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## Anti-conformism and Social Networks

Alexis Poindron

► **To cite this version:**

Alexis Poindron. Anti-conformism and Social Networks. Economics and Finance. 2016. dumas-02102411

**HAL Id: dumas-02102411**

**<https://dumas.ccsd.cnrs.fr/dumas-02102411>**

Submitted on 18 Apr 2019

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# Anti-conformism and Social Networks

Alexis Poindron

Université Paris I Sorbonne, UFR de sciences économiques 02  
Master 2 Économie Théorique et Empirique 2015-2016  
Mémoire présenté et soutenu par Alexis Poindron le 17/05/2016  
Sous la direction de Michel Grabisch et Agnieszka Rusinowska

May 17, 2016

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# 1 Introduction

## 1.1 Literature review and context of this master thesis

The literature on social learning, voting games and contagion is vast. One of the widely used models is the DeGroot's model of influence presented in his seminal paper of 1974 [15]. This is a highly tractable model to describe non-anonymous influence processes. In this model, agents start from an initial opinion and weight at each step the opinions of their neighbors to update their opinions. The process does not always converge but there exists sufficient conditions or necessary and sufficient conditions to ensure the convergence. For example, a process represented by an influence matrix  $T$  is convergent if and only if every set of nodes that is strongly connected and closed is aperiodic.

Literature on social learning has extended over the last twenty-five years over a broad range of approaches. Jackson [29] analysis the speed of convergence in the DeGroot model. The dynamic of influence in which agents make a 'yes'-'no' decision has been examined in different frameworks by Asavathiratham [2] and [3] and Grabisch and Rusinowska [25] and [27] where the latter apply the Markov chain theory to design a stochastic model of influence and relate their findings with Asavathiratham ones. In [24], Grabisch and Rusinowska propose a model in which rows and columns of a stochastic matrix do not correspond to agents but rather to sets of agents saying 'yes'. In Bala and Goyal [5], two technologies compete for customers. The authors examine whether conformism or diversity will prevail. Golub and Jackson [23] study learning and influence with a model in which agents receive signals about a true state of the world. Their model examines the convergence toward a consensus on the true state of the world. Morris [34] addresses the problem of contagion by a model in which agent choose to follow an action if it had been adopted by a certain proportion of their neighbors; the author examines under which conditions this behavior can spread to a whole network. López-Pintado [32] analyses how a new product or a new technology diffuses in a social complex network. On the literature on learning, we can quote also [17], [19] and in particular [18] in which agents decide which of two technologies to adopt based on the experience of their neighbors.

In DeGroot related models, imitation is assumed from the outset, it has nothing to do with rationality or irrationality. But other models of social influence have been developed in which imitation can be a rational phenomenon [6]. Social learning where agents play sequentially has been examined by the seminal papers of Banerjee [6] and [9]. Endogenous timing and strategic delay have been addressed in [12], [13], [36] and [39]. Experimental experiments in a laboratory have been designed to determine whether players play Bayesian or DeGroot [1], [11] and [14]. Bayesian learning is examined in [22].

The literature on influence have mainly focused on positive influence, which admittedly fits many economic situations, for example in models of networks externalities of different kinds that may arise from increasing returns to scale technologies or compatibility between technologies. The literature on network externalities started in particular under the impulsion of Rohlfs [35], Katz and Shapiro [30] and [31]. Of of the main focus of this literature is the coordination among consumers. Consumers want indeed to coordinate on which technology to adopt ("vote for the same technology"; this is the famous exemple of the war between VHS and Betamax) in order to benefit from network externalities.

However, the negative influence has been very little examined yet.

In this master thesis, we study a society where some agents are anti-conformists. Treating the case of anti-conformism is very natural to explain human behavior; it is probably one of the first ideas that come to our mind when thinking of influence, especially in the light of fashion trends. We will see that anti-conformism can be a tool to explain dynamic phenomena. For example, explaining the intrinsic dynamism in fashion trends is one application. More generally anti-conformism is likely to find many economic applications. For example in a network with externalities, the choice for a firm is whether to go compatible with other firms or not can be recasted in the context of anti-conformism. Models related to anti-conformism can be found in Grabisch and Rusinowska in [26], who adress the problem of negative influence of coalitions on agents, Bramoulle and al. [10] and D. López-Pintado [33] which last two articles examine anti-coordination games.

We can define anti-conformism in various ways. In a 'yes-no' decision problem, we can imagine for example agents who say 'no' (resp. 'yes') when a majority of agents say 'yes' (resp. 'no'). Or we can imagine agents who say 'no' (resp. 'yes') when all the other agents say 'yes' (resp. 'no'). Or we can, and this is the framework that we adopt here, consider positive and negative influences. This approach is more general and can easily accomodate the previous ones.

We adress the problem of anti-conformism with a simple approach, by introducing anti-conformism in an anonymous framework, as a pure imitation process. Anonymity is a peculiar social network in which all agents are potentially connected, but only the situation determines if and how an agent weights another one. We generalize the notion of ordered weighted averages in an anonymous social network to anti-conformism. OWA were introduced by Yager [37] and [38] and were applied to decision theory in Förster and al. [20], from which this master thesis takes its inspiration. Our aim is to provide an analysis of the terminal states and classes. The specificity of our focus lead us to introduce two groups of agents: the conformist agents and the anti-conformist agents. They update their opinion at each stage of the game by aggregating the opinions of the other agents. In [20], aggregation functions have two properties that characterize a certain kind of positive influence: non-decreasingness and boundary conditions. These assumptions are not specific to anonymity: in [27] provides a complete study of the terminal classes under those aggregation functions where in general there is no anonymity. But our anonymous framework with anti-conformism cannot assume non-decreasingness and monotonicity. To adress this seemingly difficult problem, we simplify the framework. Instead of assuming, as Förster and al did, that all agents have a different weight vector, we start our analysis by taking the same weigh vector for everyone, and then, in another section, we assume that agents of the same type (conformists or anti-conformists) have the same weigh vector. We provide analogous results on convergence and examine in details the structure of the terminal classes defined by the societies considered. At the end of this master thesis, we explain how our work can explain some dynamical phenemona in fashion trends. Social learning has indeed already been argued to be capable of explaining behavior in trends and fashion [7] and [9]. Our perspective is different: we take for granted that fashion can be a conformist or an anti-conformist action and that it is not a rational action but rather a pure imitation process (there is no "true state of the world").

