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Shuffle analogues for complex reflection groups $G(m, 1, n)$

Varvara Petrova

Internship carried out at Centre de Physique Théorique
2017 March 6 - June 23

under the supervision of
Oleg Ogievetsky

Abstract

Analogues of 1-shuffle elements for complex reflection groups of type $G(m, 1, n)$ are introduced. A geometric interpretation for $G(m, 1, n)$ in terms of « rotational » permutations of polygonal cards is given. We compute the eigenvalues, and their multiplicities, of the 1-shuffle element in the algebra of the group $G(m, 1, n)$. We report on the spectrum of the 1-shuffle analogue in the cyclotomic Hecke algebra $H(m, 1, n)$ for $m = 2$ and small n. Considering shuffling as a random walk on the group $G(m, 1, n)$, we estimate the rate of convergence to randomness of the corresponding Markov chain.

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* * *

List of symbols

$\mathbb{N}$	the set of nonnegative integers $0, 1, 2, 3, \ldots$
$\mathbb{N}^*$	the set of positive integers $1, 2, 3, \ldots$
$[\![1, n]\!]$	the set $\{1, 2, \ldots, n\}$
π_i	transposition $(i, i+1)$ in the symmetric group S_n
$\boldsymbol{\mu}_j$	the group of complex j-th roots of unity
$Ш_n$	reduced notation for shuffle element $Ш_{1,n-1}$
${}^{(m)}Ш_n$	analogue of $Ш_n$ for complex reflection group $G(m, 1, n)$
$L_{Ш_n}$	operator of the left multiplication by $Ш_n$
$G \wr S_n$	the wreath product of the group G by the symmetric group S_n
$\tilde{\Sigma}_G$	sum of all elements of the group G

Здесь стояли койки, устланные
ворсистыми, как собачья шерсть,
одеялами, с одной стороны
которых фабричным способом
было выткано слово “Ноги”.
Илья Ильф, Евгений Петров,
“Двенадцать стульев”.

Introduction

In 1988, N. Wallach considered an element of the group algebra of the symmetric group S_n which is the sum of cycles $(12\dots i)$ where i ranges from 1 to n. He discovered in [43] that the operator of the left multiplication by this element in the group algebra $\mathbb{Z}S_n$ is diagonalizable with eigenvalues

$$0, 1, 2, \dots, n-2, n. \tag{1}$$

The sum of cycles $(12\dots i)$ appears in different circumstances and is called 1-shuffle element[1], $Ш_{1,n-1}$. In particular, it describes all possible ways of removing the top card from the deck of n cards and inserting it back in the deck at a random position.

Investigating the repeated top-to-random shuffling as a random walk on S_n, P. Diaconis et al. [12] (see also R. Phatarfod [37]) found that the multiplicity of the eigenvalue i in (1) is equal to the number of permutations in S_n with i fixed points, explaining the absence of $n-1$ in (1).
The q-deformation of the result of N. Wallach for the q-analogue ${}^{q}Ш_{1,n-1}$ in the Hecke algebra $H_n(q)$ was proposed by G. Lusztig in [24]. He established that the spectrum of the operator $L_{Ш_{1,n-1}}$ of the left multiplication by $Ш_{1,n-1}$ consists of the q-numbers

$$q^{j-1}[j]_q := 1 + q^2 + q^4 + \dots + q^{2j-2}, \quad j = 0, 1, \dots, n-2, n. \tag{2}$$

Later, A. Isaev and O. Ogievetsky considered shuffle elements $Ш_{p,q}$ in the braid group ring $\mathbb{Z}B_n$. With the help of baxterized elements [22], they constructed additive and multiplicative analogues of $Ш_{p,q}$ in Hecke and Birman-Murakami-Wenzl algebras. The multiplicities of the eigenvalues in (2) have been established therein by taking the trace of $L_{Ш_{1,n-1}} : H_n(q) \to H_n(q)$, using the fact that the q-numbers (2) are linearly independent over $\mathbb{Z}$ as polynomials in q. For generic q, the multiplicities turn out to be the same as for the symmetric group.

In the present paper we propose polygonal analogues of cards that we call m-cards. We introduce elements ${}^{(m)}Ш_{1,n-1}$, which realise the analogues of top to random shuffling on m-cards. The elements ${}^{(m)}Ш_{1,n-1}$ belong to the group algebra of complex reflection groups of type $G(m,1,n)$. We adopt the approach of [12] (for details see [17]) and of [22] to compute the spectrum and the multiplicities of the eigenvalues of $L_{{}^{(m)}Ш_{1,n-1}}$. The obtained result for the multiplicities is expressed in terms of the so-called m-derangements numbers. We give a preliminary result on the spectrum of $L_{{}^{(m)}Ш_{1,n-1}}$ in the cyclotomic Hecke algebra $H(m,1,n)$, which is a deformation of the group algebra of the complex reflection group. Asymptotic convergence to the randomness in the shuffling the m-cards is briefly analysed.

[1] The definition of shuffles $Ш_{p,q}$ and their applications will be given in §2.1.

1 Complex reflection groups of type G(m,1,n)

1.1 Generalities about reflection groups

Let V be a $\mathbb{K}$-vector space of dimension n ($\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$).

Definitions. A *hyperplane* in V is a vector subspace $H \subset V$ of dimension $n-1$.
A *pseudo-reflection* (or simply *reflection*) τ is a linear transformation of V of finite order that fixes pointwise the hyperplane H (called the "mirror"). In other words, $\tau \in \mathrm{GL}_n(\mathbb{K})$ is similar to $\mathrm{diag}(1, \ldots, 1, \zeta)$ for ζ a root of unity. When $\mathbb{K} = \mathbb{R}$, the order of a non trivial τ is 2. In this case τ is a usual reflection.
A $\mathbb{K}$-*reflection group* is a finite subgroup of $\mathrm{GL}_n(\mathbb{K})$ that is generated by reflections.

In 1935, Coxeter proved that real reflection groups are exactly those finite groups which admit a presentation

$$\langle s_1, s_2, \ldots, s_n |\ (s_i s_j)^{m_{ij}} = 1,\ 1 \leq i, j \leq n \rangle$$

where $m_{ij} = m_{ji} \in \mathbb{N},\ m_{ii} = 1$ and $m_{ij} \geq 2$ if $i \neq j$. The list of Coxeter groups consists of Weyl groups, exceptional groups H_3, H_4 and the dihedral groups $I_2(p)$ (for $p = 3, 4, 6$, these are Weyl groups). Weyl groups are rational reflection groups ($\mathbb{K} = \mathbb{Q}$). They appear as "skeletons" [7] of many important mathematical objects such as algebraic groups, Hecke algebras, Artin-Tits braid groups, etc. Many of known properties of Weyl groups, and more generally of Coxeter finite groups, can be generalized to complex reflection groups ($\mathbb{K} = \mathbb{C}$) - although in most cases new methods are required.

The complex reflection groups have been classified by Shephard and Todd in 1954. The list consists of:

- an infinite family $G(me, e, n)$ depending on 3 positive integer parameters. $G(me, e, n)$ is the group of all $n \times n$ complex permutation matrices (that have exactly one nonzero entry in each row and column), with entries in $\boldsymbol{\mu}_{me}$, the product of all non-zero entries being in $\boldsymbol{\mu}_m$;
- 34 exceptional groups denoted $G_4, \ldots, G_{37}$.

The group $G(me, e, n)$ is a normal subgroup of index e in $G(me, 1, n)$.
The groups $G(me, e, n)$ give rise to the following real reflection groups (in their matrix representation) [27]:

$$\begin{array}{lcll} G(1,1,n) & = & A_{n-1} & \text{(symmetric groups)}, \\ G(2,1,n) & = & B_n & \text{(hyperoctahedral groups)}, \\ G(2,2,n) & = & D_n & \text{(even signed-permutation groups)}, \\ G(r,r,2) & = & I_2(r) & \text{(dihedral groups)}. \end{array}$$

Complex reflection groups play an important role in the study of algebraic groups. As Weyl groups, they give rise to braid groups and generalized Hecke algebras which govern representation theory of finite reductive groups [7].

1.2 Groups G(m,1,n)

Definition 1 (The wreath product $G \wr S_n$ [26]). Let G be a finite group and $G^n = G \times G \times \cdots \times G$ direct product of n copies of G. The symmetric group S_n acts on G^n by permuting the factors:

$$\sigma(g_1, \ldots, g_n) = \left(g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}\right) \quad \forall \sigma \in S_n.$$

In particular, for all transpositions $\pi_i = (i, i+1)$,

$$\pi_i(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) = (g_1, \ldots, g_{i+1}, g_i, \ldots, g_n)\,.$$

The wreath product $G \wr S_n$ is the semi-direct product[2] $G^n \rtimes S_n$ defined by this action, i.e. it is the group whose underlying set is $G^n \times S_n$, with multiplication defined by:

$$(g, \sigma)(g', \sigma') = (g\sigma(g'), \sigma\sigma') \quad \forall g, g' \in G^n,\ \forall \sigma, \sigma' \in S_n.$$

The group $G_n := G \wr S_n$ is isomorphic to the group of generalized permutation matrices with entries in G, the matrix corresponding to (g, σ) having (i, j) element $g_i \delta_{i,\sigma(j)}$, where $g = (g_1, \ldots, g_n)$. In other words, (g, σ) corresponds to the matrix $\mathrm{diag}(g_1, \ldots, g_n)P_\sigma$ where P_σ is the permutation matrix representing σ.
For $n = 0$, G_0 is the group of one element (identity); $G_1 = G$. The order of G_n is $|G|^n n!$ $\forall n \geq 0$.

Let $m \in \mathbb{N}^*$. Abstractly, the group $G(m, 1, n)$ is generated by the elements t, s_1,..., s_{n-1} with the defining relations [35]:

$$\begin{cases} s_{i+1}s_is_{i+1} = s_is_{i+1}s_i & \text{for } i = 1, \ldots, n-2\ , \\ s_is_j = s_js_i & \text{for } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1\ , \\ s_i^2 = 1 & \text{for } i = 1, \ldots, n-1 \end{cases} \tag{3}$$

and

$$\begin{cases} s_1ts_1t = ts_1ts_1\ , \\ s_it = ts_i & \text{for } i > 1\ , \\ t^m = 1\ . \end{cases} \tag{4}$$

Let $t_1 := t$ and, inductively, $t_{i+1} := s_it_is_i$ for $i = 1, \ldots, n-1$. The following holds:

$$\begin{cases} t_i^m = 1 & \text{for } i = 1, \ldots, n\ , \\ t_it_j = t_jt_i & \text{for } i, j = 1, \ldots, n\ , \\ s_it_j = t_{\pi_i(j)}s_i & \text{for } i = 1, \ldots, n-1\,,\ j = 1, \ldots, n. \end{cases} \tag{5}$$

The relations (4) are included in the relations (5). It follows from the Schreier coset graph of the group $G(m, 1, n)$ (given in §1.4.2) that:

$$\langle s_1, \ldots, s_{n-1}, t |\ (3),(4)\rangle = \langle s_1, \ldots, s_{n-1}, t_1, \ldots t_n |\ (3),(5)\rangle.$$

Proposition 1. *The group $G(m, 1, n)$ is isomorphic to $C_m \wr S_n$, the wreath product of the cyclic group with m elements, C_m, by the symmetric group S_n. Its order is equal to $m^n n!$.*

Proof. Let γ be a generator of the group C_m. Consider a map f from $G(m, 1, n)$ to a group A of generalized permutation matrices with entries in C_m such that $\forall i = 1, \ldots, n-1$, $\forall j = 1, \ldots, n$

$$f(s_i) = P_{\pi_i}, \quad f(t_j) = \mathrm{diag}(1, \ldots, 1, \gamma, 1 \ldots, 1)$$

[2]Let H and K be two groups and $\varphi : K \to \mathrm{Aut}(H)$ a group homomorphism from K to the group of automorphisms of H. Recall that the *semi-direct product* $H \rtimes_\varphi K$ is the cartesian product $H \times K$ equipped with the multiplication law:

$$(h, k)(h', k') = (h\varphi(k)(h'), kk').$$

The inverse of an element (h, k) is $(h, k)^{-1} = (\varphi(k^{-1})(h^{-1}), k^{-1})$. The semi-direct product $H \rtimes_\varphi K$ is a group. Usually φ is known from the context and its symbol is omitted in $\rtimes_\varphi$ if no confusion arises.

with γ at position j. Every element of A is a product of a diagonal matrix (with entries in C_m) with a permutation matrix. It is easy to see that A is generated by $f(s_1), \ldots, f(s_{n-1}), f(t_1), \ldots, f(t_n)$ that satisfy the relations (3) and (5). Thus A is a factor group of $G(m,1,n)$. Moreover, $|A| = m^n n!$. Using the normal form for elements of $G(m,1,n)$ we establish (see §1.4.3) that $|G(m,1,n)| = m^n n!$. Consequently $G(m,1,n)$ is isomorphic to $A \simeq C_m \wr S_n$. The direct isomorphism from $G(m,1,n)$ to $C_m \wr S_n$ is given by, see e.g. [35],

$$t \mapsto ((\gamma, 1, \ldots, 1), 1)\,, \quad s_i \mapsto ((1, \ldots, 1), \pi_i)\,.$$

