac{x}{1000} - \frac{1}{2}\right)^2\right).$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4n^2}{\hat{h}(x)^{10/3}},$$

or the bed slope with Darcy - Weisbach friction

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{f}{2g\hat{h}(x)^3},$$

where $n = 0.0218$ and $f = 0.043$.

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figures 6 and 7.

Example 3: A $1km$ long rectangular channel of width $T = 10m$ has a discharge of $Q = 20m^3/s$. The flow is subcritical at inflow and is supercritical at outflow. The Manning roughness coefficient for the channel is 0.02.

We have the depth

$$\hat{h}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3} \left(1 - \frac{1}{3} \tanh\left(3\left(\frac{x}{1000} - \frac{1}{2}\right)\right)\right) & 0 \leq x \leq 500 \\ \left(\frac{4}{g}\right)^{1/3} \left(1 - \frac{1}{6} \tanh\left(6\left(\frac{x}{1000} - \frac{1}{2}\right)\right)\right) & 500 < x \leq 1000, \end{cases}$$

and

$$\hat{h}'(x) = \begin{cases} -\left(\frac{4}{g}\right)^{1/3} \frac{1}{1000} \left(1 - \tanh\left(3\left(\frac{x}{1000} - \frac{1}{2}\right)\right)\right)^2 & 0 \leq x \leq 500 \\ -\left(\frac{4}{g}\right)^{1/3} \frac{1}{1000} \left(1 - \tanh\left(6\left(\frac{x}{1000} - \frac{1}{2}\right)\right)\right)^2 & 500 < x \leq 1000. \end{cases}$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4n^2}{\hat{h}(x)^{10/3}},$$

or the bed slope with Darcy - Weisbach friction

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{f}{2g\hat{h}(x)^3},$$

where $n = 0.0218$ and $f = 0.042$.

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figures 8 and 9.

Example 4: A $1km$ long rectangular channel of width $T = 10m$ has a discharge of $Q = 20m^3/s$. The flow is supercritical at inflow with depth $0.543853m$ and is subcritical at outflow with depth $1.334899m$. The Manning roughness coefficient for the channel is 0.02.

We have the depth

$$\hat{h}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3} \left(\frac{9}{10} - \frac{1}{6} \exp\left(\frac{-x}{250}\right)\right) & 0 \leq x \leq 500 \\ \left(\frac{4}{g}\right)^{1/3} \left(1 + \sum_{k=1}^3 a_k \exp\left(-20k\left(\frac{x}{1000} - \frac{1}{2}\right)\right) + \frac{4}{5} \exp\left(\frac{x}{1000} - 1\right)\right) & 500 < x \leq 1000, \end{cases}$$

and

$$\hat{h}'(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3} \frac{1}{1500} \exp\left(\frac{-x}{250}\right) & 0 \leq x \leq 500 \\ \left(\frac{4}{g}\right)^{1/3} \left(-\frac{1}{50} \sum_{k=1}^3 k a_k \exp\left(-20k\left(\frac{x}{1000} - \frac{1}{2}\right)\right) + \frac{1}{1250} \exp\left(\frac{x}{1000} - 1\right)\right) & 500 < x \leq 1000, \end{cases}$$

with $a_1 = -0.348427$, $a_2 = 0.552264$, $a_3 = -0.555580$.

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4n^2}{\hat{h}(x)^{10/3}},$$

or the bed slope with Darcy - Weisbach friction

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{f}{2g\hat{h}(x)^3},$$

where $n = 0.0218$ and $f = 0.0425$.

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figures 10 and 11.

Example 5: A rectangular channel, $0 \leq x \leq 100m$, has a discharge $q = 2m^2/s$. The flow is supercritical at inflow and supercritical at outflow.

We have the depth

$$\hat{h}(x) = \left(\frac{4}{g}\right)^{1/3} \left(1 - \frac{1}{4} \exp\left(-4\left(\frac{x}{100} - \frac{1}{2}\right)^2\right)\right),$$

and

$$\hat{h}'(x) = \left(\frac{4}{g}\right)^{1/3} \frac{1}{50} \left(\frac{x}{100} - \frac{1}{2}\right) \exp\left(-4\left(\frac{x}{100} - \frac{1}{2}\right)^2\right).$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4n^2}{\hat{h}(x)^{10/3}},$$

where $n = 0.0328$.

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figure 12.

Example 6: A rectangular channel, $0 \leq x \leq 100m$, has a discharge $q = 2m^2/s$. The flow is subcritical at inflow and supercritical at outflow.

We have the depth

$$\hat{h}(x) = \left(\frac{4}{g}\right)^{1/3} \left(1 - \frac{(x-50)}{200} + \frac{(x-50)^2}{30000}\right),$$

and

$$\hat{h}'(x) = \left(\frac{4}{g}\right)^{1/3} \left(-\frac{1}{200} + \frac{(x-50)}{15000}\right).$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4n^2}{\hat{h}(x)^{10/3}},$$

where $n = 0.0328$.

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figure 13.

Example 7: A rectangular channel, $0 \leq x \leq 100m$, has a discharge of $q = 2m^3/s$. The flow is subcritical at outflow and subcritical at inflow.

We have the depth

$$\hat{h}(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3} \left(\frac{4}{3} - \frac{x}{100}\right) - \frac{9x}{1000} \left(\frac{x}{100} - \frac{2}{3}\right) & x \leq \frac{200}{3} \\ \left(\frac{4}{g}\right)^{1/3} \left(0.674202 \left(\frac{x}{100} - \frac{2}{3}\right)^4 + 0.674202 \left(\frac{x}{100} - \frac{2}{3}\right)^3 - \right. \\ \left. -21.7112 \left(\frac{x}{100} - \frac{2}{3}\right)^2 + 14.492 \left(\frac{x}{100} - \frac{2}{3}\right) + 1.4305\right) & x > \frac{200}{3}, \end{cases}$$

and

$$\hat{h}'(x) = \begin{cases} \left(\frac{4}{g}\right)^{1/3} \left(\frac{-1}{100}\right) - \frac{9}{500} \left(\frac{x}{100} - \frac{1}{3}\right) & x \leq \frac{200}{3} \\ \left(\frac{4}{g}\right)^{1/3} \left(0.02696808 \left(\frac{x}{100} - \frac{2}{3}\right)^3 + 0.02022606 \left(\frac{x}{100} - \frac{2}{3}\right)^2 - \right. \\ \left. -0.434224 \left(\frac{x}{100} - \frac{2}{3}\right) + 0.14492\right) & x > \frac{200}{3}. \end{cases}$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4n^2}{\hat{h}(x)^{10/3}},$$

where $n = 0.0328$.

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figure 14.

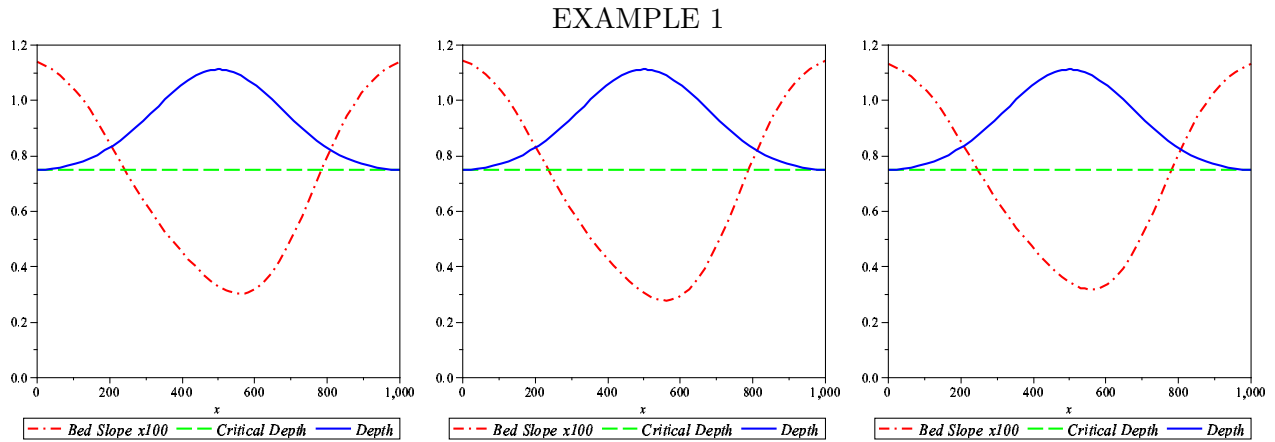


Figure 4: Depth and Bed Slope for Example 1, respectively results in 2D ($n = 0.03$), in 1D with Manning friction ($n = 0.033$) and in 1D with Darcy Weisbach friction ($f = 0.093$) from left to right.

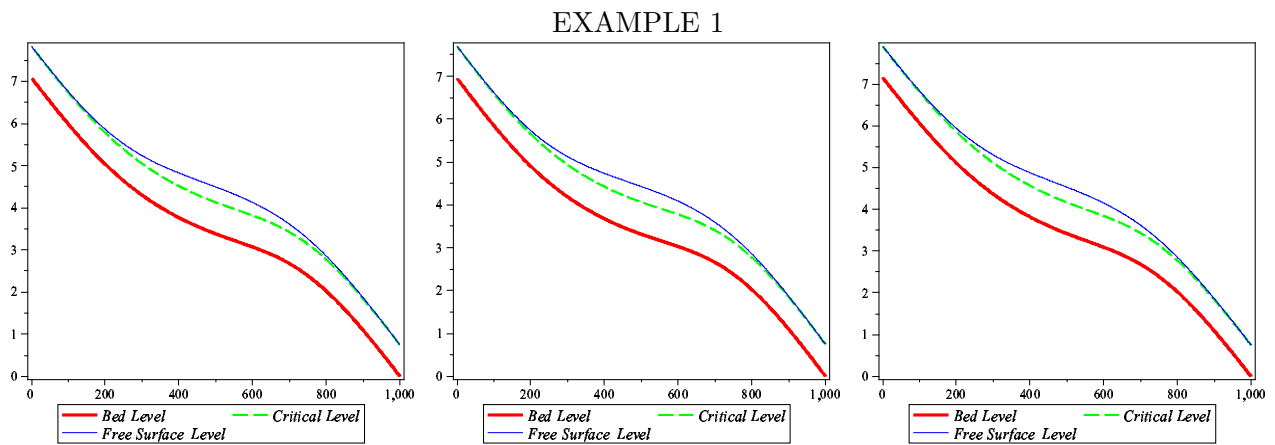


Figure 5: Surface Level and Bed Level for Example 1, respectively results in 2D ($n = 0.03$), in 1D with Manning friction ($n = 0.033$) and in 1D with Darcy Weisbach friction ($f = 0.093$) from left to right.