This master thesis is organized as follows. The end of this section recalls some basic concepts and introduces the notations used throughout this work. In Section 2 we give the definition of this new aggregator and a few exemples. Section 3 examines what we call the "homogeneous

case", that is, when all agents weigh the 'yes' and the 'no' in the same fashion (they have the same weigh vector). Section 4 examines the heterogeneous case: in this one there are two groups, the conformists and the anti-conformists, each one with a corresponding weight vector. Section 5 suggests some applications of this study. Section 6 summarizes the results and discusses them. Section 7 concludes on the applications and presents the future research that shall be conducted.

We draw the attention of the reader to the fact that we don't always explicitly state in each proposition and theorem whether the framework is the homogeneous or heterogeneous one. This is implicitly stated given the section in which they are presented.

## 1.2 Some notations and definitions

We give in this subsection definitions and notations used in this master thesis. We denote by  $|N|$  or  $n$  the cardinality of the set  $N$ . More generally, we abide by the convention of using the minuscule letters for the cardinality of the sets in capital letters. The only capital letters that we use for other objects than sets are  $A$  and  $P$  that we use in the definition of GOWA and PEWA respectively.

We omit the braces for sets. For example  $N \setminus \{i\}$  will be written  $N \setminus i$ . We write  $J \subset K$  if  $J$  is a proper subset of  $K$  and  $J \subseteq K$  if we allow for identity of sets. For  $A \subseteq B$ , we write  $[A, B] := \{A \cup K \mid K \subseteq B \setminus A\}$ . Set complementation is denoted with a tilde, i.e,  $\tilde{S} := N \setminus S$  where  $N$  is the referential set and  $S$  a subset of it.

Cycles are denoted with  $\rightarrow$ , for example  $N^a \rightarrow N^c$ . A cycle must be at least of length 2.

The minimum between  $a$  and  $b$  is denoted  $a \wedge b$ .

The maximum between  $a$  and  $b$  is denoted  $a \vee b$ . We use capital greek letters for matrices. For  $\Theta_1$  and  $\Theta_2$  two matrices of the same dimensions, we write  $\Theta_1 \leq \Theta_2$  if all entries of  $\Theta_1$  are lower or equal to the respective entries of  $\Theta_2$ .

The set of all agents denoted by  $N$ . Each player  $i \in N$  makes a 'yes' decision (denoted  $x_i = 1$ ) or a 'no' decision (denoted  $x_i = -1$ ) at each stage of the influence process. Throughout this master thesis we use the equivalent set notation, that is,  $S \subseteq N$  is the set of agents with the inclination to say 'yes'.

We denote  $1_S$  the inclination vector of all agents at the state  $S$ . It is an element of  $\{-1, 1\}^n$ .

A  $n$ -place *aggregation function* is any mapping  $A : \{-1, 1\}^n \rightarrow [-1, 1]$ .

The probability for agent  $j$  to say 'yes' at the next state is a change of scale of the output of its aggregation function:  $p_j(1_S) = \frac{1}{2}(1 + A(1_S))$ .

When the set of 'yes' agents is  $S$  we denote by  $\lambda_{S,T}$  the probability that the set  $S$  of 'yes' agents becomes at the next stage  $T$ . We can represent our aggregation model by a time-homogeneous Markov chain with transition matrix of dimension  $2^N \times 2^N$ :  $\Lambda = (\lambda_{S,T})_{S,T \subseteq N}$ . If the probabilities of saying "yes" are independent among agents and only depend on the current state, then the

probability of transition from  $S$  to  $T$  is:

$$\lambda_{S,T} = \prod_{i \in T} p_i(1_S) \prod_{i \notin T} (1 - p_i(1_S))$$

If there can be an ambiguity on the society considered, then we write  $\lambda_{S,T}^q$  the probability transition from  $S$  to  $T$  in the society defined by  $q := (w^a, w^c, n^a, n^c)$ .

A *path* from  $S$  to  $T$  with  $S, T \subseteq 2^N$  is a sequence  $j = 1, \dots, k$  of indexes such that for all states  $S = S_1, \dots, S_k = T$  we have  $p_i(1_{S_j}) \neq 0$  for all  $i \in S_{j+1}$  and  $p_i(1_{S_j}) \neq 1$  for all  $i \notin S_{j+1}$ , for all  $j = 1 \dots k$ .

A *terminal class*, denoted  $\mathbf{C}$  in this master thesis, is a collection of states  $K \subseteq 2^N$  such that for any  $S, T \in \mathbf{C}$  there exists a path from  $S$  to  $T$  and there is no path from  $S$  to  $T$  if  $S \in \mathbf{C}$  and  $T \notin \mathbf{C}$ . A *terminal state* is a terminal class that contains only one element.