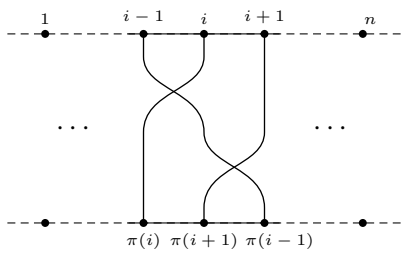
□

1.3 Rotational permutations of m-cards

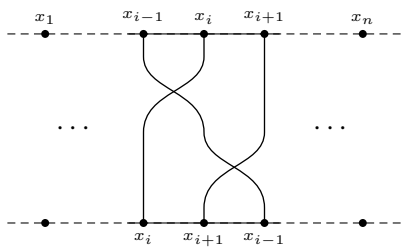
In a deck of n cards bought in a boutique the cards are placed in some standard way. We label cards by their original positions. Then an arbitrary deck $\bar{x}$ of n cards is a sequence of numbers $(x_1, \ldots, x_n)$: a card at position j has the label x_j. We identify $\bar{x}$ with a permutation

$$\begin{pmatrix} 1 & \ldots & n \\ x_1 & \ldots & x_n \end{pmatrix}.$$

We define the action of the symmetric group S_n on the set of decks. Let $\pi \in S_n$. We graphically represent π by lines joining the top segment and bottom segment, the point i of the top segment is joined to the point $\pi(i)$ of the bottom segment, as in the example:



The application of the permutation π to the deck $\bar{x}$ is a new deck $\bar{y} = (y_1, \ldots, y_n)$ that in our example is depicted as:



This means that $y_i = x_{\pi^{-1}(i)}$.

Definition 2. A *deck of n m-cards* is an n-tuple of pairs $((x_1, \rho_1), (x_2, \rho_2), \ldots (x_n, \rho_n))$ where $(x_1, \ldots, x_n)$ is a permutation of $(1, \ldots, n)$ and $\rho_i \in C_m,\ i = 1, \ldots, n$.
The deck $((x_1, \rho_1), \ldots (x_n, \rho_n))$ will be denoted by $((x_i, \rho_i), i \leq n)$ if no confusion arises.

Let M be an m-gon with a distinguished vertex. We interpret $(x, 1)$ as an m-gon M, on which the number x is printed, with the distinguished vertex pointing to the north. The m-card (x, γ^k) is the card $(x, 1)$ rotated clockwise by an angle $\frac{2\pi}{m}k$ in the plane of M.

Abstractly, the 1-card is a point and the 2-card is a segment with two vertices. In the context of card shuffling, we shall slightly transgress this picture and conceive of 1-cards as the classical cards, 2-cards as classical cards but asymmetrically pictured, and m-cards with $m \geq 3$ as asymmetrically pictured m-gonal cards, as illustrated in figure 1.

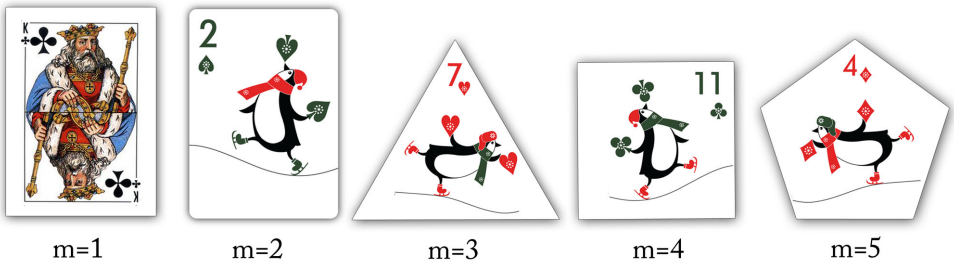


FIG. 1: m-cards

The proposition that follows shows that a deck of n m-cards can be identified with an element of $G(m,1,n)$.
Let E be a set $\{(x,\rho) \mid x \in [\![1,n]\!],\ \rho \in C_m\}$. Define the action of C_m on this set by:

$$\rho \cdot (x,\rho') = (x,\rho\rho'), \quad \forall \rho,\rho' \in C_m,\ \forall\, x \in [\![1,n]\!].$$

Let $\mathsf{Perm}(E)$ be the group of permutations of the set E. Denote by $\mathsf{Perm}^0(E)$ the subgroup of $\mathsf{Perm}(E)$ consisting of elements $\pi \in \mathsf{Perm}(E)$ such that (see e.g. [35]):

$$\pi(x,\rho) = \rho \cdot \pi(x,1), \quad \text{for } \rho \in C_m \text{ and } x \in [\![1,n]\!].$$

Proposition 2. *The group $G(m,1,n)$ is isomorphic to $\mathsf{Perm}^0(E)$.*

Proof. To specify an element π of $\mathsf{Perm}^0(E)$, it is enough to give the images under π of the elements of the set $\{(x,1) \mid x \in [\![1,n]\!]\}$. Let ϕ be the map from the set of generators of $G(m,1,n)$ to $\mathsf{Perm}^0(E)$ defined by:

$$\begin{aligned} &\phi(t)(x,1) = \begin{cases} (x,\gamma) & \text{if } x = 1\ , \\ (x,1) & \text{otherwise}\ ; \end{cases} \\ &\phi(s_i)(x,1) = (\pi_i(x),1) \quad \text{for } x \in [\![1,n]\!] \text{ and } i = 1,\dots,n-1 \end{aligned} \tag{6}$$

(recall that π_i is the transposition $(i,i+1)$). The permutations $\phi(t),\phi(s_1),\dots,\phi(s_{n-1}) \in \mathsf{Perm}^0(E)$ satisfy the defining relations of the group $G(m,1,n)$, so the map ϕ: $G(m,1,n) \to \mathsf{Perm}^0(E)$ is the group homomorphism.

- ϕ is injective: let $g \in G(m,1,n)$, such that $\phi(g) = id \in \mathsf{Perm}^0(E)$. Since $G(m,1,n) \simeq C_m \wr S_n$, we can identify g with an element $(\sigma,\mathbf{v}) \in S_n \ltimes C_m^n$, where $\mathbf{v} = (v_1,\dots,v_n)$. Then, by hypothesis, $\forall (x,\rho) \in E$, $\phi(\sigma,\mathbf{v})(x,\rho) = (x,\rho)$. But $(\sigma,\mathbf{v}) = (1,\mathbf{v})(\sigma,\mathbf{1})$ (notice that $(\sigma,\mathbf{1})(1,\mathbf{v}) = \big(\sigma,(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(n)})\big) \neq (\sigma,\mathbf{v})$ for $\mathbf{v} \neq (1,\dots,1)$). Since ϕ is a homomorphism, we have:

  $$(x,\rho) = \phi(1,\mathbf{v})\phi(\sigma,\mathbf{1})(x,\rho) = \phi(1,\mathbf{v})(\sigma(x),\rho) = (\sigma(x),v_{\sigma(x)}\rho)\ \ \forall (x,\rho) \in E.$$

  This implies $\sigma = 1$ and $\mathbf{v} = \mathbf{1}$. In other words, $\mathrm{Ker}(\phi) = \{1\}$.

- ϕ is surjective: this is the consequence of the fact that

$$\phi(t_j)(x,1) = \begin{cases} (x,\gamma) & \text{if } x = j \ , \\ (x,1) & \text{otherwise} \end{cases}$$

for any $j = 1, \ldots, n$. $\square$

We identify a deck $((x_i, \rho_i), i \leq n)$ with a "rotational" permutation of the form:

$$g = \begin{pmatrix} (1,1) & \ldots & (n,1) \\ (x_1,\rho_1) & \ldots & (x_n,\rho_n) \end{pmatrix}. \tag{7}$$

The group $\mathsf{Perm}^0(E)$ describes $G(m,1,n)$ as the group of all possible rotational permutations of m-cards in the deck $((i,1), i \leq n)$.
The element g in (7) is a permutation if $\rho_j = 1$ for all $j = 1, \ldots, n$. The application of a permutation $\sigma \in G(m,1,n)$ on an arbitrary deck $((x_i,\rho_i), i \leq n)$ generates a deck $\Big((x_{\sigma^{-1}(i)}, \rho_i), i \leq n\Big)$. In particular, for $i = 1, \ldots, n-1$ the generator s_i permutes the i-th and $(i+1)$-st m-cards without any turn, as depicted in figure 2.

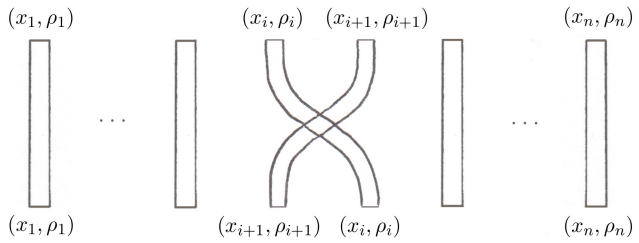


FIG. 2: Action of $s_i \in G(m,1,n)$ on a deck $((x_j,\rho_j), j \leq n)$.

For $i = 1, \ldots, n$, the element t_i rotates the i-th m-card clockwise by an angle $\frac{2\pi}{m}$. We graphically represent the action of t_i by ribbons, the i-th ribbon being turned clockwise around its axis, as illustrated in figure 3; m turns boil down to the trivial action.

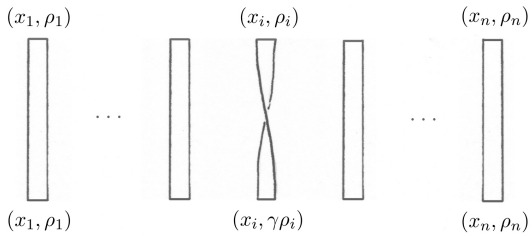


FIG. 3: Action of $t_i \in G(m,1,n)$ on a deck $((x_j,\rho_j), j \leq n)$.

In figure 4 we compare the action of the left and the right hand side terms of several defining relations for $G(m,1,n)$ with m at least 2. Only non trivial action is depicted. In figure 5 we check the equality $s_2 s_1 t s_1 s_2 = s_2 t_2 s_2 = t_3$.

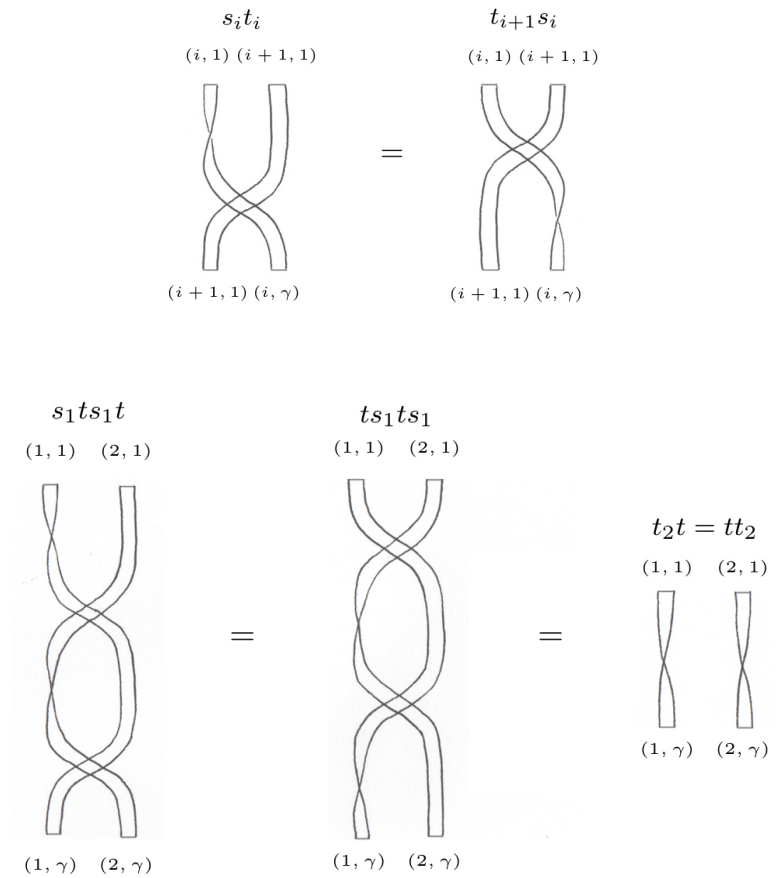

$s_i t_i$
$(i,1)$ $(i+1,1)$
$(i+1,1)$ (i,γ)
$=$
$t_{i+1} s_i$
$(i,1)$ $(i+1,1)$
$(i+1,1)$ (i,γ)
$s_1 t s_1 t$
$(1,1)$ $(2,1)$
$(1,\gamma)$ $(2,\gamma)$
$=$
$t s_1 t s_1$
$(1,1)$ $(2,1)$
$(1,\gamma)$ $(2,\gamma)$
$=$
$t_2 t = t t_2$
$(1,1)$ $(2,1)$
$(1,\gamma)$ $(2,\gamma)$