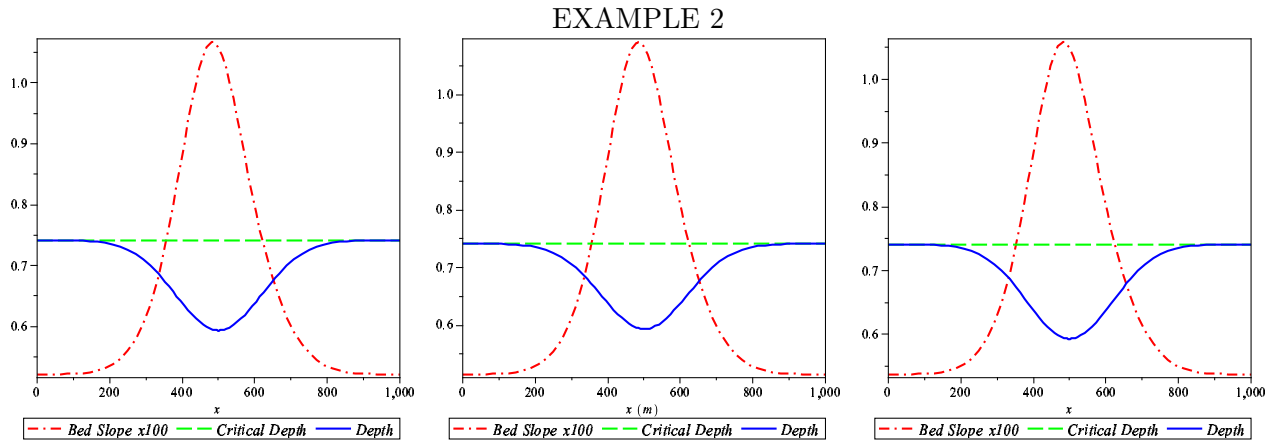


Figure 6: Depth and Bed Slope for Example 2, respectively results in 2D ($n = 0.02$), in 1D with Manning friction ($n = 0.0218$) and in 1D with Darcy Weisbach friction ($f = 0.043$) from left to right.

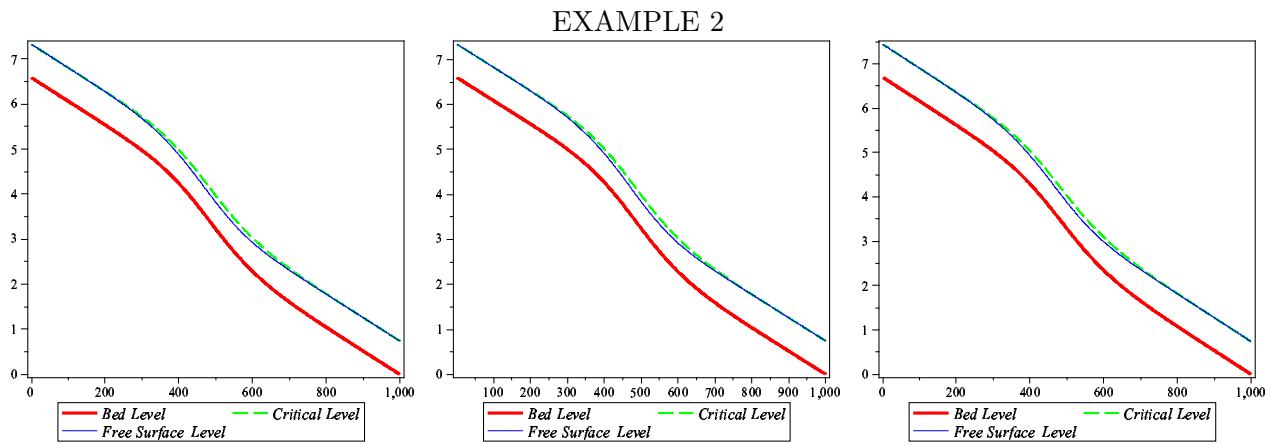


Figure 7: Surface Level and Bed Level for Example 2, respectively results in 2D ($n = 0.02$), in 1D with Manning friction ($n = 0.0218$) and in 1D with Darcy Weisbach friction ($f = 0.043$) from left to right.

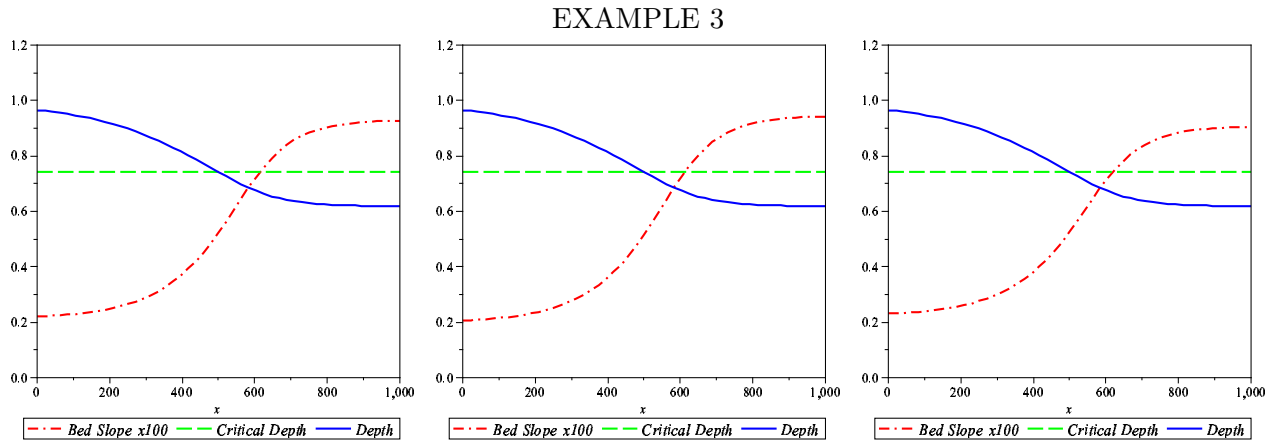


Figure 8: Depth and Bed Slope for Example 3, respectively results in 2D ($n = 0.02$), in 1D with Manning friction ($n = 0.0218$) and in 1D with Darcy Weisbach friction ($f = 0.042$) from left to right.

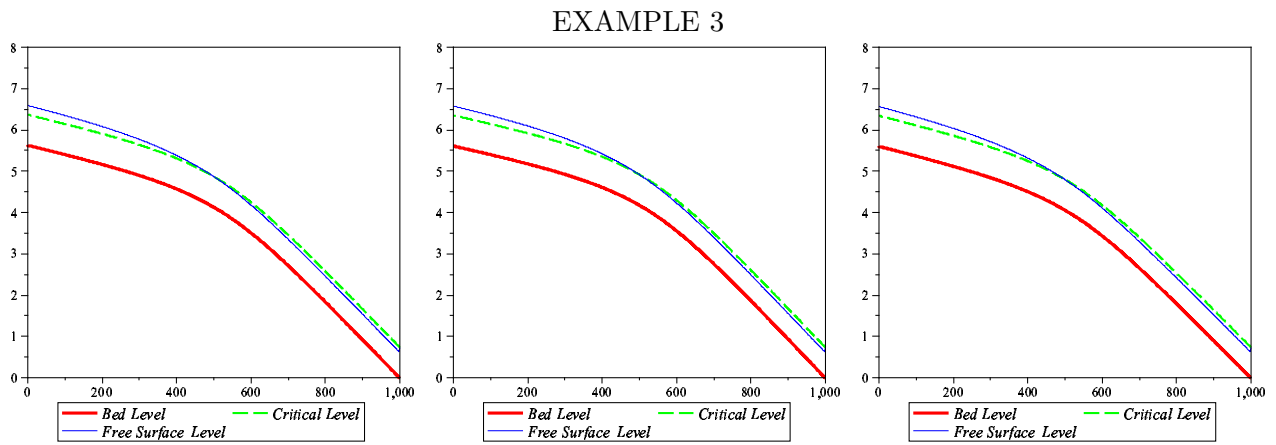


Figure 9: Surface Level and Bed Level for Example 3, respectively results in 2D ($n = 0.02$), in 1D with Manning friction ($n = 0.0218$) and in 1D with Darcy Weisbach friction ($f = 0.042$) from left to right.

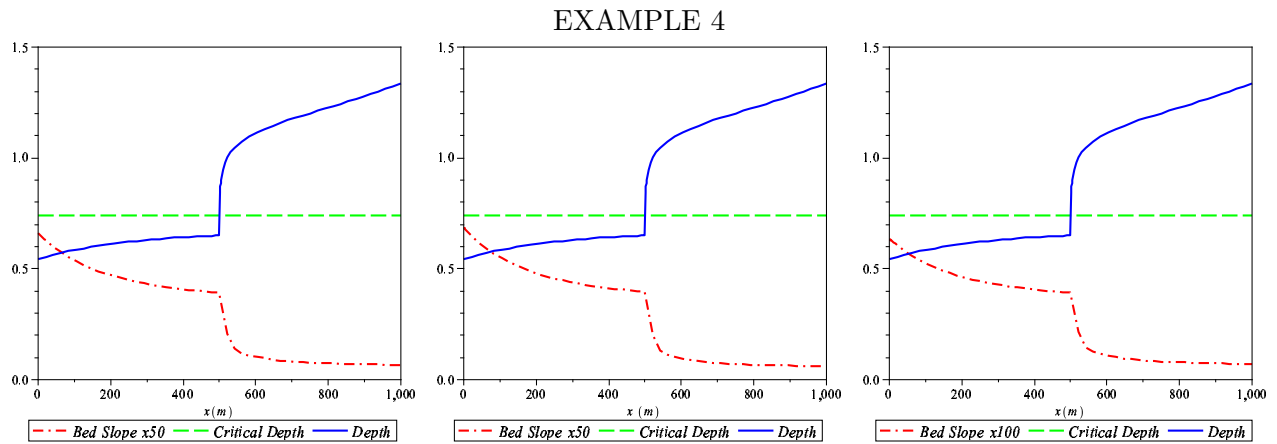


Figure 10: Depth and Bed Slope for Example 4, respectively results in 2D ($n = 0.02$), in 1D with Manning friction ($n = 0.0218$) and in 1D with Darcy Weisbach friction ($f = 0.0425$) from left to right.