## 2 Generalized OWA

### 2.1 GOWA: definitions and exemples

**Definitions :** We say that a  $n$ -place aggregation function  $A$  is a generalized ordered weighted average  $A$ =GOWA with a weigh vector  $w$ , i.e  $0 \leq w_i \leq 1$  for  $i = 1, \dots, n$  and  $\sum_{i=1}^n w_i = 1$  if  $A(x) = \alpha \sum_{i=1}^n w_i x_{(i)}$ , for all  $x \in \{-1, 1\}^n$ , where  $x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(n)}$  are the ordered components<sup>1</sup> of  $x$  and  $\alpha \in [-1, 1]$  is the coefficient of conformism of the agent. If  $\alpha > 0$ , the agent is said to be *conformist*. If  $\alpha < 0$ , he is said to be *anti-conformist*.  $\alpha = 1$  and  $\alpha = -1$  are the *pure degrees of conformism*. An agent with a pure degree of (anti)-conformism is *pure*.

We write  $p_a(1_S)$  for the probability for an anti-conformist agent to say 'yes', and  $p_c(1_S)$  for the probability for an conformist agent to say 'yes'. Sometimes we will focus on the probability for pure (anti)-conformists to say 'yes'. We write  $p_{a^\pi}(1_S)$  for the probability, at the state  $S$ , for the pure anti-conformists to say 'yes'. We use the underscript  $p_{c^\pi}(1_S)$  for the probability, at the state  $S$ , of the pure conformists to say 'yes'.

Each set  $S$  can be written as the union of a set of conformists and a set of anti-conformists:  $S = S^a \cup S^c$ . In terms of cardinality we have  $s = s^a + s^c$ . It is straightforward to check that  $s$  is such that  $x_{(s)} = 1$  and  $x_{(s+1)} = -1$ . We write  $K_s := \sum_{i=1}^s w_i$  for  $s \in [1, n]$  and  $K_0 := -1$ . Since  $\sum_i w_i = 1$ , we have  $\sum_i w_i x_{(i)} = \sum_{i=1}^s w_i - \sum_{i=s+1}^n w_i = K_s - (1 - K_s) = 2K_s - 1$ .

*Fact 1:* (i)  $p_j(1_S) \neq 0 \Leftrightarrow A_j(1_S) \neq -1$  and (ii)  $p_j(1_S) \neq 1 \Leftrightarrow A_j(1_S) \neq 1$ . (iii) If  $\alpha = 1$ ,  $p_j(1_S) = \sum_{i=1}^s w_i = K_s$  (iv) More generally:  $p_j(1_S) = \frac{1}{2}(1 + \alpha(2K_s - 1))$ . (v)  $K_s = K_{s-1} + 2w_s$

Throughout this master thesis, all agents are assigned to GOWA. The aggregation functions of agents different in  $w$  and  $\alpha$ . This aggregation model is stochastic. From the output of GOWA, a change of scale in the output of its aggregation function gives the probability for an agent to say 'yes' after one step of influence.

---

<sup>1</sup>All the "1" will be 'on the left' and the -1 'on the right'.



As we can notice from the definition of GOWA, the nature of agents saying 'yes' does not play a role in determining the probability at the next step that anti-conformists and conformists say 'yes'. In this sense, agents are 'blind'. This plays an central role in the dynamism of GOWA.

By definition of GOWA, the probabilities of saying "yes" are independent among agents and only depend on the current state. Therefore the probability of transition from  $S$  to  $T$  is:

$$\lambda_{S,T} = \prod_{i \in T} p_i(1_S) \prod_{i \notin T} (1 - p_i(1_S))$$

*Notation:*  $G(S) := \{T | \lambda_{S,T} \neq 0\}$ .

With GOWA agents only weight the 'yes' and 'no' inclinations from the agents of the society, not whether these agents are conformists or anti-conformists. This is what we mean by saying that agents are "blind". This blindness is at the root of the lack of symmetry that starts appearing in Theorem 1 between conformism and anti-conformism, despite the illusory symmetry in the definition of GOWA.

### Examples of GOWA

- If  $\alpha = 0$ , the agent tosses a coin whatever the situation. We can see with the next examples that the closer  $\alpha$  becomes to 0, the closer he gets to tossing a coin.
- If  $\alpha = -1$ ,  $w = (\frac{2}{3}, \frac{1}{3}, 0, 0, 0)$  the agent says "yes" with probability  $\frac{1}{2}(1 + (-1) \times (\frac{2}{3} - \frac{1}{3} - 0)) = \frac{1}{3}$  if only one agent says "yes" and he says "yes" for sure if all agents say "no". He says "no" for sure if everyone says "yes". We can see that in general  $\emptyset$  and  $N$  will not be terminal states.
- If  $\alpha = -1/2$ ,  $w = (\frac{2}{3}, \frac{1}{3}, 0, 0, 0)$  the agent says "yes" with probability  $\frac{1}{2}(1 + (-\frac{1}{2}) \times (\frac{2}{3} - \frac{1}{3} - 0)) = \frac{5}{12}$  if only one agent says "yes".
- If  $\alpha = -1$ ,  $w = (\frac{3}{4}, \frac{1}{4}, 0, 0, 0)$  the agent says "yes" with probability  $\frac{1}{2}(1 + (-1) \times (\frac{3}{4} - \frac{1}{4} - 0)) = \frac{1}{4}$  if only one agent says "yes" and he says "yes" for sure if all agents say "no". He says "no" for sure if everyone says "yes".
- If  $\alpha = -1$ ,  $w = (\frac{1}{4}, \frac{3}{4}, 0, 0, 0)$  the agent says "yes" with probability  $\frac{1}{2}(1 + (-1) \times (\frac{1}{4} - \frac{3}{4} - 0)) = \frac{3}{4}$  if only one agent says "yes" and he says "yes" for sure if all agents say "no". He says "no" for sure if everyone says "yes".
- If  $\alpha = 1/2$ ,  $w = (\frac{3}{4}, \frac{1}{4}, 0, 0, 0)$  the agent says "yes" with probability  $\frac{1}{2}(1 + \frac{1}{2} \times (\frac{3}{4} - \frac{1}{4} - 0)) = \frac{5}{8}$  if only one agent says "yes".
- For  $\alpha = 1$ , GOWA gives the same results as OWA since it is a change of scale (see e.g [20]).