FIG. 4

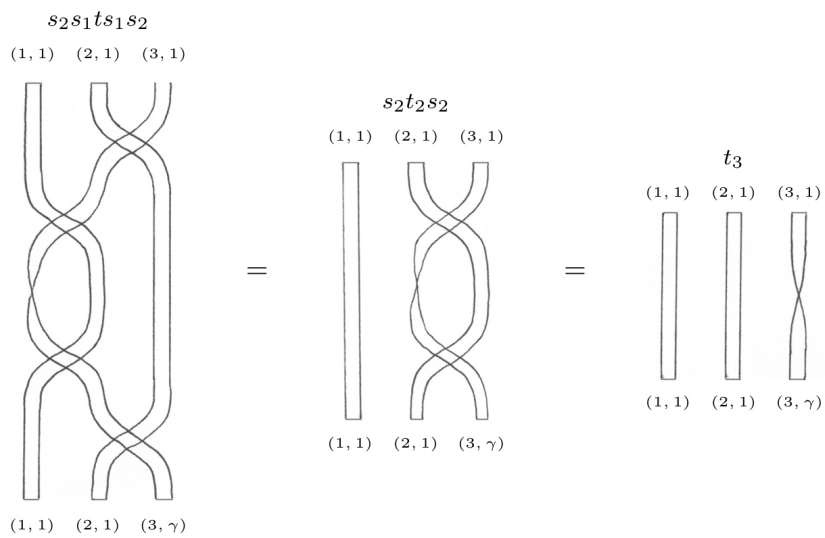

$s_2 s_1 t s_1 s_2$
$(1,1)$ $(2,1)$ $(3,1)$
$(1,1)$ $(2,1)$ $(3,\gamma)$
$=$
$s_2 t_2 s_2$
$(1,1)$ $(2,1)$ $(3,1)$
$(1,1)$ $(2,1)$ $(3,\gamma)$
$=$
t_3
$(1,1)$ $(2,1)$ $(3,1)$
$(1,1)$ $(2,1)$ $(3,\gamma)$

FIG. 5

1.4 Todd-Coxeter algorithm for the chain of groups G(m,1,n)

To understand the structure of a group from its presentation is, in general, a difficult task. A basic method for finitely presented groups is the *Todd-Coxeter algorithm*, created by J.A. Todd and H.S.M. Coxeter in 1936. Given a subgroup H of a group G, it attempts to find the index $|G : H|$, by enumerating cosets of H in G in a systematic trial and error procedure. If $|G : H|$ is finite then the Todd-Coxeter algorithm terminates in a finite (but unknown) number of steps. It returns the index $|G : H|$, together with a complete *coset table*, which is equivalent to the permutation representation of G on the space of the cosets. The Todd-Coxeter method has a wide range of applications, in particular, it may be used to construct a normal form for group elements [35], to test whether H is a normal subgroup in G or to determine a minimal presentation for a given concrete group[3] [11, 19]. Sometimes, it solves the so called word problem[4]. The procedure of enumerating the cosets is sufficiently mechanical and can be carried out by computer (first implementation in 1953).

1.4.1 Todd-Coxeter coset enumeration procedure

Let $G = \langle E|\mathcal{R}\rangle$ be a finite presentation for a group G where $E = \{g_1, \dots, g_n\}$ is a set of generators and $\mathcal{R} = \{r_1 = e, \dots, r_k = e\}$ is a set of defining relations. Each "relator" r_i is a word in the alphabet $\overline{E} = \{g_1, \dots, g_n, g_1^{-1}, \dots g_n^{-1}\}$. Let $H = \langle h_1, \dots h_p\rangle$ be a subgroup of G generated by a finite set of words in $\overline{E}$. At each step, three types of partially defined tables are reactualized [10, 40]:

- A *coset table*: this is a matrix M whose rows are labelled by cosets of H existing at this step, and columns by elements of $\overline{E}$. The entries, if defined, are cosets, $M(k, g) = \ell$ if ℓ is such that $kg = \ell$.
- A *relation table* for each relator $r = g_{i_1} g_{i_2} \cdots g_{i_t}$: this is a matrix M_r with t columns; rows are labelled by existing cosets. The entry $M_r(k, a)$, if defined, is the coset $kg_{i_1} \cdots g_{i_a}$. Since $r = e$ in G, we have $M_r(k, t) = k$.
- A *subgroup table* for each generator $h = g_{j_1} \cdots g_{j_t}$ of H: this is matrix S_h with only one row corresponding to the coset $1 := H$ and t columns. The entry $S(1, a)$ is the coset $1g_{j_1} \cdots g_{j_a}$. Since $Hh = H$, we have $S_h(1, t) = 1$.

After the tables have been initialized, the Todd-Coxeter procedure consists of listing cosets of H, starting with $1 = H$, then proceeding e.g. with $2 := 1g_1^{\pm 1}$, $3 := 1g_3^{\pm 1}$, or $3 := 2g_1^{\pm 1}$ etc. with the only rule that a coset ℓ_α must be defined by an equation $\ell_\alpha = \ell_\beta g,\ \beta < \alpha,\ g \in \overline{E}$, if the place of $\ell_\beta g$ in the coset table is still unoccupied. As soon as the coset ℓ_α has been defined, the α-th rows of the coset table and the relation tables are initialized. This definition and its trivial consequence $\ell_\alpha g^{-1} = \ell_\beta$ are then filled into all possible vacant places of the various tables. The aim of the process is to obtain information about equality or inequality of cosets that have been given different numbers. When a row in a relation table or a subgroup table is completed, we get an extra piece of information of the kind $\ell_\alpha g = \ell_\beta$ for some cosets ℓ_α, ℓ_β and $g \in \overline{E}$. This is called *deduction*. If the

[3]The starting point of Todd and Coxeter was to prove that a set of relations for a given concrete group is complete.

[4]The problem of deciding whether two words in a finitely presented group define the same element is one of three fundamental decision problems formulated by Max Dehn in 1911. P. Novikov (1955) and W. Boone (1958) constructed finitely presentable groups for which the question is undecidable in the sense that it cannot be answered by a recursive algorithm.

entry $M(\ell_\alpha, g)$ is not yet defined then we fill the entries $M(\ell_\alpha, g)$, $M(\ell_\beta, g^{-1})$, and insert this information into all other relevant places in the other tables. If $M(\ell_\alpha, g)$ is already filled with a coset, say ℓ_γ, different from that given by the deduction then we realise that ℓ_α and ℓ_γ denote the same coset of H. This is called *coincidence*. When a coincidence is found, we leave only one coset, usually the one that appeared earlier. We propagate this information to other cosets that have been defined as $\ell_\alpha g$ or $\ell_\gamma g$ for some $g \in \overline{E}$. In turn, this may lead to further deductions and coincidences. The process terminates when all entries of all tables are filled. The obtained coset table corresponds to a permutation representation of G.

Example. Consider the group $G = \langle g_1, g_2 \mid g_1^2 = g_2^2 = (g_1 g_2)^3 = e \rangle$ and its subgroup $H = \langle g_2 \rangle$ of order at most 2. Since both generators square to identity, we have $\overline{E} = E$. Also, the relation $g_1^2 = g_2^2$ tells us that while defining new cosets, we should multiply previous ones alternatingly by g_1 and g_2. This information and that from the subgroup table will be recorded in the coset table (CT), so we set up additionally only the relation table (RT) corresponding to $(g_1 g_2)^3 = e$. After the definition of the cosets $1 := H$ and $2 := 1g_1$, we obtain:

CT	g_1	g_2
1	2	1
2	1	

RT	g_1	g_2	g_1	g_2	g_1	g_2
1	2			2	1	1
2	1	1	2			2

At this stage we define $3 := 2g_2$. From RT we read the equality $3g_1g_2 = 2$, which implies $3g_1 = 3$. We complete all the entries and conclude that $|G : H| = 3$.

CT	g_1	g_2
1	2	1
2	1	3
3	3	2

RT	g_1	g_2	g_1	g_2	g_1	g_2
1	2	3	3	2	1	1
2	1	1	2	3	3	2
3	3	2	1	1	2	3

We obtain the group homomorphism $\phi : G \to S_3^{op}$ such that $\pi_i = \phi(g_i) = (i, i+1)$ for $i = 1, 2$ where S_3^{op} is the opposite group[5] of the symmetric group on the set $\{1, 2, 3\}$ of cosets. This homomorphism is surjective: as is well known, the transpositions π_i generate S_3. Thus $|G| \geq 6$. But $|G| \leq 6$ since $|H| \leq 2$ and $|G : H| = 3$. Finally, the group G is isomorphic to the group S_3.

Remarks. (i) Given a finite presentation $G = \langle E | \mathcal{R} \rangle$, it is not necessary to rewrite the relations $\mathcal{R}$ into the format $\mathcal{R} = \{r_1 = e, \ldots, r_k = e\}$. The table corresponding to the relation e.g. $xy^2 = y^3x$ may as well be presented as two side by side independent tables with the common left column and identical last columns (for defined entries):

[5]The opposite group of the group $(X, \cdot)$ is the group X^{op} whose underlining set is X and the group law is defined by:

$$\forall\, x, y \in X, \quad x \cdot_{X^{op}} y = y \cdot x.$$

The group X is isomorphic to X^{op} *via* $X \to X^{op}$, $x \mapsto x^{-1}$.

	x	y	y	$=$	y	y	y	x
1			ℓ					ℓ
$\vdots$								$\vdots$
k			ℓ'					ℓ'
$\vdots$								$\vdots$

(ii) Making small changes to the presentation that do not change the group may have a significant impact on the amount of time or memory needed to complete the enumeration. Also, the above description does not determine in which order a definition or a deduction is inserted into various places in the tables. In fact this order does not affect the result of the Todd-Coxeter procedure although it may influence the efficiency [10]. This nondeterministic behaviour is a consequence of the unsolvability of the word problem for groups.

(iii) We have presented here the classical version of the Todd–Coxeter algorithm. Its various improvements and alternative methods have been suggested (e.g. HLT strategy, Knuth–Bendix procedure [19, 41]).

We shall now investigate the permutation representation of

$$G(m,1,n) = \langle t, s_1, \ldots, s_{n-1} |\ (3),(4) \rangle$$

on the right cosets of its subgroup $W := \langle t, s_1, \ldots s_{n-2} \rangle$. Since the number of relations and thus of required tables increases considerably with n and, to a lesser extent, with m, let us first consider a case with m and n small, say, $n = m = 3$, for which $W = \langle t, s_1 \rangle$. Since s_1 and s_2 square to identity, we have $\overline{E} = \{t, t^{-1}, s_1, s_2\}$. We will maintain only three relation tables corresponding to $s_2 s_1 s_2 = s_1 s_2 s_1$, $s_2 t = t s_2$ and $s_1 t s_1 t = t s_1 t s_1$ and the coset table, other tables being irrelevant here. With the sequence of definitions $1 := W$, $2 := 1s_2$, $3 := 2s_1$, $4 := 3t$, $5 := 3t^{-1}$, $6 := 4s_1$, $7 := 5s_1$, $8 := 6s_2$, $9 := 7s_2$, neither deductions nor coincidences occur; all tables close simultaneously, which means that all cosets have been enumerated. The figure 6 shows the obtained coset table and its diagrammatic representation, called *Schreier coset graph*.

	t	t^{-1}	s_1	s_2
1	1	1	1	$\underline{2}$
2	2	2	$\underline{3}$	1
3	$\underline{4}$	$\underline{5}$	2	3
4	5	3	$\underline{6}$	4
5	3	4	$\underline{7}$	5
6	6	6	4	$\underline{8}$
7	7	7	5	$\underline{9}$
8	8	8	8	6
9	9	9	9	7

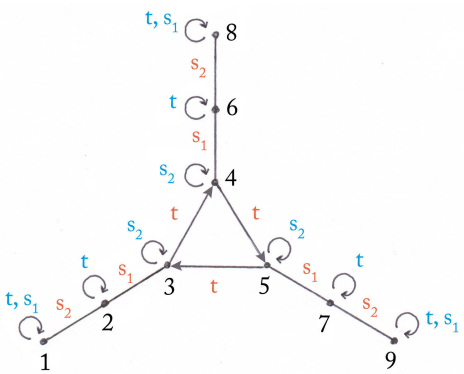


FIG. 6: Coset table and Schreier graph of $G(3,1,3)$ with respect to W. In the coset table, a number is underlined if it has been defined in this place.