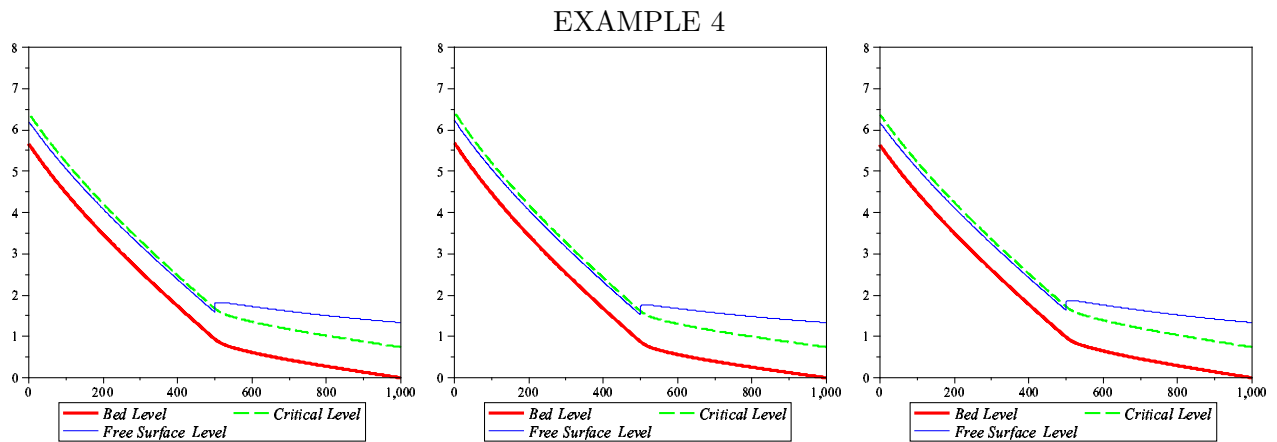


Figure 11: Surface Level and Bed Level for Example 4, respectively results in 2D ($n = 0.02$), in 1D with Manning friction ($n = 0.0218$) and in 1D with Darcy Weisbach friction ($f = 0.0425$) from left to right.

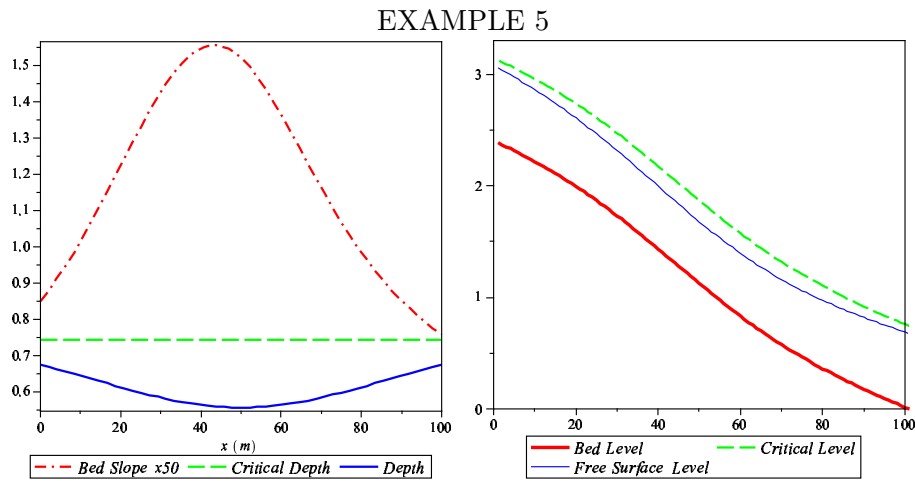


Figure 12: Bed Slope and Bed Level for Example 5, the result is given in 1D with Manning friction $n = 0.0328$.

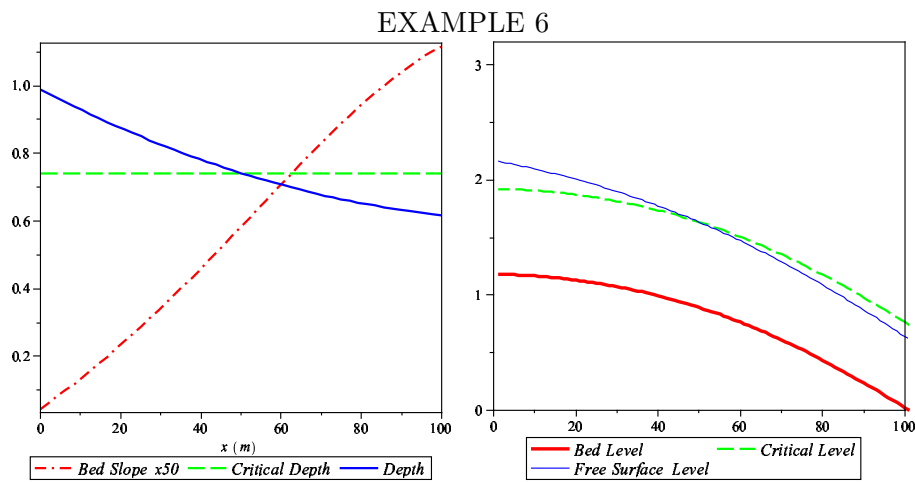


Figure 13: Bed Slope and Bed Level for Example 6, the result is given in 1D with Manning friction $n = 0.0328$.

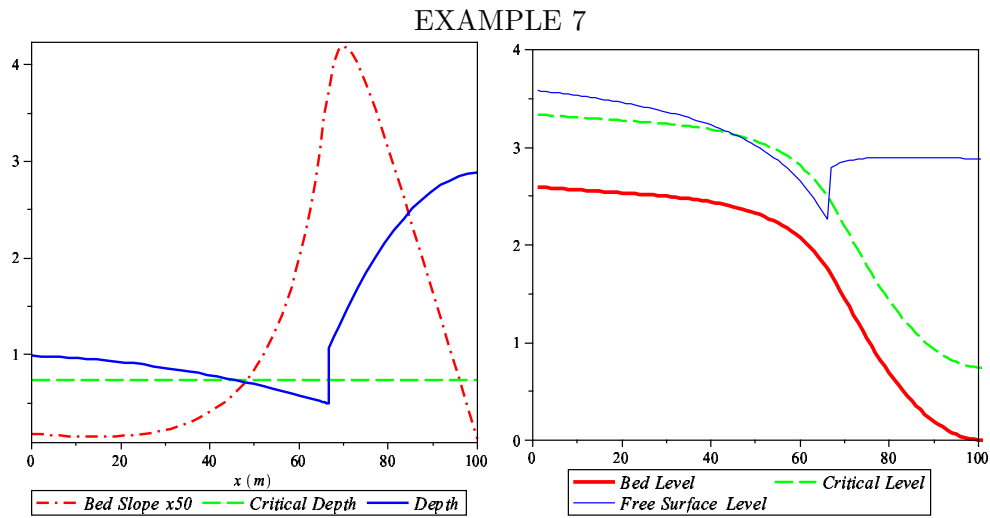


Figure 14: Bed Slope and Bed Level for Example 7, the result is given in 1D with Manning friction $n = 0.0328$.

2.1 Manning friction in 1D and 2D

When we use Manning coefficient n in 1D and in 2D are the same, the result of bed level in 1D is smaller than in 2D. Because formulars of friction slope S_f in 1D and 2D are different. We can see the solution for example 1, 2, 3 and 4 in Figures 15, 16, 17 and 18. Where the left figure is bed level in 1D and in 2D, the right figure is the error that compare between two results of bed level in 1D and 2D.

Besides, Figures 19 and 20 also show error between results of bed level in 1D and 2D for example 1, 5, 6 and 7 when using some values of n which is approximated from equaton (1.16). We see that the value $n = 0.0333$ is better for example 1, but $n = 0.0328$ is better for example 5, 6 and 7. Similarly, Figures 21 and 22 show error between results of bed level in 1D and in 2D for example 2, 3 and 4 . We see that the value $n = 0.0217$ is better for example 3, but $n = 0.0218$ is better for example 2 and 4.

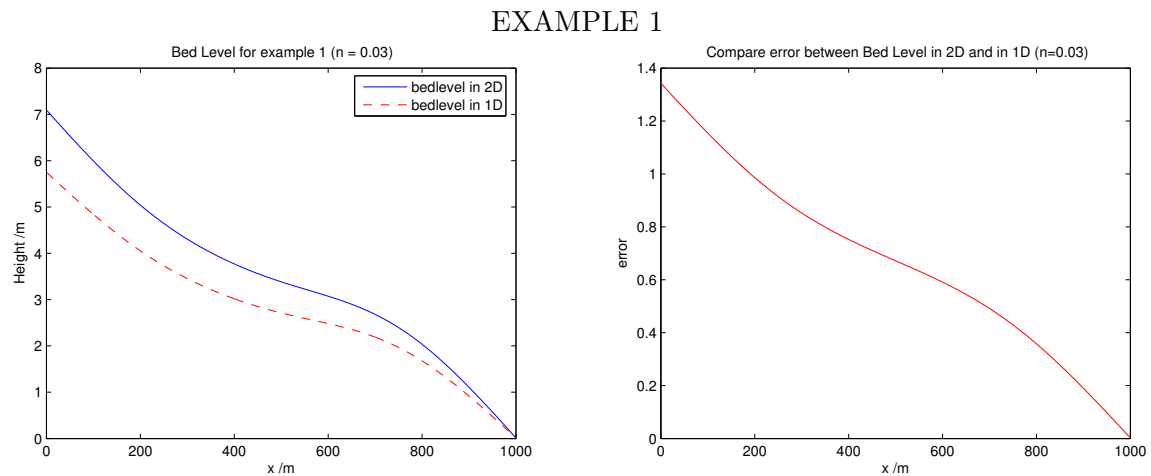


Figure 15: Bed Level for Example 1 with Manning coefficient $n = 0.03$, respectively results in 2D: $z(0) = 7.07287m$ and in 1D: $z(0) = 5.73402m$.