Taking  $\alpha \notin \{-1, 1\}$  (not pure agents) can be thought of allowing some trembling hand in the process of decision. When  $\alpha$  is close to 0, the agent almost tosses an unbiased coin.

Before treating the case of two groups (anti-conformists and conformists) defined by  $(\alpha^a, w^a, n^a)$ , where  $n^a$  and  $n^c$  denote respectively the number of anti-conformists and the number of conformists,

and  $(\alpha^c, w^c, n^c)$  ( $\alpha^a < 0$  and  $\alpha^c > 0$ ), we treat in the next section the case where conformists and anti-conformists share the same  $w$  (homogeneous framework).

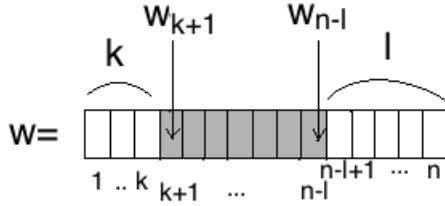
**Definitions :** We say that a weigh vector  $w$  is *left-null of order  $k$*  when it is of the form  $w = (\underbrace{0, \dots, 0}_k, *, \dots, *)$  and  $w_{k+1} > 0$ . We say that a weigh vector is *right-null of order  $p$*  when it is of the form  $w = (*, \dots, *, \underbrace{0, \dots, 0}_l)$  and  $w_{n-l} > 0$ . We refers to it as a  $(k, l)$  weigh vector.

When  $w$  is left-null of order  $k$  and right-null of order  $l$ , we say that  $w$  is *lower-null* of order  $k \wedge l$  and *upper-null* of order  $k \vee l$ .  $w$  is said to be *lower-null* (*upper-null*) if he is lower-null (*upper-null*) of order at least 1.

In this master thesis the letters  $k$  and  $l$  will always denote respectively the left and right-orders of nullity.

*Fact 2:*  $\min\{i | w_i \neq 0\} = k + 1$  and  $\max\{i | w_i \neq 0\} = n - l$ .

By convention when we draw weigh vectors, we put blanks for the left and right zeros. We put a gray zone for the coefficients included between the indexes  $k + 1$  and  $n - l$  (which are strictly positive by definition), inclusively. The  $w_i$  such that  $i$  is strictly between  $k + 1$  and  $n - l$  can possibly be 0.



In the next section (homogenous case, ie same  $w$  for everyone), three parameters  $(w, n^a, n^c)$  are sufficient to determine a terminal class. We call this set of parameters a *society*. In section 4 (heterogeneous framework, ie a weigh vector  $w^a$  for anti-conformists and a weigh vector  $w^c$  for conformists) a society will be pinned down by four parameters:  $(w^a, w^c, n^a, n^c)$ . Put it formally, we could define a correspondance that associates to each society one several terminal classes.

## 2.2 Discussion on GOWA

In this master thesis, we will only make a study of terminal classes and states of societies (exhaustive list of possible terminal classes, unicity of terminal classes or states, symetries...). Since only the terminal classes are of interest for us, what matters is only  $w_{k+1}$  and  $w_{n-l}$  ( $w_{k+1}^a, w_{n-l}^a, w_{k+1}^c$  and  $w_{n-l}^c$  in the heterogeneous framework). The probabilities of transition that we consider are 0, 1 and  $\lambda \notin \{0,1\}$ ; we don't need more precisions to get the reduced graph and the terminal classes deduced from the reduced graphs (the only exception will be Lemma 3.bis, presented in Appendix 1). In particular the  $w_i$  for  $i \in (k + 1, n - l)$  are uninteresting for us (except in the case of Lemma 3.bis that we present in Appendix). For the same reason, what matters for in only who, in  $S$ , is pure. We can notice that when an agent is not pure, whether  $\alpha > 0$  or  $\alpha < 0$  plays no role in defining the reduced graphs of transition between states, and therefore the terminal classes.

It is interesting in our definition of GOWA to introduce not pure agents, rather than allowing only either  $\alpha = 1$  or  $-1$ . We will only address briefly this problem, in Section 3.3 (homogeneous framework) and Section 4.8 (heterogeneous framework). We will show the link between the notion of anti-conformism and not-purity and we will discuss the implications of these results in the concluding section. We won't extend much on this vast topic that we keep for a future research. Let's mention briefly that the notion of not-purity, that we introduced as a trembling-hand, can be thought of as being equivalent to mixed strategies. We don't use the word "mixed" that we keep for a situation where the agent chooses  $\alpha = 1$  or  $-1$  with some probability. Though we don't examine this issue in this master thesis, it is possible to recast the notion of not-purity in terms of mixed strategies.

### 3 Terminal states and classes with GOWA: homogeneous framework

In this section and all the propositions exposed, all agents decide according to a GOWA aggregation function. They all have the same  $w$  but we allow for different  $\alpha$  among agents.

Let  $N$  be a group of agents of cardinality  $n$  such that  $N = N^a \cup N^c$  where the subset  $N^a$  of cardinality  $n^a$  is the set of anti-conformist agents and  $N^c$  of cardinality  $n^c$  the set of conformist agents, so that  $n = n^a + n^c$ . By default these sets can be empty.

#### 3.1 Preliminary results

This subsection introduces the basic results whose underlying mechanisms are reused in the whole master thesis under more sophisticated forms.