1.4.2 Schreier coset graph for the chain of groups $G(m,1,n)$

Definition 3. Given a finitely presented group $G = \langle E|\mathcal{R}\rangle$ and a subgroup H of finite index in G, a Schreier coset graph Γ of (G,H) is a directed graph whose vertices are the right cosets of H and whose edges are of the form (Hg, Hgx) for $x \in E$. By convention, the edge (Hg, Hgx) has label x.
When $H = \{e\}$, Γ is called Cayley graph.

The Schreier graph depicts the action of G on the space of cosets. For a given edge label g, every vertex has exactly one outgoing edge and one incoming edge with that label. A path $g_1 \cdots g_k$ from a vertex s has each g_i in $\overline{E}$, and we follow the path in left-to-right order, so that $sg_1 \cdots g_k$ is a vertex at the end of the path. Since the map $Hg \mapsto (Hg)^{-1} = g^{-1}H$ defines a bijection between the right cosets and left cosets of H, Γ encodes as well the action of G by left multiplication on the left cosets. Given a path $g_1^{-1} \cdots g_k^{-1}$ in left-to-right order from a vertex s, $g_k^{-1} g_{k-1}^{-1} \cdots g_1^{-1} s$ corresponds then to a left coset at the end of the path.

The Schreier graph may be obtained from the coset table or, directly, by a graphical procedure [9, 28] equivalent to the Todd-Coxeter algorithm.

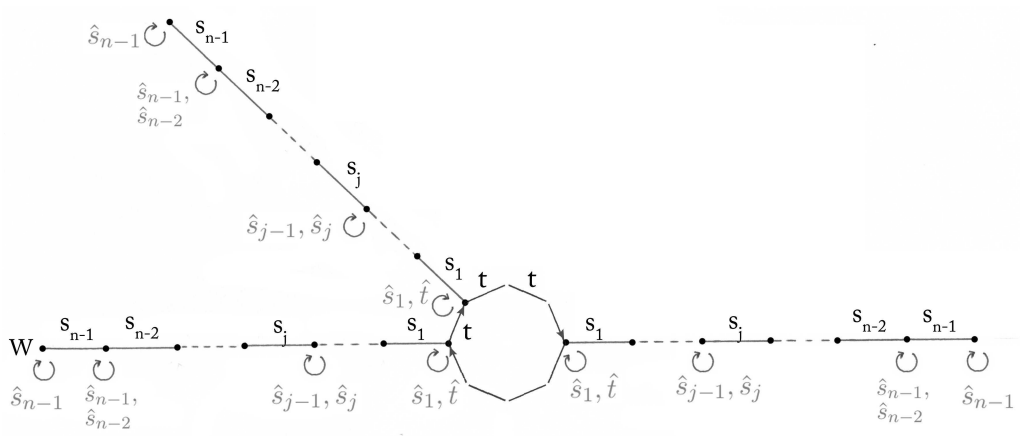


FIG. 7: Schreier coset graph for $G(m,1,n)$ with respect to W [35].

Figure 7 is the Schreier graph for $G(m,1,n)$ with respect to $W = \langle t, s_1, \ldots s_{n-2}\rangle$. An unoriented edge represents a pair of edges with opposite directions. If a generator leaves a coset invariant, the corresponding edge is a loop. The symbol $\hat{x}$ for $x \in E$ stands for the set $E\backslash\{x\}$. In our figure, $\hat{x}$ is the generator of $G(m,1,n)$ that does *not* stabilize the corresponding coset. At each vertex of the oriented m-gon in the middle of the figure starts a tail with $n-1$ edges. All tails are identically (edges-)labelled.

1.4.3 Normal form for elements of G(m,1,n)

A normal form for a group G with generating set E is a choice of one reduced word[6] in the alphabet $\overline{E}$ for each element of G. By convention, the normal form of the identity is the empty word. A normal form allows to test the equality of two words. The Schreier coset graph for $G(m,1,n)$ provides us with a normal form with respect to W:

Proposition 3. *Any element $x \in G(m,1,n)$ can be written in the form:*

$$x = ws_{n-1} \cdots s_1 t^{\alpha} s_1 \cdots s_j \tag{8}$$

[6] i.e. a word in which occurrences of kind xx^{-1} and $x^{-1}x$ are omitted.

where $j \in \{0, \ldots, n-1\}$, $\alpha \in \{0, \ldots, m-1\}$ and $w \in W \simeq G(m, 1, n-1)$ (by convention, the empty product is equal to 1).

It follows from proposition 3 that $|G(m, 1, n)| = |G(m, 1, n-1)|mn = m^n n!$.
Starting with a normal form (powers of t) for $G(m, 1, 1) \simeq C_m$, we build recursively the global normal form [35] for elements of $G(m, 1, n)$ for any n. We have an ascending chain:

$$G(m, 1, 0) \subset G(m, 1, 1) \subset G(m, 1, 2) \subset \ldots$$

Remark. The group $G(m, 1, n)$ admits an antiautomorphism[7], identical on generators. The application of this antiautomorphism yields another normal form:

$$x = s_j s_{j-1} \cdots s_1 t^\alpha s_1 \cdots s_{n-1} w \tag{9}$$

with the same notations as in (8). Also, (9) can be obtained from the Schreier graph viewed as the representation of $G(m, 1, n)$ in its action by left multiplication on the left cosets of W.
Instead of the Todd-Coxeter algorithm, one can use the diamond lemma to establish a Gröbner base and a normal form for the algebra $\mathbb{Z}G(m, 1, n)$ (see [4] or, more adapted to the situation here, [31]).

1.4.4 Rewriting rules for the chain of groups G(m,1,n)

In symbolic computation with finitely presented groups the defining relations are interpreted as *rewriting rules*. These are the instructions to replace the left hand side by the right hand side. The procedure of replacing subwords which are left sides of rewriting rules with the corresponding right sides is called *rewriting*. A word is said to be *reduced*, with respect to the given set of rewriting rules, if no rewriting can be performed on it.

Consider an element $s_2 s_1 s_2 t \in G(2, 1, 3)$. There are two possibilities to rewrite it by applying the defining relations $\mathcal{R} = \{(3), (4)\}$:

$$s_2 s_1 s_2 t \begin{cases} \nearrow s_2 s_1 t s_2 \\ \searrow s_1 s_2 s_1 t. \end{cases}$$

The words $s_1 s_2 s_1 t$ and $s_2 s_1 t s_2$ are reduced with respect to $\mathcal{R}$. Supplementary instructions are required to equalize them.

More generally, we are faced to the already mentioned word problem (see footnote 4). For a given presentation of the group the word problem is solved if one finds an algorithm that takes a word and rewrites it in a unique way, called a *canonical form* for the corresponding element of the group. The normal form given in previous paragraph guarantees the existence of such an algorithm for the group $G(m, 1, n)$.

In order to build an algorithm that takes as input a word in generators of $G(m, 1, n+1)$ and their inverses, and returns its normal form as in (8) one only needs to know the normal form of the words $\psi_{kj}^{(n)} x = s_n \cdots s_1 t^k s_1 \cdots s_j x$ for all generators x of $G(m, 1, n+1)$, $k = 0, \ldots, m-1$ and $j = 0, \ldots, n$. Below is the list of rewriting rules we recursively established and implemented in *Mathematica* (the equalities are to be understood as "replace by").

[7] Given two groups G and G', an antihomomorphism is a map $\varphi : G \to G'$ that reverses the order of multiplication: $\forall x, y \in G,\ \varphi(xy) = \varphi(y)\varphi(x)$. If $G' = G$ and φ is bijective, then φ is an antiautomorphism.

Proposition 4. *The rewriting rules for $G(m,1,n+1)$ with $n>1, m\geq 1$ include (additionally to the defining relations) the rewriting rules for $G(m,1,n)$ and the following:*

$$\begin{cases} s_n s_{n-1}\cdots s_j s_n = s_{n-1}s_n s_{n-1}\cdots s_j, & j=1,...,n-2,\\ \psi_{kj}^{(n)} s_n = s_{n-1}\psi_{kj}^{(n)}, & k=0,...,m-1,\ j=0,1,...,n-2. \end{cases}$$

For $G(m,1,2)$ we have:

$$s_1 t^k s_1 t = t s_1 t^k s_1, \quad \forall\, k=1,...,m-1.$$

One can check these equalities in the following way: given two words, follow the Schreier graph of the pair $(G(m,1,n),W)$ from the basepoint W to see whether they end at the same vertex. To find relations appearing at step $n+1$, consider only those paths (possibly including loops) that do not exist in the graph at step n.

Listed below are the instructions for $G(m,1,5)$ (the numbers $1,2,3,4$ denote the generators s_1,s_2,s_3,s_4 and $k=1,\ldots,m-1$):

		$4324=3432$
$1t^k1t = t1t^k1$		$43214=34321$
	$3213=2321$	$4321t^k4=34321t^k$
$21t^k2=121t^k$	$321t^k3=2321t^k$	$4321t^k14=34321t^k1$
$21t^k121=121t^k12$	$321t^k13=2321t^k1$	$4321t^k124=34321t^k12$

2 Shuffle analogues in G(m,1,n) group algebra

Notation. For an element $w\in S_n$ we denote by $w^{\uparrow\ell}$ the image of w under the homomorphism $S_n\to S_{n+\ell}$ sending $\pi_i=(i,i+1)$ to $\pi_{i+\ell}$, $i=1,\ldots,n-1$. In the sequel, the notation is extended to the group $G(m,1,n)$.

2.1 Symmetrizers and shuffles

Definition 4 (Symmetrizer). Given a finite group G and a field $\mathbb{F}$ of characteristic 0, the symmetrizer Σ_G is a non zero element of the group algebra $\mathbb{F}[G]$ that satisfies:

(i) $g\Sigma_G=\Sigma_G$, $\forall g\in G$;

(ii) $\Sigma_G^2=\Sigma_G$ (idempotency).

The symmetrizer is uniquely defined by (i) and (ii); it is equal to $\Sigma_G=\dfrac{1}{|G|}\tilde{\Sigma}_G$ where $\tilde{\Sigma}_G=\sum_{g\in G} g$. In particular, $\Sigma_G\cdot g=\Sigma_G$, $\forall g\in G$.

Let V be an $\mathbb{F}$ – vector space and $\rho:\ G\to GL(V)$ a representation of the group G. The symmetrizer projects V on the space V^G of G-invariants: $\forall g\in G$, $\forall v\in V$:

$$\rho(g)\left(\rho\left(\Sigma_G\right)(v)\right)=\rho\left(g\Sigma_G\right)(v)=\rho\left(\Sigma_G\right)(v).$$

In the symmetric group S_n, the symmetrizer $\Sigma_n=\frac{1}{n!}\sum_{\sigma\in S_n}\sigma$ is the Young symmetrizer associated to the Young diagram (n), corresponding to the trivial representation of S_n (see e.g. [16]).

Definition 5 ((p,q)-shuffle). For $p,q \in \mathbb{N}$, a (p,q)-*shuffle* is a permutation $\sigma \in S_{p+q}$ of the set of integers $\{1,\dots,p+q\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.
When $p=1$ or $q=1$, the (p,q)-shuffle is called 1-shuffle.

The (p,q)-shuffle gets its name from the fact that it describes a possible way of building a pack of $p+q$ cards by shuffling a deck of p cards through a deck of q cards, while retaining the order on the two sub-decks.
The sum over all (p,q)-shuffles (with p,q fixed) is called shuffle element of the group algebra of the symmetric group and is commonly denoted by the symbol $Ш_{p,q}$. Since $Ш_{p,q}$ contains $\binom{p+q}{p}$ terms, it is considered as analogue of binomial coefficient [18], the sum of all elements of the symmetric group S_n being itself analogue of $n!$.

Shuffles $Ш_{p,q}$ can be defined inductively by any of the recurrence relations (analogues of the Pascal rule) [22]:

$$\begin{aligned} Ш_{p,q} &= Ш_{p-1,q} + Ш_{p,q-1}\pi_{p+q-1}\cdots\pi_q \\ &= Ш^{\uparrow 1}_{p,q-1} + Ш^{\uparrow 1}_{p-1,q}\pi_1\cdots\pi_q, \end{aligned}$$

with $Ш_{0,q} = Ш_{q,0} = 1$ for any $q \geq 0$.
Shuffles $Ш_{p,q}$ are intimately related with symmetrizers:

Property 5. *In the symmetric group rings $\mathbb{Z}S_n$:*

$$\tilde{\Sigma}_{q+p} = Ш_{p,q}\tilde{\Sigma}_q\tilde{\Sigma}_p^{\uparrow q}. \tag{10}$$

In particular,

$$\tilde{\Sigma}_n = Ш_{1,n-1}\tilde{\Sigma}_{n-1}. \tag{11}$$

Certain analogues of shuffle elements exist for braid groups and Hecke algebras. By their combinatorial structure and relationship with the structure of products, shuffles come into many constructions in homotopy theory and higher category theory. In particular, they are involved in the product of simplices [25]. In exterior algebra, the wedge product of a p-form and a q-form can be defined as a sum over (p,q)-shuffles weighted by their signatures[8]. Shuffle elements are also involved in the bialgebras of type 1 [29], in the construction of the Hopf algebra structure on the tensor algebra $T(V)$ of a vector space V [38], in the construction of the standard complex for quantum Lie algebras [20], in the study of multiple polylogarithms [5, 6], in the q-deformations of the Cayley-Hamilton theorem [30], etc.