EXAMPLE 2

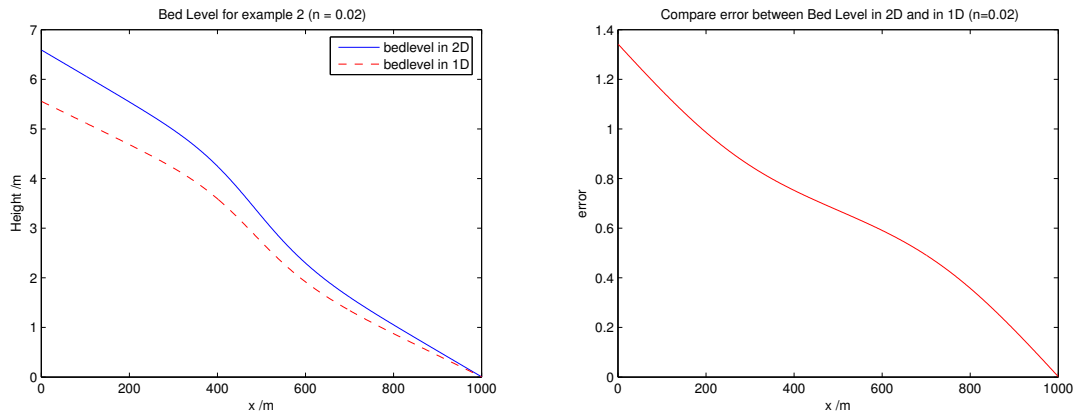


Figure 16: Bed Level for Example 2 with Manning coefficient $n = 0.02$, respectively results in 2D: $z(0) = 6.58152m$ and in 1D: $z(0) = 5.54832m$.

EXAMPLE 3

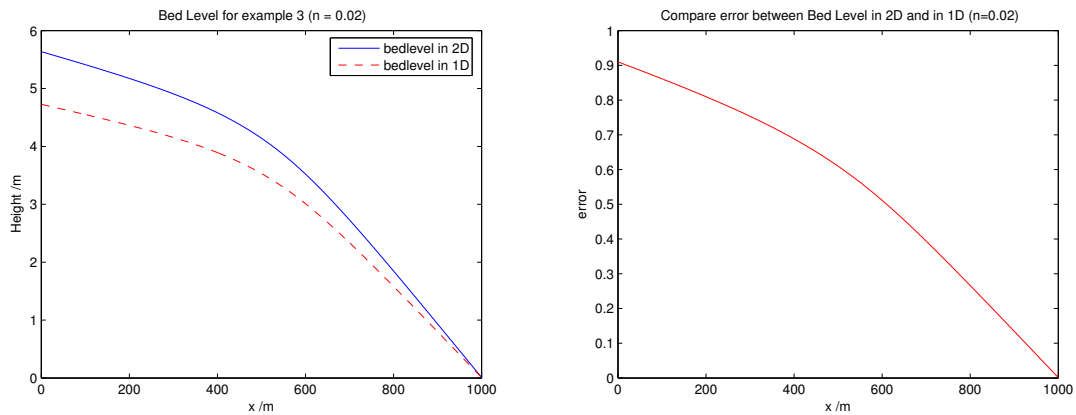


Figure 17: Bed Level for Example 3 with Manning coefficient $n = 0.02$, respectively results in 2D: $z(0) = 5.6245m$ and in 1D: $z(0) = 4.71637m$.

EXAMPLE 4

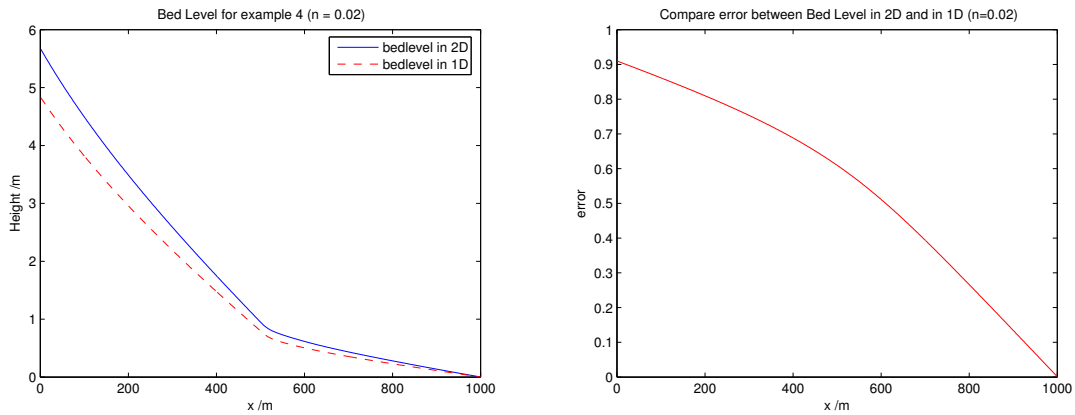


Figure 18: Bed Level for Example 4 with Manning coefficient $n = 0.02$, respectively results in 2D: $z(0) = 5.65672m$ and in 1D: $z(0) = 4.82113m$.

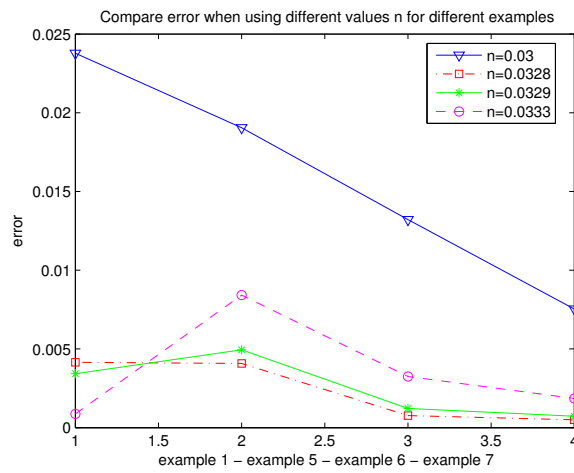


Figure 19: Compare errors between the results of bed level in 1D and 2D when using different Manning coefficients for example 1, 5, 6 and 7.

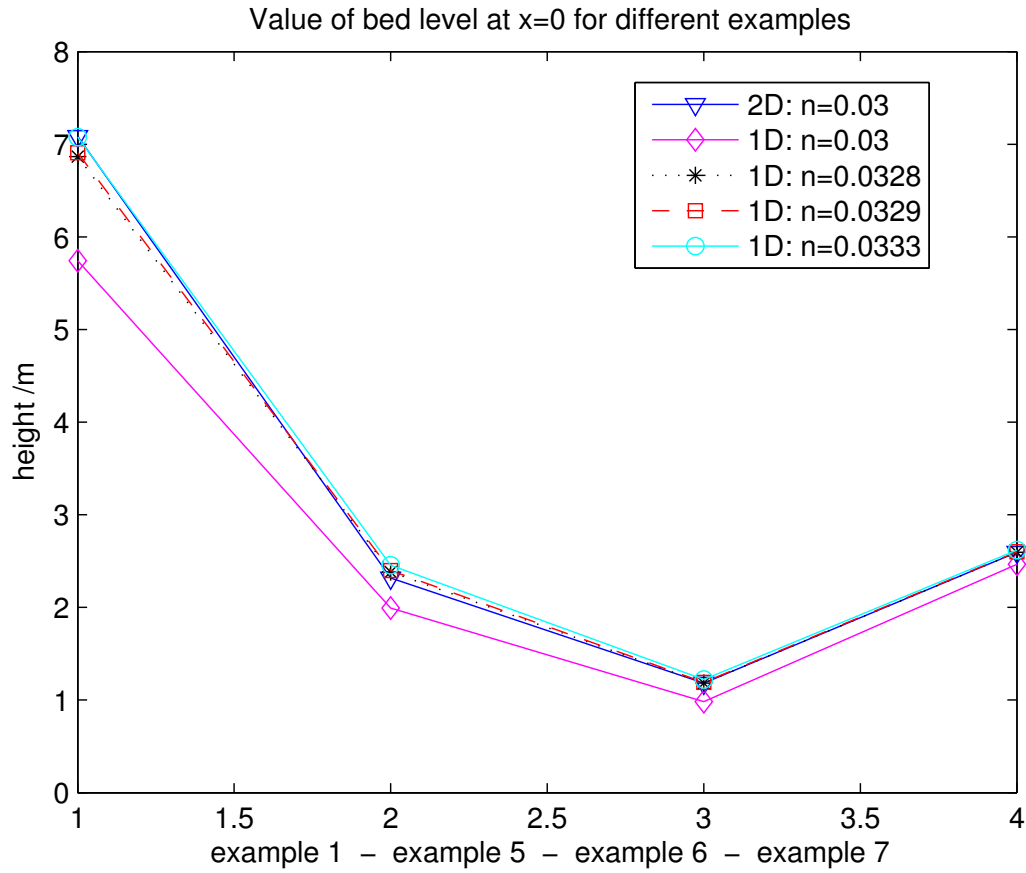


Figure 20: Compare values of bed level at $x = 0$ when using different Manning coefficients for example 1, 5, 6 and 7.