*Fact 3:* Assume that  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$ . We denote  $S$  the current state. Then:

- $p_a(1_S) = 0 \Leftrightarrow p_c(1_S) = 1$ .
- $p_a(1_S) = 1 \Leftrightarrow p_c(1_S) = 0$ .

We will use extensively Fact 3 under the form  $p_a(1_S) \notin \{0,1\} \Leftrightarrow p_c(1_S) \notin \{0,1\}$ . It is essential to notice that this result holds only because conformists and anti-conformists have the same weigh vector (homogeneous framework). In section 4 in which we will build the heterogeneous case, this fact will not be true anymore.

**Proposition 1:** Assume that all agents are pure. Then  $\lambda_{N,N^c} = 1$  and  $\lambda_{\emptyset,N^a} = 1$ .

*Proof.* Let's prove that  $\lambda_{N,N^c} = 1$ :

For an agent  $j$  whose degree of conformism is  $\alpha$ ,  $p_j(1_N) = \frac{1}{2} (1 + \alpha \sum_{i=1}^n w_i x_{(i)}) = \frac{1}{2} (1 + \alpha)$ . If  $j$  is anti-conformist,  $\alpha = -1$  by assumption. Therefore  $p_j(1_N) = 0$ . If  $j$  is conformist,  $\alpha = 1$  by assumption so  $p_j(1_N) = 1$ .

Let's prove that  $\lambda_{\emptyset,N^a} = 1$ :

For an agent  $j$  whose degree of conformism is  $\alpha$ ,  $p_j(1_\emptyset) = \frac{1}{2} (1 + \alpha \sum_{i=1}^n w_i x_{(i)}) = \frac{1}{2} (1 - \alpha)$ . If  $j$  is anti-conformist,  $\alpha = -1$  by assumption. Therefore  $p_j(1_\emptyset) = 1$ . If  $j$  is conformist,  $\alpha = 1$  by assumption so  $p_j(1_\emptyset) = 0$ .  $\square$

**Corollary 1:** If  $N^a \neq \emptyset$ , then  $\emptyset$  and  $N$  are never terminal states.

*Proof.* From Proposition 1,  $\emptyset$  and  $N$  cannot be terminal states if agents are pure. They are not terminal states either if some agents are not pure since in this case, these agents never choose 'yes' or 'no' for sure.  $\square$

Proposition 1 will still be true in the heterogeneous framework. We will use this result throughout this master thesis.

**Proposition 2 :** Assume that  $w$  is such that  $w_1 \neq 0$  and  $w_n \neq 0$  and that  $N^c \neq \emptyset$  and  $N^a \neq \emptyset$ . Then the terminal class is  $2^N$ .

In other words, if  $w$  is neither left nor right-null, then the terminal class is  $2^N$ .

*Note :* No assumption of *pure* conformism or *pure* anti-conformism is needed here.

The proof of Proposition 2 relies on the following lemma.

**Lemma 1:** Assume that  $w$  is such that  $w_1 \neq 0$  and  $w_n \neq 0$  and that  $N^c \neq \emptyset$  and  $N^a \neq \emptyset$ . The current state is  $S$ . If  $S \notin \{\emptyset, N\}$  then  $\lambda_{S,T} \neq 0$  for all  $T \in 2^N$ .

*Proof.* (Lemma 1)

Under these assumptions,  $-1 < A(1_S) < 1$  for both conformists and anti-conformists. This implies that  $0 < p(1_S) < 1$  both for conformists and anti-conformists, which in turn implies  $\lambda_{S,T} \neq 0$ . Therefore any  $T$  is possible at the next period.  $\square$

*Proof.* (Proposition 2)

There are three subcases.

- (i)  $S \notin \{\emptyset, N\}$  and  $T$  unrestricted. By lemma 1,  $T$  is reachable.
- (ii)  $S = N$ . For an anti-conformist,  $p_a(1_S) < 1$ <sup>2</sup> and  $p_c(1_S) > 0$  for conformists<sup>3</sup>. At the next state, there is a non-zero probability that some conformists say 'yes' and one anti-conformist say 'no' (state  $S'$ ). By lemma 1, any  $T$  is reachable from  $S'$ .
- (iii)  $S = \emptyset$ . For an anti-conformist,  $p_a(1_S) > 0$  and  $p_c(1_S) < 1$  for conformists. At the next state, there is a non-zero probability that some conformists say 'no' and one anti-conformist say 'yes' (state  $S'$ ). By lemma 1, any  $T$  is reachable from  $S'$ .

$\square$

Propositions 3 and 4 hereafter characterize the terminal classes. In particular Proposition 4 gives conditions for their existence and uniqueness. The examples below give a flavor of the mechanisms at stake.

*Examples :*

- If  $w_n = 0$ , and the only anti-conformist agent is  $\{j\}$ , there is only one terminal state :  $N \setminus \{j\}$ .
- If  $w_1 = 0$ , and the only anti-conformist agent is  $\{j\}$ , then  $\{j\}$  is the only terminal state.

---

<sup>2</sup> $p_a(1_S) = 0$  if he is purely anti-conformist though such an extreme is not needed for the proof.

<sup>3</sup>Again, all of them will say 'yes' for sure if they are *purely* conformist, but we don't need them to be.

- If  $w_1 = w_{n-1} = w_n = 0$  and if  $j$  and  $k$  are anti-conformists and the others conformists, then the only terminal state is  $N \setminus \{jk\}$ . But  $\{j\}$  and  $\{k\}$  are not terminal states.
- If  $w_1 = w_2 = w_{n-1} = w_n = 0$  and if  $j$  and  $k$  are anti-conformists and the others conformists, terminal states are  $\{jk\}$  and  $N \setminus \{jk\}$ .