For brevity, 1-shuffle element $Ш_{1,n-1}$ will be denoted by $Ш_n$.

Proposition 6. [22] *Let $Ш_n = 1 + s_{n-1} + s_{n-2}s_{n-1} + ... + s_1\cdots s_{n-1}$ be the 1-shuffle of the symmetric group ring $\mathbb{Z}\mathbb{S}_n$. The element $Ш_n$ satisfies the following identity:*

$$Ш_n^2 = Ш_n(1 + Ш_{n-1}) \tag{12}$$

Proof. For $n=1$, there is nothing to prove ($Ш_0 = 0, Ш_1 = 1$). Now we assume that $Ш_{n-1}^2 = Ш_{n-1}(1 + Ш_{n-2})$ for some $n > 1$. Recall that

$$Ш_{n-1}^{\uparrow 1} = Ш_n - s_1 s_2 \cdots s_{n-1} \tag{13}$$

[8]The *length* of the (p,q)-shuffle $(\sigma(1),\dots,\sigma(p),\sigma(p+1),\dots,\sigma(p+q))$ is the integer $\epsilon(\sigma) = \sum_{i=1}^{p}(\sigma_i - i)$; $(-1)^{\epsilon(\sigma)}$ is the *signature* of the corresponding permutation in S_{p+q}.

and thereby $Ш_n^2 = (Ш_{n-1}^{\uparrow 1} + s_1 \cdots s_{n-1})^2$. By induction hypothesis and (13), we get successively:

$$\begin{aligned} Ш_n^2 &= Ш_{n-1}^{\uparrow 1}\left(1 + Ш_{n-2}^{\uparrow 1}\right) + A \\ &= (Ш_n - s_1 \cdots s_{n-1})(1 + Ш_{n-1} - s_1 \cdots s_{n-2}) + A \end{aligned} \tag{14}$$

where $A := Ш_{n-1}^{\uparrow 1} s_1 \cdots s_{n-1} + s_1 \cdots s_{n-1} Ш_{n-1}^{\uparrow 1} + s_1 \cdots s_{n-1} s_1 \cdots s_{n-1}$. Due to the fact that $\forall\, 1 < j < n$:

$$s_j s_1 \cdots s_{n-1} = s_1 \cdots s_{n-1} s_{j-1}, \tag{15}$$

we have $s_1 \cdots s_{n-1} s_1 \cdots s_{n-1} = s_2 \cdots s_{n-1} s_1 \cdots s_{n-2}$.
On the other hand and for the same reason,

$$Ш_{n-1}^{\uparrow 1} s_1 \cdots s_{n-1} = s_1 \cdots s_{n-1} Ш_{n-1}. \tag{16}$$

Thus, after some algebra, the Eq.(14) reduces to:

$$Ш_n^2 = Ш_n(1 + Ш_{n-1}) + \Delta$$

with $\Delta := s_1 \cdots s_{n-1}\left(Ш_{n-1}^{\uparrow 1} - 1\right) - Ш_{n-1}^{\uparrow 1} s_1 \cdots s_{n-2} + s_2 \cdots s_{n-1} s_1 \cdots s_{n-2}$. From the recurrence relation $Ш_n = 1 + Ш_{n-1} s_{n-1}$ applied to the first $Ш_{n-1}^{\uparrow 1}$ in Δ and Eq.(15), it follows that:

$$\Delta = \left(Ш_{n-2}^{\uparrow 2} - Ш_{n-1}^{\uparrow 1}\right) s_1 \cdots s_{n-2} + s_2 \cdots s_{n-1} s_1 \cdots s_{n-2}.$$

But $Ш_{n-2}^{\uparrow 2} - Ш_{n-1}^{\uparrow 1} = -s_2 \cdots s_{n-1}$, whence the result. □

The quadratic relation (12) is equivalent to:

$$Ш_n Ш_{n-1} = Ш_n\,(Ш_n - 1)\,. \tag{17}$$

2.2 Shuffle analogues in G(m,1,n) group algebra

We consider the 1-shuffle analogues in $\mathbb{Z}G(m,1,n)$ of the form

$${}^{(m)}Ш_n = Ш_n \cdot {}^{(m)}\mathcal{T}_n$$

where

$${}^{(m)}\mathcal{T}_n := 1 + t_n + t_n^2 + ... + t_n^{m-1}.$$

Proposition 7. *The element ${}^{(m)}Ш_n$ satisfies:*

$${}^{(m)}Ш_n^2 = {}^{(m)}Ш_n\left(m + {}^{(m)}Ш_{n-1}\right). \tag{18}$$

Proof. First, notice that the commutativity of elements t_i with each other implies the commutativity of elements ${}^{(m)}\mathcal{T}_i$ for all $i \geq 1$. Moreover, for given $i \geq 1$, ${}^{(m)}\mathcal{T}_i$ commutes with $Ш_{i-1}$ and satisfies:

$$\begin{cases} {}^{(m)}\mathcal{T}_i^2 & = m \cdot {}^{(m)}\mathcal{T}_i & \text{for } i \geq 1, \\ {}^{(m)}\mathcal{T}_i s_{i-1} & = s_{i-1} \cdot {}^{(m)}\mathcal{T}_{i-1} & \text{for } i > 1. \end{cases}$$

In view of this,

$$\begin{aligned}{}^{(m)}Ш_n^2 &= Ш_n \cdot {}^{(m)}\mathcal{T}_n Ш_n \cdot {}^{(m)}\mathcal{T}_n \\ &= Ш_n \cdot {}^{(m)}\mathcal{T}_n \left(1 + Ш_{n-1}s_{n-1}\right) \cdot {}^{(m)}\mathcal{T}_n\end{aligned}$$

transforms successively into:

$$\begin{aligned}{}^{(m)}Ш_n^2 &= m \cdot {}^{(m)}Ш_n + Ш_n Ш_{n-1} \cdot {}^{(m)}\mathcal{T}_n s_{n-1} \cdot {}^{(m)}\mathcal{T}_n \\ &= m \cdot {}^{(m)}Ш_n + Ш_n Ш_{n-1} s_{n-1} \cdot {}^{(m)}\mathcal{T}_n \cdot {}^{(m)}\mathcal{T}_{n-1}.\end{aligned}$$

Since $Ш_{n-1}s_{n-1} = Ш_n - 1$, by taking into account (17), we have:

$$Ш_n Ш_{n-1} s_{n-1} = Ш_n Ш_{n-1}. \tag{19}$$

Finally,

$$\begin{aligned}{}^{(m)}Ш_n^2 &= m \cdot {}^{(m)}Ш_n + Ш_n \cdot {}^{(m)}\mathcal{T}_n Ш_{n-1} \cdot {}^{(m)}\mathcal{T}_{n-1} \\ &= {}^{(m)}Ш_n \left(m + {}^{(m)}Ш_{n-1}\right),\end{aligned}$$

as claimed. □

2.3 Minimal polynomial and multiplicities of eigenvalues

In this section we determine the minimal polynomial of ${}^{(m)}Ш_n$ and the multiplicities of the corresponding eigenvalues calquing the approach developed by Garsia and Wallach in [17] for the symmetric group. We propose an alternative way to find the minimal polynomial, in the spirit of [22].

Let α and β be two words in an alphabet A. We denote by $\alpha \Cup \beta$ the formal sum of all the words that can be obtained by shuffling α and β as it is done with two decks of cards. We define for $0 \leq a \leq n$:

$$\mathbf{B}_a = \sum_{\alpha \in G(m,1,a)} \alpha \Cup \beta_{a,n} \tag{20}$$

where α is viewed as a deck of a m-cards and $\beta_{a,n}$ is the ordered deck $((a+1,1),(a+2,1),\ldots,(n,1))$. In particular, $\mathbf{B}_0 = 1$.
As an element of $\mathbb{Z}G(m,1,n)$, $\mathbf{B}_a$ may be viewed as a shuffling operator or a sum of all possible decks in which the cards with labels $a+1, a+2, ..., n$ maintain their relative order. It is easy to see that

$$\mathbf{B}_1 = (1 + s_1 + s_2 s_1 + ... + s_{n-1}s_{n-2}\cdots s_1)\left(1 + t + t^2 + ... + t^{m-1}\right). \tag{21}$$

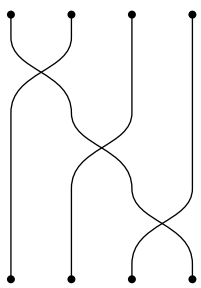


FIG. 8: $s_3 s_2 s_1$ inserts the top card into the position 4 and maintains the relative order of remaining cards.

As illustrated in figure 8, the element $s_j s_{j-1} \cdots s_1$ inserts the top card into the position $j+1$ counted from the top of the deck. For $1 \leq k < m$, the operator $s_j s_{j-1} \cdots s_1 t^k$ turns first the top card by an angle $\frac{2m}{m}k$ and then inserts it into the position $j+1$.

Notice that the element $\mathbf{B}_1$ is the shuffle ${}^{(m)}Ш_n$ in disguise. Indeed, by conjugating B_1 by the longest element in $G(1,1,n)$ defined recursively by:

$$w_n := w_{n-1}s_{n-1}s_{n-2}\cdots s_1$$

with $w_1 = 1$, one obtains, since $w_n^{-1} = w_n$ $\forall n$:

$${}^{(m)}Ш_n = w_n \mathbf{B}_1 w_n. \tag{22}$$

Proposition 8.

$$\prod_{a=0}^{n}\left({}^{(m)}\text{Ш}_n - am\right) = 0. \tag{23}$$

First proof. Due to (22), it is enough to prove the result for $\mathbf{B}_1$.
Consider an element $\mathbf{B}_a$ for an arbitrary $a = 1, ..., n-1$. It is composed of $\frac{n!}{(n-a)!}m^a$ decks that may be grouped according to the number and the angle of their top card. We get $am+1$ groups labelled by $(1,\rho), (2,\rho), ..., (a,\rho), (a+1,1)$ with $\rho \in C_m$, in the same way as the top card. Upon a multiplication by $\mathbf{B}_1$, each group (k,ρ), $1 \le k \le a$, $\rho \in C_m$, yields a term $\mathbf{B}_a$ and the group $(a+1,1)$ gives the term $\mathbf{B}_{a+1}$. This establishes the identity:

$$\forall\, 0 \le a \le n-1, \quad \mathbf{B}_1\mathbf{B}_a = am\mathbf{B}_a + \mathbf{B}_{a+1}. \tag{24}$$

Obviously, $\mathbf{B}_n$ is composed of all possible decks:

$$\mathbf{B}_n = \sum_{g \in G(m,1,n)} g. \tag{25}$$

Consequently, the further shuffling, by applying $\mathbf{B}_1$, produces no more new decks, but only replicates $\mathbf{B}_n$ mn times:

$$\mathbf{B}_1\mathbf{B}_n = nm\mathbf{B}_n. \tag{26}$$

The identity (24) may be rewritten in the form: $\forall\, 0 \le a \le n-1$

$$\mathbf{B}_{a+1} = (\mathbf{B}_1 - am)\mathbf{B}_a. \tag{27}$$

By iteration, we obtain $\forall\, 1 \le a \le n$:

$$\mathbf{B}_a = \mathbf{B}_1(\mathbf{B}_1 - m)(\mathbf{B}_1 - 2m)\cdots(\mathbf{B}_1 - (a-1)m). \tag{28}$$

The left multiplication by $(\mathbf{B}_1 - nm)$ of (28) for $a = n$ yields the identity:

$$\mathbf{B}_1(\mathbf{B}_1 - m)(\mathbf{B}_1 - 2m)\cdots(\mathbf{B}_1 - nm) = 0.$$

Second proof. Let ${}^{(m)}\tilde{\Sigma}_n$ denote the sum of all elements of the group $G(m,1,n)$ ($\mathbf{B}_n$, in the previous notation). The recurrent relation similar to (11) holds:

$${}^{(m)}\tilde{\Sigma}_n = {}^{(m)}\text{Ш}_n \cdot {}^{(m)}\tilde{\Sigma}_{n-1}. \tag{29}$$

By induction on n, let us prove that

$$\prod_{\alpha=0}^{n-1}\left({}^{(m)}\text{Ш}_n - \alpha m\right) = {}^{(m)}\tilde{\Sigma}_n. \tag{30}$$

For $n = 1$, there is nothing to prove. The quadratic relation (18) for ${}^{(m)}\text{Ш}_n$ implies that for all $j \ge 0$:

$${}^{(m)}\text{Ш}_n\left({}^{(m)}\text{Ш}_n - mj\right) = {}^{(m)}\text{Ш}_n\left({}^{(m)}\text{Ш}_{n-1} - m(j-1)\right). \tag{31}$$

Consequently,

$$\prod_{\alpha=0}^{n-1}\left({}^{(m)}\text{Ш}_n - \alpha m\right) = {}^{(m)}\text{Ш}_n \prod_{\beta=0}^{n-2}\left({}^{(m)}\text{Ш}_{n-1} - \beta m\right). \tag{32}$$

By induction hypothesis,

$$\prod_{\beta=0}^{n-2}\left({}^{(m)}\text{Ш}_{n-1}-\beta m\right)={}^{(m)}\tilde{\Sigma}_{n-1}, \tag{33}$$

whence (30). The equality ${}^{(m)}\tilde{\Sigma}_n{}^{(m)}\text{Ш}_n = mn{}^{(m)}\text{Ш}_n$ yields the result (23). □

Let $L_{{}^{(m)}\text{Ш}_n}$ be the operator of left multiplication by ${}^{(m)}\text{Ш}_n$.