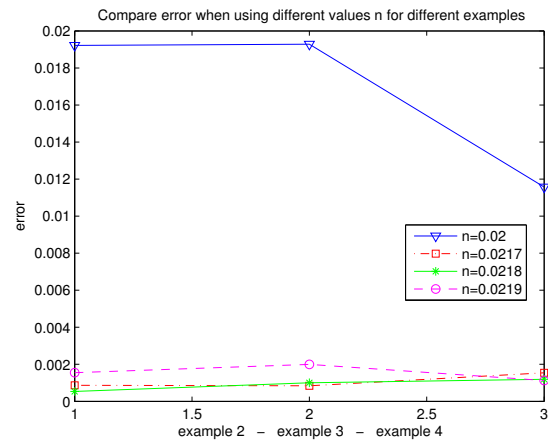


Figure 21: Compare errors between the results of bed level in 1D and 2D when using different Manning coefficients for example 2, 3 and 4.

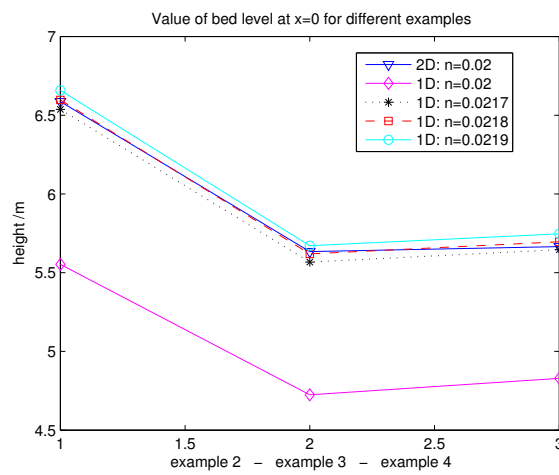


Figure 22: Compare values of bed level at $x = 0$ when using different Manning coefficients for example 2, 3 and 4.

2.2 Manning and Darcy - Weisbach friction

Now we will compare results when using Manning friction and Darcy - Weisbach friction. Using the same values of Manning coefficient $n = 0.0328$ and Darcy - Weisbach coefficient $f = 0.095$ for example 1, 5, 6 and 7. We see that Darcy - Weisbach friction is better for some examples, but Manning friction is better for another examples. However, when the value of n is fixed, we can choose some values of f to have smaller error when using Darcy - Weisbach friction. The results are shown in Figures from 23 to 27.

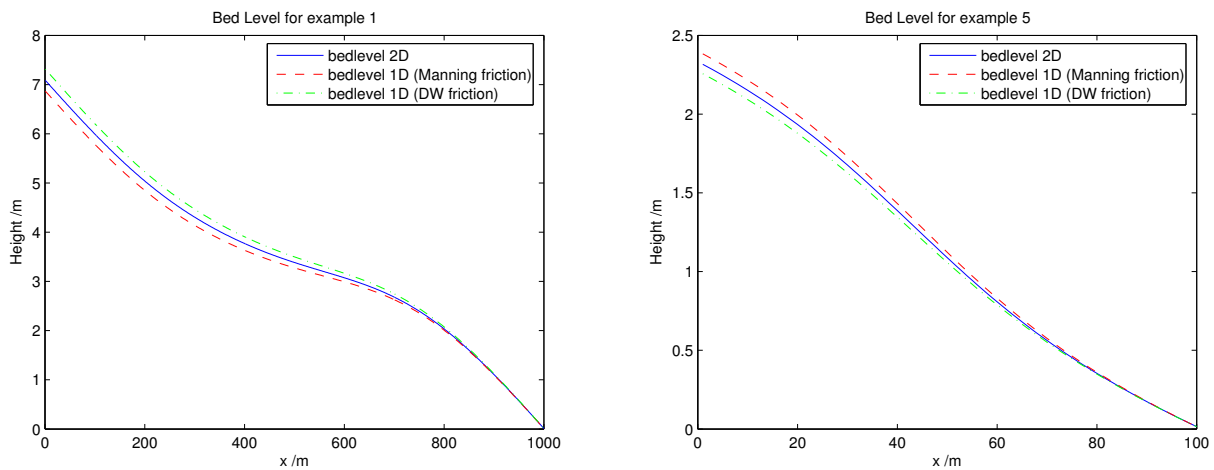


Figure 23: Bed Level for Example 1 and Example 5 with Manning coefficient $n = 0.0328$ and Darcy - Weisbach coefficient $f = 0.095$.

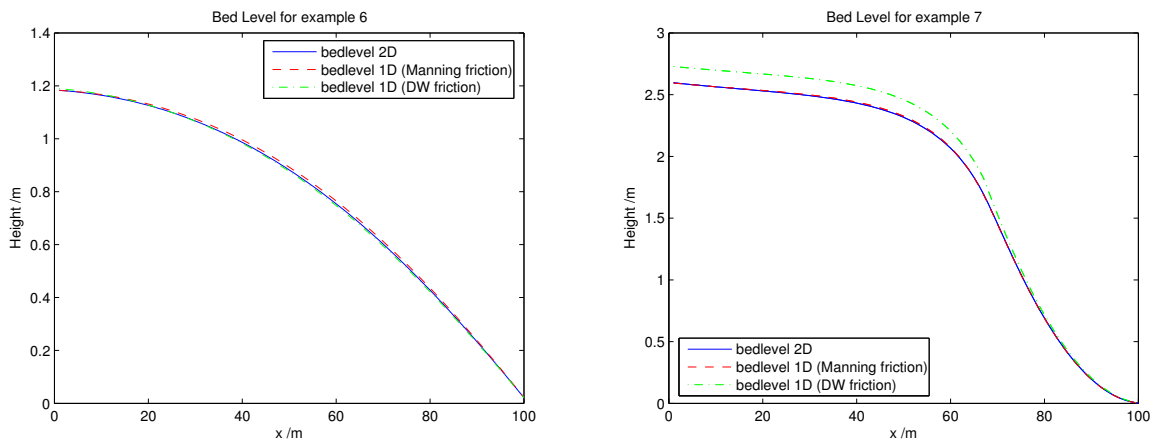


Figure 24: Bed Level for Example 6 and Example 7 with Manning coefficient $n = 0.0328$ and Darcy - Weisbach coefficient $f = 0.095$.

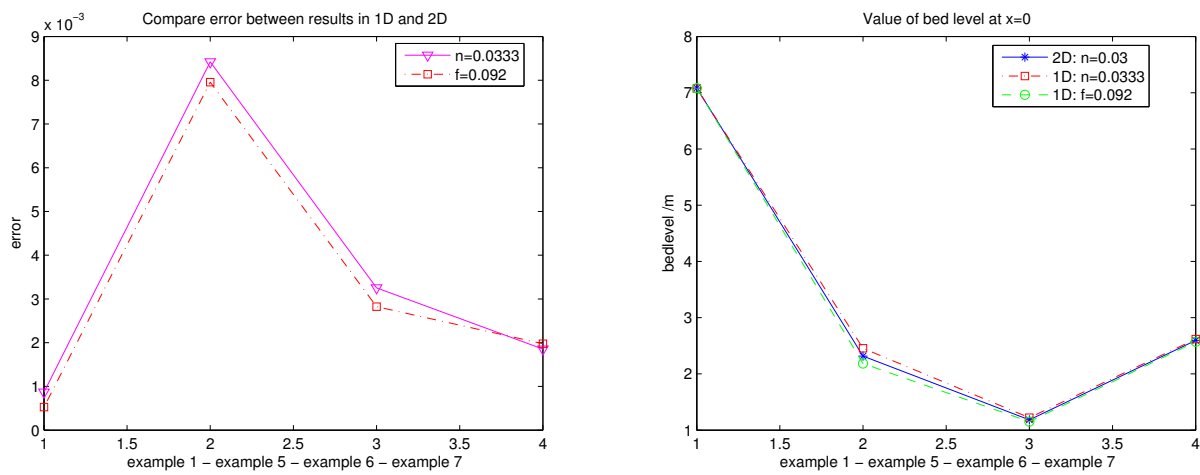


Figure 25: Check the results when using the same Manning coefficient $n = 0.0333$ and Darcy - Weisbach coefficient $f = 0.092$ for example 1, 5, 6, 7.

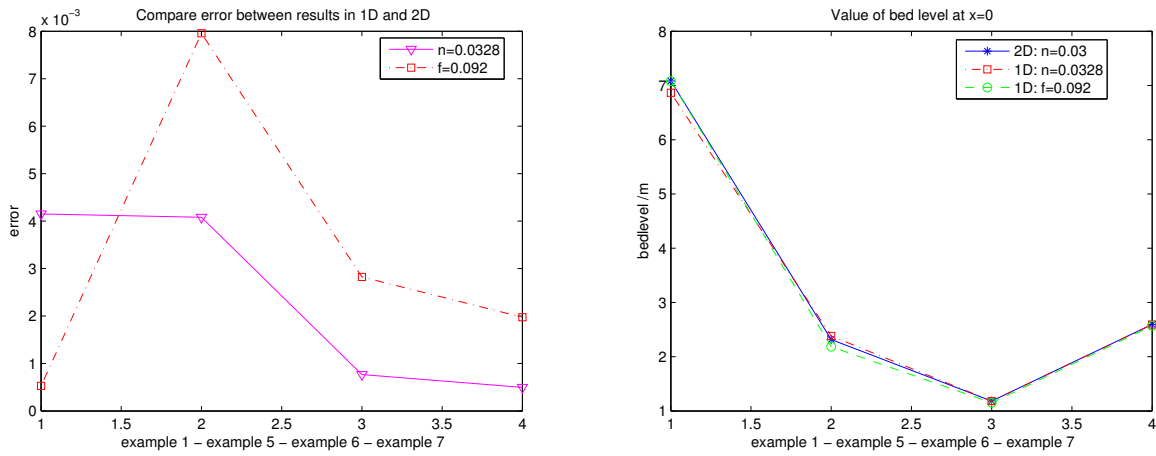


Figure 26: Check the results when using the same Manning coefficient $n = 0.0328$ and Darcy - Weisbach coefficient $f = 0.092$ for example 1, 5, 6, 7.