*Fact 4:* There is no terminal state nor cycle if at least one agent is neither purely conformist nor purely anti-conformist.

*Proof.* The current state is  $S$ . If the agent  $j$  is not purely conformist or purely anti-conformist,  $0 = \frac{1}{2}(1 - 1) < p(1_S) = \frac{1}{2}(1 + \alpha \sum_i w_i x(i)) < \frac{1}{2}(1 + 1) = 1$ . Therefore we have  $p_j(1_S) \notin \{0,1\}$ .  $\square$

**Proposition 3** (form of the terminal states): Assume that all agents are pure,  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$ . Then:

1. If there exists a terminal state, then  $w$  is lower-null or upper-null.
2. Terminal states cannot contain both types of agents.
3. Terminal states are either  $N^a$  or  $N^c$ .
4.  $N^a$  is a terminal state if and only if  $w$  is left-null of order  $k \geq n^a$ .
5.  $N^c$  is a terminal state if and only if  $w$  is right-null of order  $l \geq n^a$ .

*Proof.* .

1. This is a direct consequence of Proposition 2.
2. If all agents are pure, then all agents of the same type must have the same inclination at a terminal state. By Corollary 1,  $\emptyset$  and  $N$  are not terminal states.
3. Agents are pure so all agents of the same type of conformism must make the same choice. Therefore a strict subset of  $N^a$  or a strict subset of  $N^c$  cannot be a terminal state either. From the previous point,  $\emptyset$  and  $N$  are not terminal states either. We are left with only  $N^a$  and  $N^c$ .
4. If  $w$  is left-null of order  $k \geq n^a$ , then  $N^a$  is a terminal state. Indeed, we have  $p_j(1_{N^a}) = \frac{1}{2} (1 + \alpha (\sum_{i=1}^n w_i x(i))) = \frac{1}{2} (1 + \alpha (\sum_{i=k+1}^n w_i x(i)))$ . Since  $k \geq n^a$ , this is equal to:

$$\frac{1}{2} \left( 1 + \alpha \left( \sum_{i=k+1}^n w_i \times (-1) \right) \right) = \frac{1}{2} (1 - \alpha) = \delta_\alpha$$

where  $\delta_\alpha = 1$  if the agent  $j$  is anti-conformist and  $\delta_\alpha = 0$  if the agent  $j$  is conformist. Therefore  $N^a$  is a terminal state.

Conversely, assume that  $N^a$  is a terminal state and that there exists  $p \in [1, n^a]$  such that  $w_p \neq 0$ . Then  $\sum_{i=1}^n w_i x(i) = \sum_{i=1}^{n^a} w_i x(i) + \sum_{i=n^a+1}^n w_i x(i) \geq w_p - \sum_{i=n^a+1}^n w_i > -1$  so that for an anti-conformist :  $p_a(1_{N^a}) < 1$ . Contradiction.

5. If  $w$  is right-null of order  $l \geq n^a$ , then  $N^c$  is a terminal state. Indeed, we have  $p_j(1_{N^c}) = \frac{1}{2} (1 + \alpha (\sum_{i=1}^n w_i x_{(i)})) = \frac{1}{2} (1 + \alpha (\sum_{i=1}^{n-l} w_i x_{(i)}))$ . Since  $l \geq n^a$ , this is equal to:

$$\frac{1}{2} \left( 1 + \alpha \left( \sum_{i=1}^{n-l} w_i \times 1 \right) \right) = \frac{1}{2} (1 + \alpha) = \delta_\alpha$$

where  $\delta_\alpha = 0$  if the agent  $j$  is anti-conformist and  $\delta_\alpha = 1$  if the agent  $j$  is conformist. Therefore  $N^c$  is a terminal state.

Conversely, assume that  $N^c$  is a terminal state and that there exists  $p \in [\underbrace{n^c + 1}_{n - n^a + 1}, n]$  such that

$w_p \neq 0$ . Then  $\sum_{i=1}^n w_i x_{(i)} = \sum_{i=1}^{n^a} w_i x_{(i)} + \sum_{i=n^c+1}^n w_i x_{(i)} \leq \sum_{i=1}^{n^a} w_i - w_p < 1$  so that for a conformist :  $p_c(1_{N^c}) < 1$ . Contradiction.

□

Proposition 3 is worth being compared with OWA. We know that  $\emptyset$  and  $N$  are the terminal states with OWA [20], and we can check easily that they are also terminal states with GOWA if  $N^a \neq \emptyset$ . Therefore our terminal states  $N^a$  and  $N^c$  are robust to the limit case  $N^a = \emptyset$ . But we have two more terminal classes:  $2^N$  and the cycle  $N^a \rightarrow N^c$ . Those two terminal classes are not terminal states. Therefore they carry some dynamism. We start feeling that GOWA will explain dynamical phenomena in influence process. This was not possible with OWA, and this was possible with DeGroot only under very special cases (periodic matrices).

**Proposition 4** (existence and uniqueness of terminal states). Assume that  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$ .

1. There exists a terminal state if and only if the following two conditions are true :
  - All agents are pure.
  - $w$  is upper-null of order  $m \geq n^a$ .

This terminal state is either  $N^c$  (if and only if  $w$  is right-null of order  $p \geq n^a$ ) or  $N^a$  (if and only if  $w$  is left-null of order  $k \geq n^a$ ) or both  $N^c$  and  $N^a$  (if and only if  $w$  is lower-null of order  $k \geq n^a$ ).

2. If such a terminal state exists, then it is unique if and only if  $w$  is lower-null of order  $m < n^a$ .

*Proof.* .