Proposition 9. *For all $i=0,...,n$, the multiplicity of the eigenvalue im of the operator $L_{{}^{(m)}\text{Ш}_n}$ is given by the number:*

$$m^{n-i}\binom{n}{i}(n-i)!\sum_{a=0}^{n-i}\frac{(-1)^a}{a!m^a}. \tag{34}$$

Proof. We start by calculating $\mathbf{B}_1^k$ for $0\le k\le n$. Recall that the Stirling number of the second kind has the property:

$$x^k=\sum_{a=0}^{k}S_{k,a}x(x-1)(x-2)\cdots(x-(a-1)), \tag{35}$$

where $x\in\mathbb{C}$ and $k\in\mathbb{N}$. The substitution $x\to\frac{x}{m}$ in (35) yields the identity:

$$x^k=m^k\sum_{a=0}^{k}\frac{1}{m^a}S_{k,a}x(x-m)(x-2m)\cdots(x-(a-1)m). \tag{36}$$

Thus, by (28):

$$\mathbf{B}_1^k=m^k\sum_{a=0}^{k}\frac{1}{m^a}S_{k,a}\mathbf{B}_a. \tag{37}$$

Stirling numbers also satisfy:

$$(e^t-1)^a=\sum_{k=a}^{\infty}S_{k,a}\frac{a!}{k!}t^k \quad\text{or}\quad S_{k,a}=\frac{k!}{a!}(e^t-1)^a|_{t^k}. \tag{38}$$

where $f(t)|_{t^k}$ stands for the coefficient of t^k in the series of $f(t)$. Plugging (38) into (37) gives:

$$\mathbf{B}_1^k=m^k\sum_{a=0}^{k}\mathbf{B}_a\frac{k!}{m^a a!}(e^t-1)^a|_{t^k}. \tag{39}$$

For our purpose, it is convenient to extend the summation in the right hand side of (39) to $a=0,\ldots,n$. This yields:

$$\begin{aligned}\mathbf{B}_1^k&=m^k\sum_{a=0}^{n}\mathbf{B}_a\frac{k!}{m^a a!}(e^t-1)^a|_{t^k}\\&=m^k\sum_{a=0}^{n}\mathbf{B}_a\frac{k!}{m^a a!}\sum_{i=0}^{a}\binom{a}{i}e^{it}(-1)^{a-i}|_{t^k}\\&=m^k\sum_{a=0}^{n}\mathbf{B}_a\frac{k!}{m^a a!}\sum_{i=0}^{a}\binom{a}{i}\frac{i^k}{k!}(-1)^{a-i}\\&=m^k\sum_{a=0}^{n}\frac{1}{m^a}\mathbf{B}_a\sum_{i=0}^{a}\frac{i^k}{i!(a-i)!}(-1)^{a-i}.\end{aligned}$$

By changing the order of summation, we get:

$$\mathbf{B}_1^k = m^k \sum_{i=0}^{n} \frac{i^k}{i!} \sum_{a=i}^{n} \frac{(-1)^{a-i}}{(a-i)!m^a} \mathbf{B}_a. \tag{40}$$

Let

$$\mathbf{E}_i = \frac{1}{i!} \sum_{a=i}^{n} \frac{(-1)^{a-i}}{(a-i)!m^a} \mathbf{B}_a. \tag{41}$$

The equality (40) evaluated at $k = 0$ reads

$$\sum_{i=0}^{n} \mathbf{E}_i = 1. \tag{42}$$

Substituting into the identity

$$\mathbf{B}_1 \mathbf{B}_1^k = \mathbf{B}_1^{k+1}$$

the expressions (40) for $\mathbf{B}_1^k$ and $\mathbf{B}_1^{k+1}$ yields the equality

$$\sum_{i=0}^{n} i^k \left(\mathbf{B}_1 \mathbf{E}_i - im\mathbf{E}_i \right) = 0, \quad k = 0, \ldots, n. \tag{43}$$

Since the matrix $M_{ik} := \{i^k\}_{i,k=1,\ldots,n}$ is invertible, we find

$$\mathbf{B}_1 \mathbf{E}_i = im\mathbf{E}_i, \quad i = 1, \ldots, n. \tag{44}$$

For $k = 0$, the equality (43) reads:

$$\sum_{i=0}^{n} \left(\mathbf{B}_1 \mathbf{E}_i - im\mathbf{E}_i \right) = 0, \tag{45}$$

wherefrom

$$\mathbf{B}_1 \mathbf{E}_0 = 0.$$

Let $\mathcal{B}$ the algebra of polynomials in $\mathbf{B}_1$. By (28), $\mathcal{B}$ is a linear span of $\mathbf{B}_a$, $a = 0, \ldots, n$. By (41), $\mathcal{B}$ is a linear span of $\mathbf{E}_a, a = 0, \ldots, n$. Therefore $\mathbf{E}_i$ and $\mathbf{E}_j$ commute. We have

$$\mathbf{B}_1 \mathbf{E}_i \mathbf{E}_j = mi\mathbf{E}_i \mathbf{E}_j = mj\mathbf{E}_i \mathbf{E}_j, \quad 0 \le i, j \le n,$$

whence

$$\mathbf{E}_i \mathbf{E}_j = 0, \quad 0 \le i \ne j \le n.$$

Multiplying (42) by $\mathbf{E}_i$, we find that

$$\mathbf{E}_i^2 = \mathbf{E}_i, \quad i = 0, \ldots, n.$$

The multiplicity of the eigenvalue im is thus given by the trace of the matrix $L_{\mathbf{E}_i}$:

$$tr(L_{\mathbf{E}_i}) = \frac{1}{i!} \sum_{a=i}^{n} \frac{(-1)^{a-i}}{(a-i)!m^a} tr(L_{\mathbf{B}_a}). \tag{46}$$

Since the trace of any element except the identity vanishes in the regular representation, (46) reduces to:

$$\begin{aligned} tr(L_{\mathbf{E}_i}) &= \frac{1}{i!}\sum_{a=i}^{n}\frac{(-1)^{a-i}}{(a-i)!m^a}m^n n! \\ &= \frac{n!m^{n-i}}{i!}\sum_{a=0}^{n-i}\frac{(-1)^a}{m^a a!} \\ &= \binom{n}{i}m^{n-i}(n-i)!\sum_{a=0}^{n-i}\frac{(-1)^a}{m^a a!}. \end{aligned}$$

Operators $\mathbf{E}_i$, $i = 0, \ldots, n$, are therefore the orthogonal projections onto the eigenspaces of $\mathbf{B}_1$. □

The combinatorial meaning of the multiplicities is explained in the next paragraph.

2.4 m-derangement numbers

An *m-derangement* (or simply *derangement*) in $G(m, 1, n)$ is a rotational permutation under which each card changes either its position or angle. An element $g \in G(m, 1, n)$ has i fixed points if it fixes i m-cards. For given n we call *m-derangement number* the number $d_{m,n}$ of elements in $G(m, 1, n)$ with no fixed points. The number of elements in $G(m, 1, n)$ with i fixed points is equal to $\binom{n}{i}d_{m,n-i}$.

A deck of $n + 1$ m-cards can be deranged in the following ways:

- By rotating the $(n+1)$-st card at its place and by deranging the remaining n cards. There are $(m-1)d_{m,n}$ derangements of this type.
- By inserting the $(n+1)$-st card at the j-th position, $1 \le j \le n$, with an arbitrary turn, the j-th card being moved to the $(n+1)$-st place without any turn, and by deranging the remaining $n-1$ cards. There are $nmd_{m,n-1}$ derangements of this type.
- By inserting the $(n+1)$-st card at the j-th position, $1 \le j \le n$, with an arbitrary turn, and by deranging the remaining cards with the restriction that the j-th card is not allowed to go to $(n+1)$-st place without a turn. The number of such cases is clearly the same as mn times the number of m-derangements of n, that is, $mnd_{m,n}$.

Thus (we assume that $d_{m,0} = 1$)

$$d_{m,n+1} = (mn + m - 1)d_{m,n} + mnd_{m,n-1}\ . \tag{47}$$

Now let us define the function

$$f(t) := \sum_{n=0}^{\infty}\frac{t^n}{n!}d_{m,n+1} \tag{48}$$

and substitute $d_{m,n+1}$ by its expression in (47). This yields:

$$m^{-1}f(t) = tf(t) + \left(1 + t - m^{-1}\right)g(t) \tag{49}$$

with

$$g(t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} d_{m,n}. \tag{50}$$

Since $g'(t) = f(t)$, (49) becomes a differential equation:

$$g'(t) + \left(1 - \frac{1}{m^{-1} - t}\right) g(t) = 0$$

with the initial condition $g(0) = 1$. The solution is given by:

$$g(t) = \frac{e^{-t}}{1 - mt},$$

or

$$g(t) = \sum_{n=0}^{\infty} c_n t^n \quad \text{with } c_n = m^n \sum_{k=0}^{n} \frac{(-1)^k}{k! m^k}.$$

By identification with (50) we finally get:

$$d_{m,n} = m^n n! \sum_{k=0}^{n} \frac{(-1)^k}{k! m^k}.$$

Corollary. *The multiplicity of the eigenvalue im of $L_{(m)Ш_n}$ is equal to $\binom{n}{i} d_{m,n-i}$, the number of elements of $G(m,1,n)$ with i fixed points.*

Remarks.

(i) The spectrum of $L_{(1)Ш_n}$ is $\{0, 1, \ldots, n-2, n\}$, because $d_{1,1} = 0$; the spectrum has a gap at $n-1$. For $m > 1$, the gap disappears.

(ii) The Eq. (47) admits a second solution which is $d_{m,n} = m^n n!$.

2.5 Cyclotomic algebras

The multiplicities of the eigenvalues of shuffle elements $Ш_{1,n-1}$ have been derived in a different way in [22]. That approach relies on the Hecke algebra $H_n(q)$, the deformation of the group algebra of the symmetric group.

The Hecke algebra of $G(m,1,n)$ denoted by $H(m,1,n)$ was introduced independently by Ariki and Koike [3], and Broué and Malle [8], see [2] for further details.

In this section, we describe some preliminary calculations of the spectrum of the analogue of ${}^m Ш_{1,n-1}$ for the algebra $H(m,1,n)$ with $m = 2$.

Definition 6 (Cyclotomic Hecke algebra). Let $m > 0$ and $n \geq 0$. The cyclotomic Hecke algebra $H(m,1,n)$ is the associative algebra over $\mathbb{C}\left[q^{\pm}, v_1^{\pm}, \ldots, v_m^{\pm}\right]$ generated by $\tau, \sigma_1, \ldots, \sigma_{n-1}$ with the defining relations:

$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for all } i = 1, \ldots, n-2\ , \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for all } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1, \\ \tau \sigma_1 \tau \sigma_1 = \sigma_1 \tau \sigma_1 \tau & \\ \tau \sigma_i = \sigma_i \tau & \text{for } i > 1\ , \\ \sigma_i^2 = (q - q^{-1}) \sigma_i + 1 & \text{for all } i = 1, \ldots, n-1\ , \\ (\tau - v_1) \ldots (\tau - v_m) = 0. & \end{cases}$$

The operators $J_1 := \tau,\ J_{i+1} := \sigma_i J_i \sigma_i,\ 1 \leq i \leq n-1$ are called *Jucys-Murphy elements*[9].