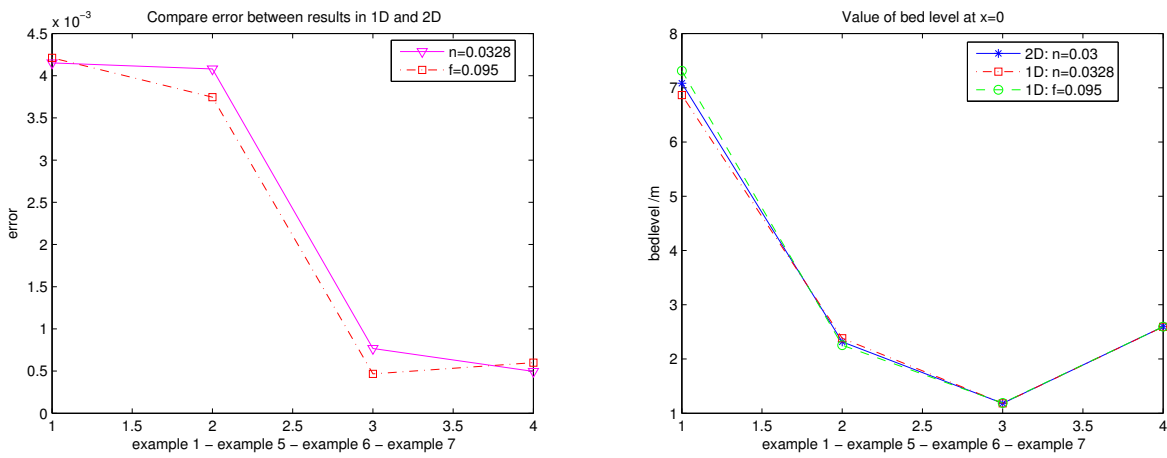


Figure 27: Check the results when using the same Manning coefficient $n = 0.0328$ and Darcy - Weisbach coefficient $f = 0.095$ for example 1, 5, 6, 7.

2.3 Rain

The results of bed slope and bed level will change when it is rain $R \neq 0$.

For instance, when $R = 0.001m/s$, the discharge will be $q = Rx + q_0$ with $q_0 = Q/T = 2m^2/s$. The bed slope in example 1 and 2 are given by

$$S_0 = \left(1 - \frac{(Rx + 2)^2}{gh^3}\right) h'(x) + \frac{2(Rx + 2)R}{gh^2} + \frac{n^2(Rx + 2)^2}{h^{10/3}}.$$

The results are shown in Figures 28 and 29.

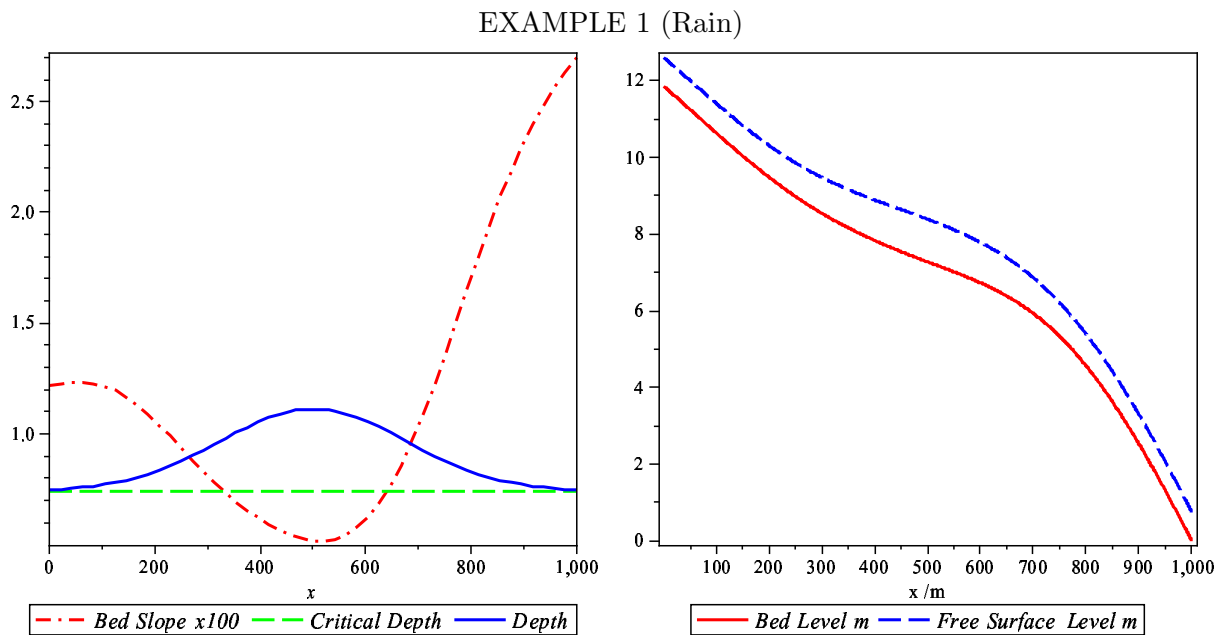


Figure 28: Bed Slope and Bed Level for Example 1 with Rain ($R = 0.001m/s$), the result is given in 1D with Manning friction $n = 0.033$.

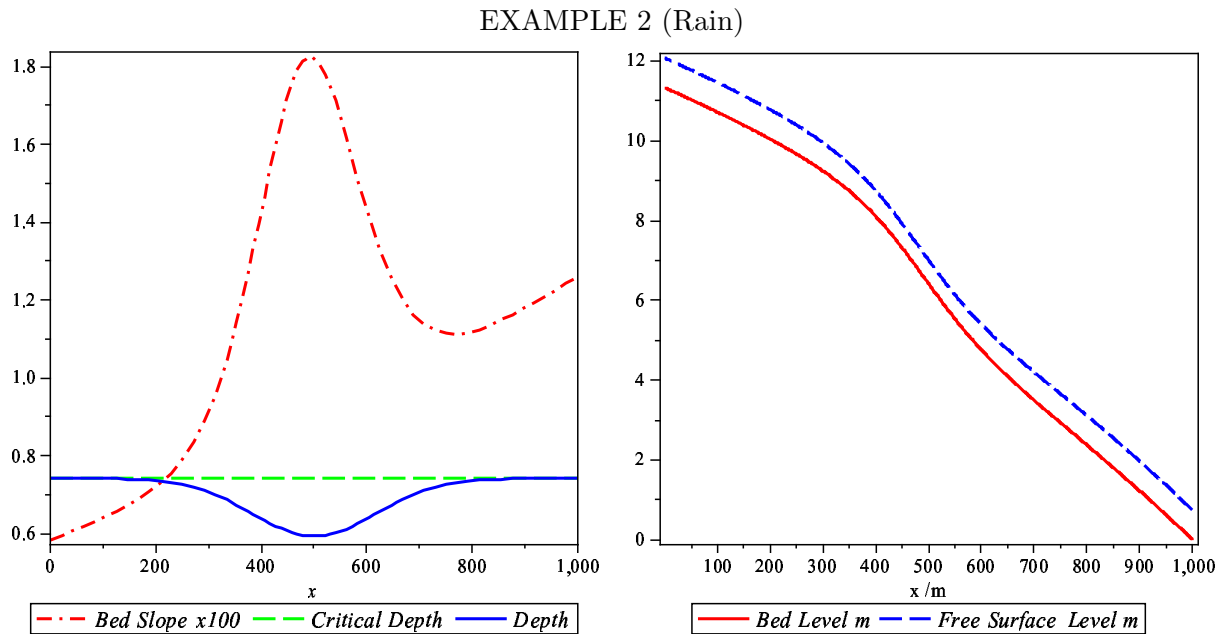


Figure 29: Bed Slope and Bed Level for Example 2 with Rain ($R = 0.001m/s$), the result is given in 1D with Manning friction $n = 0.0217$.

2.4 Depth is periodic functions

Example 8: A 5km long rectangular channel of width $T = 10m$ has a discharge of $Q = 20m^3/s$. The flow is subcritical at inflow and is subcritical at outflow with depth 1.125m. The Manning roughness coefficient for the channel is 0.03.

We have the depth

$$\hat{h}(x) = \frac{9}{8} + \frac{1}{4} \sin\left(\frac{\pi x}{500}\right),$$

and

$$\hat{h}'(x) = \frac{\pi}{2000} \cos\left(\frac{\pi x}{500}\right).$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3}\right) \hat{h}'(x) + \frac{4n^2}{\hat{h}(x)^{10/3}},$$

for the case raining with $R = 0.001m/s$ or $R = -0.001m/s$

$$S_0 = \left(1 - \frac{(Rx + 2)^2}{g\hat{h}^3}\right) \hat{h}'(x) + \frac{2(Rx + 2)R}{g\hat{h}^2} + \frac{n^2(Rx + 2)^2}{\hat{h}^{10/3}}.$$

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figures 30, 31, 32.

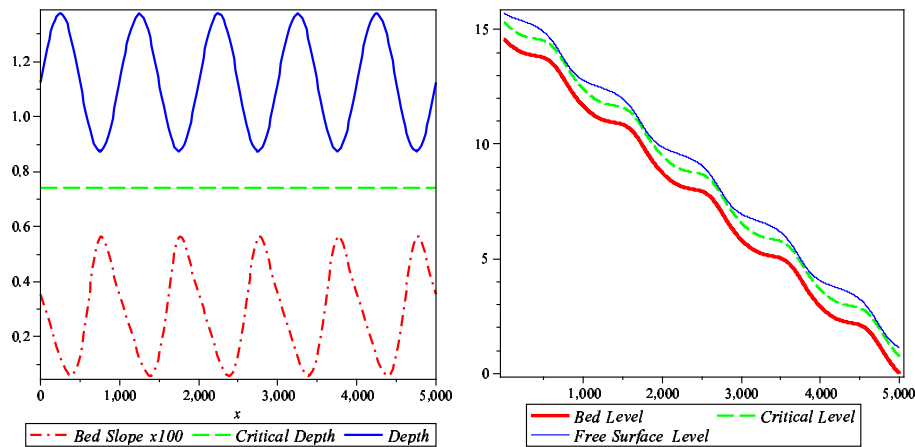


Figure 30: Bed Slope and Bed Level for Example 8, the result is given in 1D with Manning coefficient $n = 0.03$.