1. This follows directly from Proposition 3.3, 3.4 and 3.5.

2.  $\Rightarrow$

If  $w$  is lower-null of order  $m \geq n^a$ , then both  $N^a$  and  $N^c$  are terminal states so uniqueness does not hold.

$\Leftarrow$

By proposition 3.3 we know that if there are several terminal states, then these terminal states are  $N^c$  and  $N^a$ . We apply Proposition 3.4 and 3.5: if  $N^a$  (resp.  $N^c$ ) is a terminal state then  $w$  is left-null (resp. right-null) of order  $m \geq n^a$ . Therefore if there are two terminal states, then  $w$  is lower-null of order  $m \geq n^a$ . Conversely if  $w$  is lower-null of order  $m \geq n^a$  then there are two terminal states ( $N^a$  and  $N^c$ ).

□

### 3.2 List of possible terminal classes and unicity

Proposition 5 is the most important proposition of this section since it summarizes all the possible dynamics of the societies in the homogeneous framework. It will be generalized by Theorem 1.

**Lemma 2:** Assume that all agents are pure and that  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$ .

1. There exists no society  $q = (n^a, n^c, w^a, w^c)$  such that  $J \subseteq [\emptyset, N^a]$ ,  $J \neq N^a$  is a terminal class.
2. There exists no society  $q = (n^a, n^c, w^a, w^c)$  such that  $J \subseteq [\emptyset, N^c]$ ,  $J \neq N^c$  is a terminal class.

*Proof.*

We already know that  $\emptyset$  cannot be a terminal state.  $\emptyset \subset J \subseteq [\emptyset, N^a]$  with  $J \neq N^a$  cannot be a terminal class, for it would imply that there exists a state in which  $\sum_{i=1}^n w_i x_{(i)} \notin \{-1, 1\}$ , which would imply that conformists have a non-zero probability to say 'yes'.

We already know that  $\emptyset$  cannot be a terminal state.  $\emptyset \subset J \subseteq [\emptyset, N^c]$  with  $J \neq N^c$  cannot be a terminal class, for it would imply that there exists a state in which  $\sum_{i=1}^n w_i x_{(i)} \notin \{-1, 1\}$ , which would imply that anti-conformists have a non-zero probability to say 'yes'.  $\square$

Actually, Lemma 2.2 can be extended: even if some agents (conformists or anti-conformists) are not pure, then  $J \subseteq [\emptyset, N^c]$ ,  $J \neq N^a$  cannot be a terminal class. This is still true in the heterogeneous framework. However in the heterogeneous framework,  $[\emptyset, N^a]$  can be a terminal class (in the sense that there exists a society  $q$  such that it is a terminal class), both in the case where all agents are pure and in the case where some are not (see Proposition 13).

**Proposition 5 :** Assume that  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$  and that all agents are pure. Then the terminal classes are either  $N^a$ ,  $N^c$ ,  $N^a \rightarrow N^c$  or  $2^N$ .

*Proof.* Let  $\mathbf{C}$  be a terminal class different from  $2^N$  and  $K \notin \mathbf{C}$ .

1. *Case 1.* There exists a state  $H \in \mathbf{C}$  that contains both conformists and anti-conformists. There are two (non mutually excludable) subcases :
  - (a) *Subcase 1.* In the state  $H$ ,  $\sum_{i=1}^n w_i x_{(i)}$  is different from  $-1$  and  $1$ . Since choices of agents are independent from each others,  $K$  is reachable from  $\mathbf{C}$ . Contradiction.
  - (b) *Subcase 2.* In the state  $H$ ,  $\sum_{i=1}^n w_i x_{(i)} = -1$  (respectively  $1$ ). That is, the next state is  $S = N^a$  (respectively  $N^c$ ). Assume without loss of generality that it is  $N^a$ . Two sub-subcases:
    - i. *Sub-subcase 1:*  $w$  is such that  $p_j(1_{N^a}) = \delta_\alpha$  where  $\delta_\alpha = 0$  if  $j$  is conformist and  $1$  if  $j$  is anti-conformist. That is,  $N^a$  is a terminal state, which contradicts  $\mathbf{C}$  being a terminal class which contains  $H$ .
    - ii. *Sub-subcase 2:*  $w$  is such that  $p_j(1_{N^a}) = \delta_\alpha$  where  $\delta_\alpha = 1$  if  $j$  is conformist and  $0$  if  $j$  is anti-conformist. The next state is  $N^c$ . Since  $H \in \mathbf{C}$ ,  $N^c$  is not a terminal state. We can draw two sub-sub-subcases or notice that the same reasoning as the subcase 2 holds. We get that  $N^a \rightarrow N^c$  must be a cycle (necessary condition). This contradicts  $H$  belonging the terminal class.

2. *Case 2.*  $\mathbf{C}$  contains only  $\emptyset$  and/or  $N$ .

When the current state is  $\emptyset$ , we have  $\sum_{i=1}^n w_i x_{(i)} = -1$ . At the next state, conformists say 'no' for sure and anti-conformist 'yes' for sure (because they are all pure). This means that the next state is  $N^a$ , which does not belong to  $\mathbf{C}$ . Contradiction.

When the current state is  $N$ , we have  $\sum_{i=1}^n w_i x_{(i)} = 1$ . At the next state, conformists say 'yes' for sure and anti-conformists say 'no' for sure (because they are all pure). This means that the next state is  $N^c$ , which does not belong to  $\mathbf{C}$ . Contradiction.

3. *Case 3.* There exists a state  $H \subset N^a$  such that  $H \in \mathbf{C}$ .

By Lemma 2.1, there must be a state of  $\mathbf{C}$  that contain both conformists and anti-conformists. We are driven back to Case 1 and we get a contradiction.