The algebra $H(m,1,n)$ is the flat deformation of the complex reflection group algebra $\mathbb{C}G(m,1,n)$; the specialisation $q = \pm 1,\ v_i = \zeta^i,\ i = 0,\dots,m-1$ where ζ is a primitive m-th root of unity, yields the group algebra $\mathbb{C}G(m,1,n)$.
The algebras $H(m,1,n)$ form an ascending chain of algebras with respect to n:

$$H(m,1,0) \subset H(m,1,1) \subset H(m,1,2)\dots \tag{51}$$

As explained in [35], $H(m,1,n)$ is spanned by elements $\sigma_j^{-1}\sigma_{j-1}^{-1}\cdots\sigma_1^{-1}\tau^\alpha\sigma_1\cdots\sigma_{n-1}w$ with $j \in \{0,\dots,n-1\},\ \alpha \in \{0,\dots,m-1\}$ and $w \in \widetilde{W}$ where $\widetilde{W}$ is the subalgebra generated by $\tau,\sigma_1,\dots,\sigma_{n-2}$. This is an analogue of the normal form discussed in §1.4.3. We have $\dim(H(m,1,n)) = m^n n!$.

The representation theory of the cyclotomic Hecke algebra has been constructed in [3] and reobtained in [32] from the analysis of the spectrum of the Jucys-Murphy elements[10]. A set of common eigenvalues of $J_1,\dots,J_n$ turns out to be related to a Young m-tableau.

It is well known that irreducible representations of the symmetric group are labelled by Young diagrams. The dimension of the representation V_λ, corresponding to the Young diagram λ, equals the number of standard tableaux of the shape λ. The Bratteli diagram for the chain of the symmetric groups is called the Young graph. The Bratteli diagram for the chain (51) of the cyclotomic Hecke algebras corresponds to the m-th power of the Young graph. Figure 9 is an example of such a diagram.

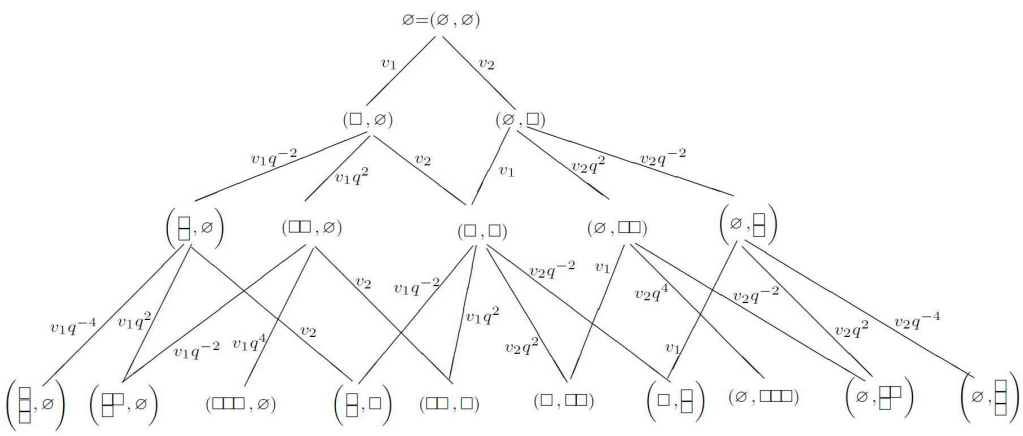


FIG. 9: Bratteli diagram for the chain $H(2,1,0) \subset H(2,1,1) \subset H(2,1,2) \subset H(2,1,3)$ from [32].

A Young m-diagram, or m-partition, $\lambda^{(m)}$ is an m-tuple of Young diagrams $\lambda^{(m)} = (\lambda_1,\dots,\lambda_m)$. The size of a Young diagram λ is the number of nodes of the diagram and is denoted by $|\lambda|$. By definition the size of an m-tuple $\lambda^{(m)} = (\lambda_1,\dots,\lambda_m)$ is

$$|\lambda^{(m)}| := |\lambda_1| + \cdots + |\lambda_m|.$$

[9]The Jucys-Murphy elements form a maximal commutative set in the algebra $H(m,1,n)$. Besides, $J_i\sigma_k = \sigma_k J_i$ if $k \notin \{i-1,i\}$.

[10]The approach to the representation theory of the symmetric groups using the spectrum of Jucys-Murphy elements has been developed by [36]. It was generalized to the Hecke algebras in [21], to the groups $G(m,1,n)$ in [34] and to the algebras $H(m,1,n)$ in [32, 33].

An m-node $\alpha^{(m)}$ is a pair (α, p) corresponding to the cell α in the Young diagram λ_p, $1 \leq p \leq m$. The number p is referred to as a "position" of the m-node $\alpha^{(m)}$. An m-partition $\lambda^{(m)}$ of size n filled with numbers $1, \ldots, n$ is an m-tableau of size n denoted by $X_{\lambda^{(m)}}$. If $X_{\lambda^{(m)}}$ is composed of standard Young tableaux (entries in each row and column of all tableaux λ_i of $X_{\lambda^{(m)}}$ increase in the right and down directions), then $X_{\lambda^{(m)}}$ is said to be *standard m-tableau.* Additionally, we associate to each m-node $\alpha^{(m)}$ of $X_{\lambda^{(m)}}$ a number (the "content") which is $v_k q^{2(s-r)}$ where k, r, s correspond to the position, the line and column of the node $\alpha^{(m)}$ respectively. Here is an example from [32] of a standard Young 2-tableau of size 10 filled with the content:

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ v_1 & v_1 q^2 & v_1 q^4 \\ \hline 6 & 9 & \\ v_1 q^{-2} & v_1 & \\ \cline{1-2} 7 & & \\ v_1 q^{-4} & & \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|} \hline 3 & 8 & 10 \\ v_2 & v_2 q^2 & v_2 q^4 \\ \hline 5 & & \\ v_2 q^{-2} & & \\ \cline{1-1} \end{array} \right)$$

The content of the m-node carrying the number i will be denoted by $c\left(X_{\lambda^{(m)}}|i\right)$. By definition, for any permutation $\pi \in S_n$, the m-tableau $X^{\pi}_{\lambda^{(m)}}$ is obtained from the m-tableau $X_{\lambda^{(m)}}$ by applying the permutation π to the numbers occupying the m-nodes of $X_{\lambda^{(m)}}$,

$$c\left(X^{\pi}_{\lambda^{(m)}}|i\right) = c\left(X_{\lambda^{(m)}}|\pi^{-1}(i)\right), \quad \forall i = 1, \ldots, n.$$

Now let $U_{\lambda^{(m)}}$ be a vector space spanned by vectors labelled by the standard m-tableaux $X_{\lambda^{(m)}}$ of size n. The basis vectors will be generically denoted by $\mathcal{X}_{\lambda^{(m)}}$. There is the representation $\rho_{\lambda^{(m)}} : H(m,1,n) \to \mathrm{End}(U_{\lambda^{(m)}})$, with the following formulas for the action of $\tau, \sigma_1, \ldots, \sigma_{n-1}$ on the basis vectors $\mathcal{X}_{\lambda^{(m)}}$:

$$\begin{aligned} \rho_{\lambda^{(m)}}(\sigma_i) : \mathcal{X}_{\lambda^{(m)}} \mapsto & \frac{(q-q^{-1})c\left(X_{\lambda^{(m)}}|i+1\right)}{c\left(X_{\lambda^{(m)}}|i+1\right) - c\left(X_{\lambda^{(m)}}|i\right)} \mathcal{X}_{\lambda^{(m)}} \\ & + \frac{qc\left(X_{\lambda^{(m)}}|i+1\right) - q^{-1}c\left(X_{\lambda^{(m)}}|i\right)}{c\left(X_{\lambda^{(m)}}|i+1\right) - c\left(X_{\lambda^{(m)}}|i\right)} \mathcal{X}^{\pi_i}_{\lambda^{(m)}}, \end{aligned} \tag{52}$$

$$\rho_{\lambda^{(m)}}(\tau) : \mathcal{X}_{\lambda^{(m)}} \mapsto c\left(X_{\lambda^{(m)}}|1\right) \mathcal{X}_{\lambda^{(m)}} \tag{53}$$

where π_i is the transposition $(i, i+1)$. In (52), $\mathcal{X}^{\pi_i}_{\lambda^{(m)}}$ is set to zero if $X^{\pi_i}_{\lambda^{(m)}}$ is not a standard m-tableau.

Using the formulas (52) and (53), we have examined for $m = 2$ and small n the spectrum of the deformation of the 1-shuffle in various representations[11] of $H(2,1,n)$. Namely, consider the cyclotomic Hecke algebra $H(2,1,n)$ with $v_1 = p$ and $v_2 = -p^{-1}$. Let elements τ_i be recursively defined by:

$$\tau_i = \sigma_{i-1}^{-1} \tau_{i-1} \sigma_{i-1}, \quad i = 2, \ldots, n \tag{54}$$

with $\tau_1 = \tau$. We propose the following q-deformation of the 1-shuffle:

$$^{(2)}Ш_{1,n-1;q} := \left(1 + q\sigma_{n-1} + q^2 \sigma_{n-2}\sigma_{n-1} + \cdots + q^{n-1}\sigma_1 \cdots \sigma_{n-1}\right)(1 + p\tau_n), \tag{55}$$

[11] The spectrum of the q-analogue of $Ш_{1,n-1}$ for certain representations of the Hecke algebra $H_n(q)$ is found in [14].

Our calculations lead to the following conjecture.

Conjecture 10. *The spectrum of the left multiplication by the 1-shuffle* ${}^{(2)}Ш_{1,n-1;q}$ *in the algebra* $H(2,1,n)$ *consists of the values:*

$$\left(p^2 + q^{2(j-\ell)}\right) q^{\ell-1}[\ell]_q \tag{56}$$

where $q^{\ell-1}[\ell]_q$ *are the q-numbers defined in (2),* $\ell = 0, 1, \ldots, n$ *and* $j = \ell, \ell+1, \ldots, n-2, n$.

Remark. We numerically observed (for $n \leq 5$) that the multiplicities of the eigenvalues whose classical limit (obtained by taking $q = p = 1$) corresponds to the eigenvalues $0, 2(n-1), 2n$ still corresponds to the number of elements of $G(2,1,n)$ with $0, (n-1)$ and n fixed points respectively. This is not the case for the other eigenvalues.

2.6 Asymptotic analysis of top-to-random shuffling

In this section, we use the results of the basic limit theory of finite Markov chains to estimate how close k repeated shuffles, treated as a random walk on $G(m,1,n)$, get the deck of n m-cards to being randomized.
We start by reminding the basic concepts of Markov chains. For the more detailed discussion and proofs see e.g. [13, 15], that we summarily follow here.

A set of random variables $\{X_t, t \in T\}$ taking values in a set $\mathcal{S} = \{E_i\}_{i \in I}$, where T is a set of indices representing *time*, X_t is the *state* of the system at time t and $\mathcal{S}$ is the *set of states*, is a stochastic process (i.e. a random process in time) called a Markov chain, if it is memoryless: the outcome of any trial (experience) depends on the outcome of the directly preceding trial and only on it. Formally,

$$\begin{aligned}\mathbb{P}\left(X_{t+1} = E_j \mid X_0 = E_{i_0}, X_1 = E_{i_1}, \ldots, X_{t-1} = E_{i_{t-1}}, X_t = E_i\right) \\ = \mathbb{P}\left(X_{t+1} = E_j | X_t = E_i\right)\end{aligned} \tag{57}$$

for all $E_{i_0}, \ldots, E_{i_{t-1}}, E_i, E_j \in \mathcal{S}$ and $t \in T$. Here we assume that the Markov chain (MC) is discrete time, that is, that T is discrete. The MC is said to be time-homogeneous if the conditional probabilities in (57) only depend on state E_i and E_j and not on t, i.e.

$$\mathbb{P}\left(X_{t+1} = E_j | X_t = E_i\right) = \mathbb{P}\left(X_1 = E_j | X_0 = E_i\right) = p_{ij} \quad \forall t \geq 0. \tag{58}$$

The Markov chain is said to be *finite* if the set of states $\mathcal{S}$ is finite. Finally, p_{ij} is called the *probability of transition from* E_i *to* E_j. The transition probabilities are arranged in a *matrix of transition probabilities*

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly P is a square matrix with non-negative entries and unit row sums:

$$\sum_j P_{ij} = 1$$

for all i. Such a matrix (finite or infinite) is called a stochastic matrix. If in addition the entries of any column sum to 1, i.e. for all j it is $\sum_i P_{ij} = 1$ then the matrix P is called

doubly stochastic.

Given the transition matrix P, the ij-th entry of the matrix P^k gives the conditional probability $p_{ij}^{(k)} := \mathbb{P}(X_k = E_j \mid X_0 = E_i)$ that the Markov chain, starting in state E_i enters in state E_j in k steps; $p_{ij}^{(k)}$ is the sum of the probabilities of all possible paths $E_i E_{i_1} \dots E_{i_{k-1}} E_j$ of length k starting at E_i and ending at E_j. In particular $p_{ij}^{(1)} = p_{ij}$. The identity

$$p_{ij}^{(k+\ell)} = \sum_{\nu} p_{i\nu}^{(k)} p_{\nu j}^{(\ell)},$$

which is nothing else but the equation $P^{k+\ell} = P^k P^\ell$, holds for all $k, \ell \geq 0$ provided that $p_{ij}^{(0)} := \delta_{ij}$.