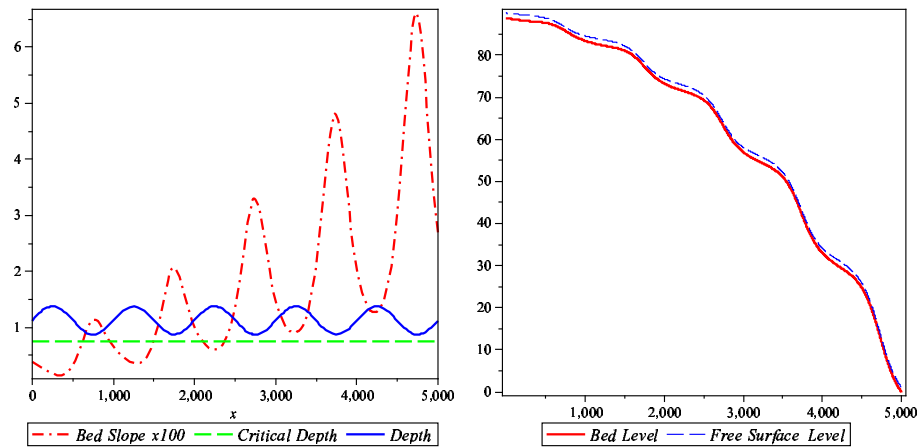


Figure 31: Bed Slope and Bed Level for Example 8 with Rain $R = 10^{-3}$ m/s, the result is given in 1D with Manning coefficient $n = 0.03$.

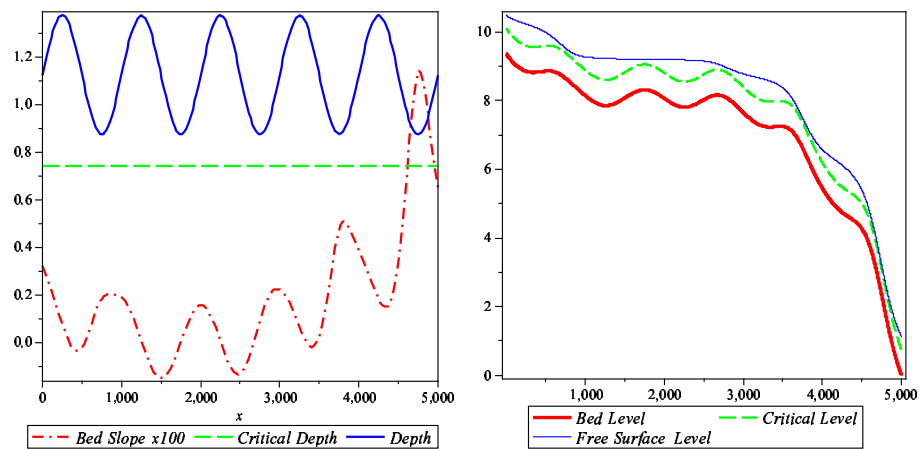


Figure 32: Bed Slope and Bed Level for Example 8 with Rain $R = -10^{-3}$ m/s, the result is given in 1D with Manning coefficient $n = 0.03$.

2.5 Example with small depth

Example 9: A rectangular channel of length $L = 4m$ has a discharge $q = 4 * 10^{-5}m^2/s$. The flow is subcritical at inflow and is subcritical at outflow. The Manning roughness coefficient for the channel is 0.03.

We have the depth

$$\hat{h}(x) = \frac{1}{500} \left(\frac{4}{g} \right)^{1/3} \left(1 + \frac{1}{2} \exp \left(-16 \left(\frac{x}{4} - \frac{1}{2} \right)^2 \right) \right),$$

and

$$\hat{h}'(x) = \left(\frac{4}{g} \right)^{1/3} \frac{1}{1000} (-2x + 4) \exp \left(-16 \left(\frac{x}{4} - \frac{1}{2} \right)^2 \right).$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3} \right) \hat{h}'(x) + \frac{16 \cdot 10^{-10} n^2}{\hat{h}(x)^{10/3}},$$

for the case raining with $R = 10^{-5}m/s$ or $R = -10^{-5}m/s$

$$S_0 = \left(1 - \frac{(Rx + 4 * 10^{-5})^2}{g\hat{h}^3} \right) \hat{h}'(x) + \frac{2(Rx + 4 * 10^{-5})R}{g\hat{h}^2} + \frac{n^2(Rx + 4 * 10^{-5})^2}{\hat{h}^{10/3}}.$$

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figures 33, 34 and 35.

Example 10: A rectangular channel of length $L = 4m$ has a discharge $q = 4 * 10^{-5}m^2/s$. The flow is supercritical at inflow and is supercritical at outflow. The Manning roughness coefficient for the channel is 0.02. We have the depth

$$\hat{h}(x) = \frac{1}{500} \left(\frac{4}{g} \right)^{1/3} \left(1 - \frac{1}{5} \exp \left(-36 \left(\frac{x}{4} - \frac{1}{2} \right)^2 \right) \right),$$

and

$$\hat{h}'(x) = - \left(\frac{4}{g} \right)^{1/3} \frac{1}{1000} \left(-\frac{9}{2}x + 9 \right) \exp \left(-36 \left(\frac{x}{4} - \frac{1}{2} \right)^2 \right).$$

Then the bed slope with Manning friction is given by

$$S_0(x) = \left(1 - \frac{4}{g\hat{h}(x)^3} \right) \hat{h}'(x) + \frac{16 \cdot 10^{-10} n^2}{\hat{h}(x)^{10/3}},$$

for the case raining with $R = 10^{-5}m/s$ or $R = -10^{-5}m/s$

$$S_0 = \left(1 - \frac{(Rx + 4 * 10^{-5})^2}{g\hat{h}^3} \right) \hat{h}'(x) + \frac{2(Rx + 4 * 10^{-5})R}{g\hat{h}^2} + \frac{n^2(Rx + 4 * 10^{-5})^2}{\hat{h}^{10/3}}.$$

The solution for this problem is given by $h(x) \equiv \hat{h}(x)$ and is shown in Figures 36, 37 and 38.

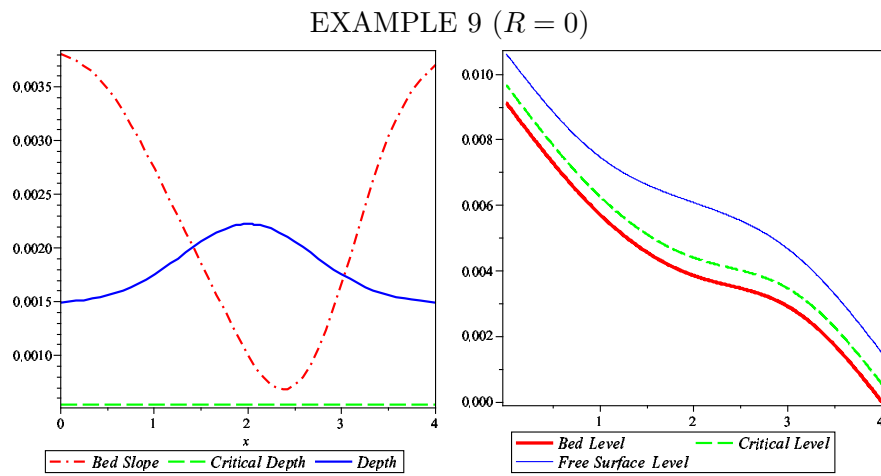


Figure 33: Bed Slope and Bed Level for Example 9, the result is given in 1D with Manning friction $n = 0.03$.

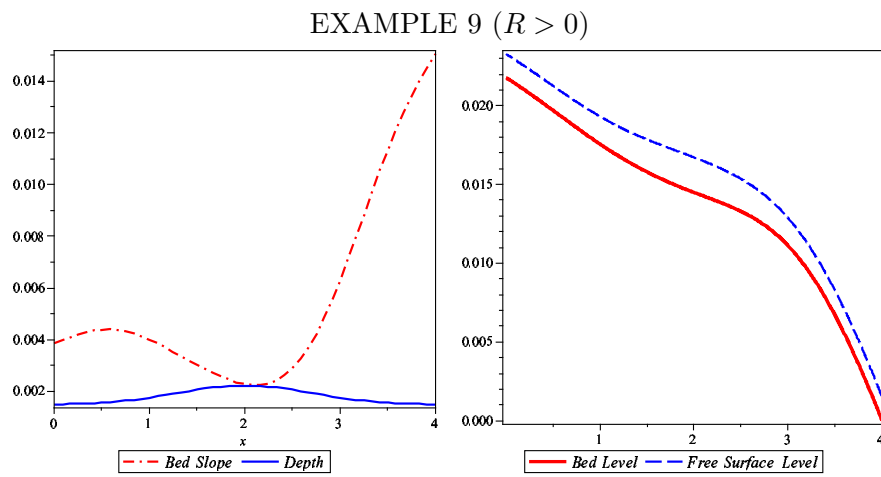


Figure 34: Bed Slope and Bed Level for Example 9 with Rain $R = 10^{-5} m/s$, the result is given in 1D with Manning friction $n = 0.03$.

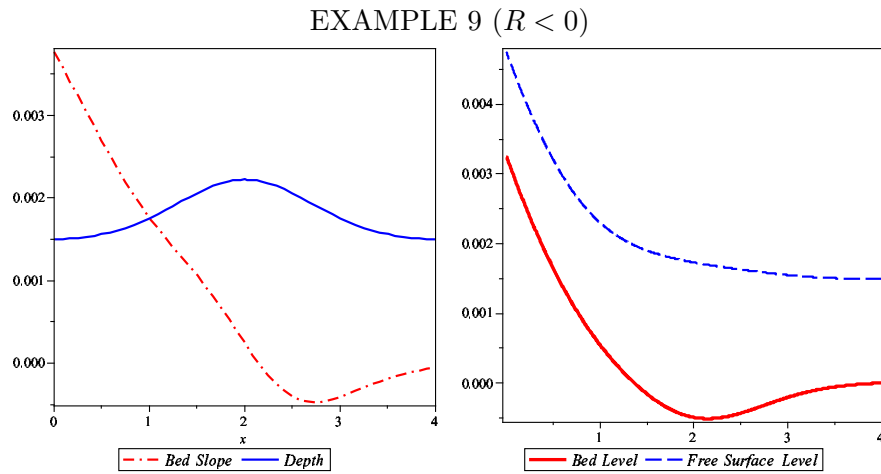


Figure 35: Bed Slope and Bed Level for Example 10 with Rain $R = -10^{-5}m/s$, the result is given in 1D with Manning friction $n = 0.03$.