Let $a = (a_i,\, i \in I)$ stand for the initial distribution where $a_i = \mathbb{P}(X_0 = E_i)$ is the probability that the MC starts out in state E_i. The absolute probability of entering E_j at the k-th step is then

$$a_j^{(k)} = \mathbb{P}(X_k = E_j) = \sum_i \mathbb{P}(X_0 = E_i)\, \mathbb{P}(X_k = E_j \,|X_0 = E_i) = \sum_i a_i p_{ij}^{(k)}.$$

The row vector $a^{(k)} = \left(a_i^{(k)},\, i \in I\right)$ is the distribution of the MC at time k. We have $a^{(0)} = a$ and for all $k > 0$ $a^{(k)} = aP^k = a^{(k-1)}P$.

The transition matrix together with the initial distribution give a complete probabilistic description of a MC. Usually we let the process start from a fixed state E_i, that is, we put $a_j = \delta_{ij}$ for all j. In this case $a_j^{(k)} = p_{ij}^{(k)}$.

The states of a MC can be classified from two independent point of views: persistent/ transient states on the one hand, periodic/aperiodic on the other hand. The first is fundamental, whereas the second concerns a technical detail.
We shall denote by $f_{ij}^{(k)}$ the probability of *first visit* of the state E_j after k steps starting from the state E_i:

$$f_{ij}^{(k)} = \mathbb{P}(X_k = E_j, X_1 \neq E_j, \dots, X_{k-1} \neq E_j \mid X_0 = E_i)$$

with $f_{ij}^{(0)} := 0$, for all i, j. We define the probability f_{ij} that, starting from E_i, the system will ever pass through E_j:

$$f_{ij} = \sum_{k=1}^{\infty} f_{ij}^{(k)} \leq 1.$$

The *mean recurrence time* for the state E_j is given by:

$$\mu_j = \sum_{k=1}^{\infty} k f_{jj}^{(k)} \leq \infty.$$

Definition 7. A state E_j to which a return is certain i.e. such that $f_{jj} = 1$ is called *persistent*. A state E_j for which $f_{jj} < 1$ is called *transient*. A persistent state E_j is called *null-persistent* if its mean recurrence time $\mu_j = \infty$. A persistent state E_j with $\mu_j < \infty$ is called *non null-persistent*.

Remark: A transient state will only be visited a finite number of times.

Definition 8. The state E_j has *period* $t > 1$ if $p_{jj}^{(k)} = 0$ unless $k = \nu t$ is a multiple of t, and t is the largest integer with this property. The state E_j is *aperiodic* if no such $t > 1$ exists.

It means that the MC visits a periodic state only at times which are non zero multiples of t. Every state E_i for which $p_{ii} > 0$ and every state E_j to which no return is possible (for which $p_{jj}^{(k)} = 0$ for all $k > 0$) are necessarily aperiodic.

Definition 9. Two states are said to be *of the same type* if they share all characteristics: they have the same period or they are both aperiodic; both are transient or both are persistent; in the latter case either both mean recurrence times are infinite, or else both are finite.

The long-term behaviour of a MC is related to how often states are visited and how accessible the states are from each other. We shall say that *E_j can be reached from E_i (we write $E_i \to E_j$) if there exists some $n \geq 0$ such that $p_{ij}^{(n)} > 0$. A state E_i communicates with a state E_j (we write $E_i \leftrightarrow E_j$) if $E_i \to E_j$ and $E_j \to E_i$.* Communication is an equivalence relation on the state space that thus can be partitioned into (disjoint) equivalence classes, called communication classes.

Definition 10. A MC is called *irreducible* if, and only if, there exist no communication class other then the state space $\mathcal{S}$.

Theorem 11. *All states of an irreducible MC are of the same type.*

Corollary. *In a finite irreducible MC all states are non nul-persistent.*

Definition 11. An irreducible MC whose states are all aperiodic non null-persistent is called *ergodic*.

Definition 12. Let $(X_0, X_1, \dots)$ be a MC with the state space $\mathcal{S}$ and the transition matrix P. A distribution $u = (u_j,\, j \in I)$ over $\mathcal{S}$ is said to be *invariant* or *stationary* if

$$u = uP$$

i.e. $u_j = \sum_i u_i P_{ij}$ for all j.

If the initial distribution is stationary then $X_0, X_1, \dots$ is a sequence of identically distributed random variables. The stationary distribution does not necessarily exist, nor is necessarily unique.

Lemma 12. *The uniform distribution of a finite MC is stationary if its transition probability matrix P is doubly stochastic.*

Indeed, if $u = (u_j,\, j \in I)$ is the distribution over the state space $\mathcal{S}$ such that $|\mathcal{S}| = N$ and $u_j = \frac{1}{N}$ for all j, and if P is doubly stochastic then $(uP)_j = \sum_i u_i P_{ij} = \frac{1}{N} \sum_i P_{ij} = \frac{1}{N} = u_j$ for all j.

Definition 13. Let $(X_0, X_1, \dots)$ be a MC with the state space $\mathcal{S}$ and the transition matrix P. A distribution $u = (u_j,\, j \in I)$ over $\mathcal{S}$ is said to be limiting if

$$\lim_{k \to \infty} p_{ij}^{(k)} = u_j \quad \text{for all } i.$$

This definition is equivalent to: u is a limiting distribution $\Leftrightarrow$ for any initial distribution a, we have $aP^k \underset{k \to \infty}{\to} u$.

A limiting distribution does not necessarily exist, but if it exists, then it is unique. If u is a limiting distribution, then it is stationary.
The following theorem is important in the theory of Markov chains.

Theorem 13. *An irreducible aperiodic Markov chain possesses an invariant probability distribution $u = (u_j,\, j \in I)$ if, and only if, it is ergodic. In this case u is the limiting distribution of the MC, and $u_j = \mu_j^{-1}$ for all j.*

We now return to our original problem. For a given m we shall investigate the matrix of the left multiplication by the 1-shuffle ${}^{(m)}Ш_{1,n-1}$. We shall see that such a matrix serves, if suitably normalized, as a matrix of transition probabilities. The bottom-to-random shuffling rule encoded in ${}^{(m)}Ш_{1,n-1}$ is effective in the sense that the randomization of the deck is achieved, yet slowly, and the underlying Markov chain is easy to analyse mathematically.

Let X_k be the state of the deck after $k \geq 0$ shuffles randomly picked among the set $\{s_j s_{j+1} \dots s_{n-1} t_n^r,\ j = 0, \dots, n-1,\ r = 0, \dots m-1\}$ of elements of $G(m,1,n)$ whose sum is ${}^{(m)}Ш_{1,n-1}$. The sequence of random variables $X_0, X_1, X_2, \dots$ taking values in $G(m,1,n)$ is a (time-homogeneous discrete time) finite Markov chain with the transition probabilities

$$p_{ij} = \begin{cases} \frac{1}{mn} & \text{if } E_j = s_\ell s_{\ell+1} \cdots s_{n-1} t_n^r E_i \ \text{ for some } \ \ell < n,\ r < m; \\ 0 & \text{elsewise} \end{cases}$$

Consider an operator $L_{{}^{(m)}Ш_{1,n-1}}$ of the left multiplication by the 1-shuffle ${}^{(m)}Ш_{1,n-1}$:

$$\begin{aligned} L_{{}^{(m)}Ш_{1,n-1}} \colon \mathbb{Z}G(m,1,n) &\to \mathbb{Z}G(m,1,n) \\ g &\mapsto {}^{(m)}Ш_{1,n-1} \cdot g. \end{aligned}$$

Let A be its matrix in the basis $(E_i)_{i \in I}$ where $I = [\![1, m^n n!]\!]$. Up to a multiplicative constant, the matrix P of transition probabilities is the transpose of the matrix A:

$$P = \tfrac{1}{mn} A^t. \tag{59}$$

In virtue of proposition 8, the matrix P admits $n+1$ eigenvalues: $1,\ \frac{n-1}{n},\ \frac{n-2}{n}, \dots,\ \frac{1}{n},\ 0$. Because it belongs to the convex envelope of the set of permutation matrices, the matrix P is doubly stochastic. More intuitively, when the bottom card is rotated and placed at random position in the deck (this is the shuffling rule described by ${}^{(m)}Ш_{1,n-1}$), there are mn potential new states of the deck. Also, each state can be reached in one step from mn states with the probability $\frac{1}{mn}$ from each.

Since every state can be reached from every other state (in a finite number of steps), the chain is irreducible.

A sufficient condition for an irreducible MC to be aperiodic is that $P_{ii} > 0$ for some i. But $P_{11} > 0$, our chain is thus aperiodic. Finally, the group $G(m,1,n)$ being finite and the chain irreducible, all states are non nul-persistent. It follows that the chain is ergodic. By the theorem (13), the chain possesses a unique invariant probability distribution $(u_j, j \in I)$. It satisfies

$$\lim_{k \to \infty} p_{ij}^{(k)} = u_j = \frac{1}{m^n n!}, \quad \forall i, j \in I. \tag{60}$$

The eq.(60) simply tells us that as the number of shuffles tends to infinity, all possible states of the deck become equally likely. (Also, every state is expected to be revisited every $m^n n!$ steps). Needless to say, the conclusion for the top-to-random shuffling described by $\mathbf{B_1}$ in (21) will be the same. Using the spectral representation formula for the k-step transition matrix [23, 39], one can show that the convergence to the stationarity occurs

exponentially fast with the rate of convergence governed by the second largest eigenvalue: there exists a constant $\alpha > 0$ such that:

$$\forall k \geq 1, \quad |P_{ij}^k - u_j| \leq \begin{cases} \alpha \left(\frac{n-2}{n}\right)^k & \text{if} \quad m = 1, \\ \alpha \left(\frac{n-1}{n}\right)^k & \text{if} \quad m > 1. \end{cases}$$

The speed of convergence is the same for m-cards with $m > 1$ and is slower compared to the classical cards.

The asymptotic rate of convergence only tells a part of the story of a Markov chain's path to stationarity. In particular, it misses the so-called cut-off phenomenon. As it has been analysed in [1] (by means of the notion of strong stationary time), the randomization occurs abruptly after $\sim n \log_2 n$ shuffles (for classical cards).
The shuffling process can also be considered from the point of view of information theory based on the the Shannon entropy $U = -\sum_{i=1}^{n!} p_i \log_2 p_i$ where $p_i = \frac{1}{n!}$ $\forall i$ and the information $I = \log_2(n!) - U$. The latter is a measure of randomisation for the case $m = 1$. Shuffles remove the information from the deck until asymptotically all the information is gone [42]. According to the numerical analysis in [42], $\sim \log_2 n$ shuffles would be enough to achieve this.

Conclusion

The Todd-Coxeter algorithm applied to the chain of complex reflection groups $G(m, 1, n)$ provides the normal form for the elements of $G(m, 1, n)$. We have presented the set of instructions needed to rewrite any word in its normal form. We have interpreted the group $G(m, 1, n)$ as a group of rotational permutations of m-cards.

We suggested analogues of 1-shuffles for the group rings $\mathbb{Z}G(m, 1, n)$. We have established a quadratic relation involving ${}^{(m)}Ш_n$ and ${}^{(m)}Ш_{n-1}$. We derived the minimal polynomial of the shuffle ${}^{(m)}Ш_n$ in two different ways. The first one appeals to the m-card interpretation of the group $G(m, 1, n)$. The second one is based on the above mentioned quadratic relation. We have given an explicit expression for the eigenprojectors of the shuffle ${}^{(m)}Ш_n$, mimicking the argument in [43] for the symmetric group. The multiplicities of the corresponding eigenvalues are calculated by taking traces of the eigenprojectors. The eigenvalues of $\frac{1}{mn}{}^{(m)}Ш_n$ are of the form $\frac{j}{n}$, $j = 0, \ldots, n$. The multiplicity of the eigenvalue $\frac{j}{n}$ turns out to correspond to the number of elements of $G(m, 1, n)$ that have j fixed points in the m-card representation. For $m = 1$ the eigenvalue $\frac{n-1}{n}$ is absent. For $m > 1$ the eigenvalue $\frac{n-1}{n}$ appears.

We have introduced the 1-shuffle analogues for the Hecke algebras of the hyperoctahedral groups $G(2, 1, n)$. By experimenting with the representation theory of $H(m, 1, n)$, we conjectured the spectrum of the left multiplication by ${}^{(2)}Ш_{1,n-1;q}$. In future, we hope to investigate closer the structure of its spectrum and generalize the result to all cyclotomic algebras $H(m, 1, n)$.

Finally, considering repeated shuffling as a random walk on $G(m, 1, n)$, we have found that the rate of the asymptotic convergence to randomness is the same for all m-cards with $m > 1$ and is slower compared to classical cards.

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