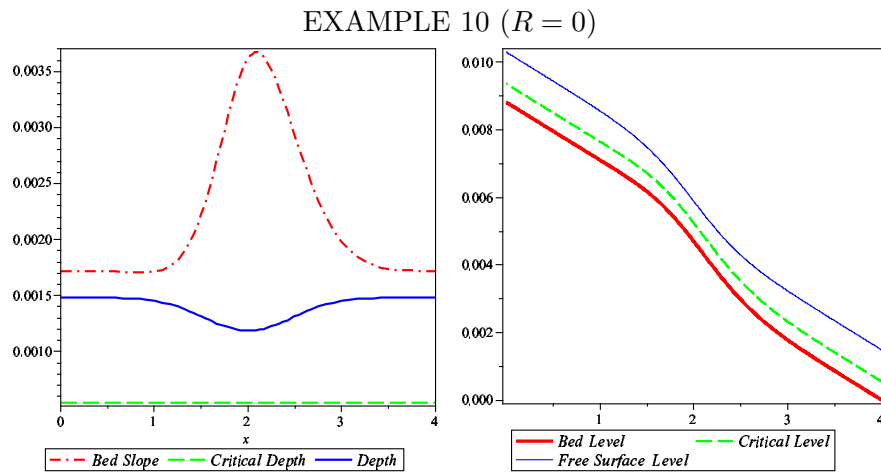


Figure 36: Bed Slope and Bed Level for Example 10, the result is given in 1D with Manning friction $n = 0.03$.

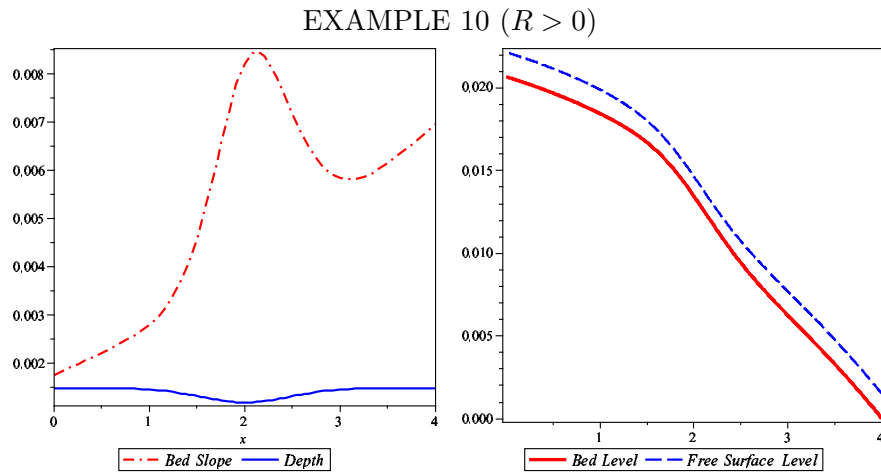


Figure 37: Bed Slope and Bed Level for Example 10 with Rain $R = 10^{-5}m/s$, the result is given in 1D with Manning friction $n = 0.03$.

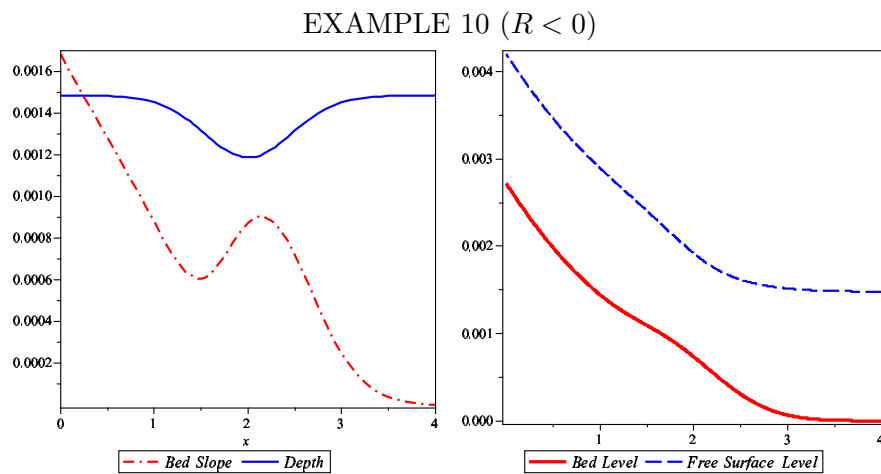


Figure 38: Bed Slope and Bed Level for Example 10 with Rain $R = -10^{-5}m/s$, the result is given in 1D with Manning friction $n = 0.03$.

2.6 Bed Slope is constant

When the bed slope $S_0 = -\partial_x z$ is constant, the bed level is a line. If S_0 is given, we have to find a function $h(x)$ such that the value of S_0 that is computed from it is a constant. We can use numerical method to find $h(x)$ from equation (1.19) or (1.20) when $q, \partial_x z, n$ or f and $h(0)$ is given.

For example, we use a uniform grid, $x_i = i\Delta x, i = 0, 1, \dots, N$, of spacing $\Delta x = \frac{L}{N}$, where $N \in \mathbb{Z}^+$. From equation (1.20), we have

$$\left(1 - \frac{q^2}{gh_i^3}\right) \frac{h_{i+1} - h_i}{\Delta x} + \partial_x z + \frac{fq^2}{8gh_i^3} = 0,$$

$i = 0..N$ with $h_0 = h(0)$.

Example 11: A rectangular channel of length $L = 4m$ has a discharge $q = 4 * 10^{-5} m^2/s$. The bed slope is given by $S_0 = -\partial_x z = 0.05$ and the Darcy - Weisbach coefficient $f = 0.04$. The flow is subcritical at inflow with depth $0.01m$ and is subcritical at outflow. We can approximate a function

$$h(x) = 0.0098 + 0.05x.$$

Then, we compute again the values of bed slope S_0 and bed level z . The result is shown in Figures 39 and 40.

Besides, the values of S_0 and z don't depend on choosing values of Manning or Darcy Weisbach coefficient. If it is rain, the bed level also have no change in this case.

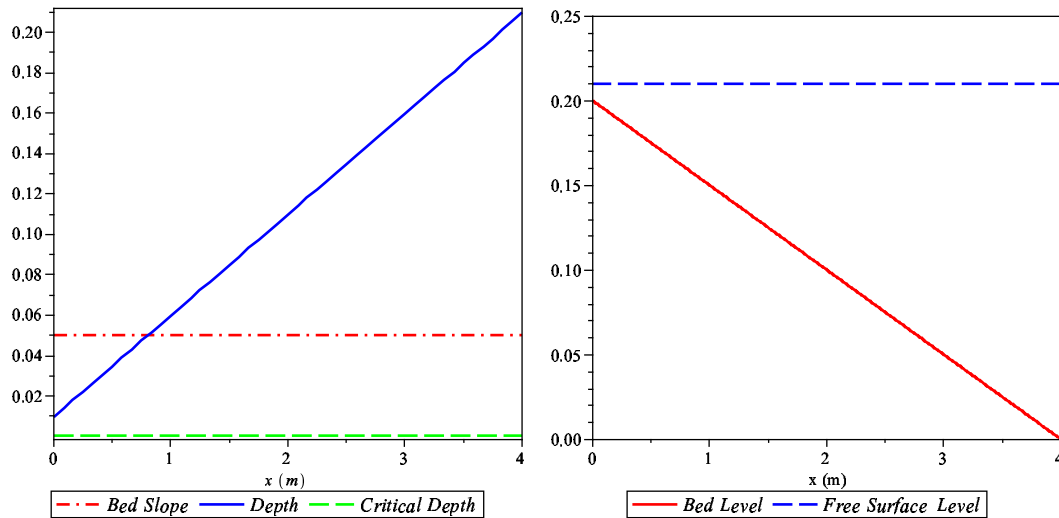


Figure 39: Bed Slope and Bed Level for Example 11, the result is given in 1D with Darcy Weisbach coefficient $f = 0.04$.

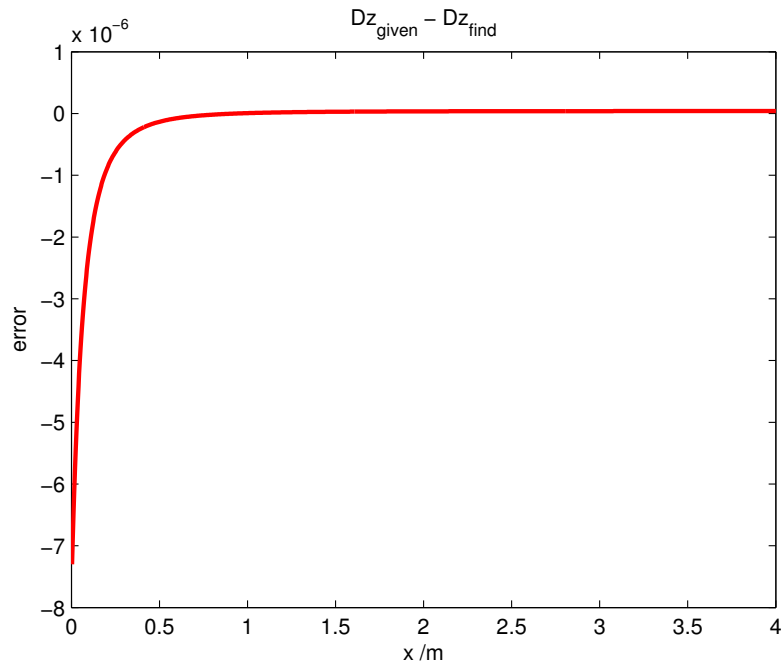


Figure 40: Comparison between values of $\partial_x z = -0.05$ and values of $\partial_x z$ is computed from $h(x)$.

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