4. *Case 4.* There exists a state  $H \subset N^c$  such that  $H \in \mathbf{C}$ .

By Lemma 2.2, there must be a state of  $\mathbf{C}$  that contain both conformists and anti-conformists. We are driven back to Case 1 and we get a contradiction.

5. *Case 5.*  $N^a \in \mathbf{C}$ .

There are three sub-cases:

(a)  $N^a$  is a terminal state.

(b)  $N^a \rightarrow N^c$  with probability 1. If in addition to this  $N^c \rightarrow N^a$  with probability 1 then  $N^a \rightarrow N^c$  is a cycle. If it is not true that  $N^c \rightarrow N^a$  with probability 1, then we examine Case 6.

(c)  $\sum_i w_i x_{(i)} \notin \{-1, 1\}$ . Then any  $K$  directly reachable from  $N^a$  and belongs to  $\mathbf{C}$ . This contradicts  $\mathbf{C}$  being different from  $2^N$ .

6. *Case 6.*  $N^c \in \mathbf{C}$ .

There are three sub-cases:

(a)  $N^c$  is a terminal state.

(b)  $N^c \rightarrow N^a$  with probability 1. If in addition to this  $N^a \rightarrow N^c$  with probability 1 then  $N^c \rightarrow N^a$  is a cycle. If it is not true that  $N^a \rightarrow N^c$  with probability 1, then we examine Case 5.

(c)  $\sum_i w_i x_{(i)} \notin \{-1, 1\}$ . Then any  $K$  directly reachable from  $N^c$  and belongs to  $\mathbf{C}$ . This contradicts  $\mathbf{C}$  being different from  $2^N$ .

We just proved that if  $\mathbf{C}$  is different from  $2^N$ , then it must be either  $N^a$ ,  $N^c$  or  $N^a \rightarrow N^c$ . To end the proof, we check that the four terminal classes announced in Proposition 5 are possible. We know from Proposition 2 that  $2^N$  can be a terminal class. We know from Proposition 3 that  $N^a$  and  $N^c$  can be terminal states. It remains to check that the cycle  $N^a \rightarrow N^c$  can be a terminal class. The transition  $N^a \rightarrow N^c$  occurs with probability 1 if and only if  $\max\{i \mid w_i \neq 0\} \equiv n - l \leq n^a$ . The transition  $N^c \rightarrow N^a$  occurs with probability 1 if and only if  $\min\{i \mid w_i \neq 0\} \equiv k + 1 > n^c$ . The terminal class is a cycle  $N^a \rightarrow N^c$  if and only if  $n^c < k + 1 \leq n - l \leq n^a$ , that is, if and only if  $n^c \leq k, l \leq n^a$ . This is possible to design (for example  $n^c = 1$ ,  $n^a = 3$  and  $w = (0, *, *, 0)$ )<sup>4</sup>. □

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<sup>4</sup>This will be generalized to the heterogeneous framework in Proposition 11.



Corollary 2 hereafter means that the society must be sufficiently stable to allow two equilibria (two terminal classes) and that anti-conformists make the society unstable if they are too many.

**Corollary 2:** Assume that  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$ . If  $n^c \leq n^a$  there cannot be several terminal classes.

*Proof.* By Proposition 5, the only case of multiplicity of terminal classes is when the states  $N^a$  and  $N^c$  are terminal states.

Assume that  $w$  is left-null of order  $k \geq n^a$ . Then  $n - k \leq n - n^a = n^c \leq n^a$ . Therefore  $w$  cannot be right-null of order  $l \geq n^a$ . Finally it cannot be lower-null of order  $m \geq n^a$ . The result follows from Proposition 4.2. □

Proposition 6 examines the case all agents are anti-conformists. In this case  $N^c \rightarrow N^a$  collapses into  $\emptyset \rightarrow N$ .

**Proposition 6 :** If  $N = N^a$ :

1. There are no terminal states.
2. If all agents are pure, the terminal class is  $\emptyset \rightarrow N$  and it is the only one.

*Proof.* By assumption all agents are anti-conformists.

1. If at least one agent is not pure, this is obvious. Now we assume that all agents are pure. Since agents are pure and of the same type, they must make the same choice at each state: either they all say 'yes' with probability 1, or they all say 'no' with probability 1, or they all have a probability  $p_a(1_S) \notin (0,1)$  to say 'yes'. The last case is impossible by definition of a terminal state. This makes two candidates to examine:  $\emptyset$  and  $N$ . Neither of them is a terminal state.
2. If  $p_a(1_S) \notin (0,1)$ , the state after some iterations will be either  $\emptyset$  or  $N$ . If all agents are pure we have a cycle  $\emptyset \rightarrow N$ . This happens with any vector  $w$  and it is the only possible terminal class. □

### 3.3 Introduction to not pure agents in the homogeneous framework

We write  $M^a \subseteq N^a$  the set of anti-conformists that are not pure ( $m^a := |M^a|$ ). We write  $M^c \subseteq N^c$  the set of conformists that are not pure ( $m^c := |M^c|$ ). Let:  $P^a := N^a \setminus M^a$  of cardinality  $p^a$  and  $P^c := N^c \setminus M^c$  of cardinality  $p^c$ , so that  $n^a = m^a + p^a$  and  $n^c = m^c + p^c$ .

*Fact 5:*  $\emptyset \rightarrow [P^a, N^a]$  with probability 1 and  $N \rightarrow [P^c, N^c]$  with probability 1.

*Note:*  $[P^a, N^a]$  (resp.  $[P^c, N^c]$ ) collapses into  $N^a$  (resp.  $N^c$ ) when  $m^a = 0$  (resp.  $m^c = 0$ ), which is consistent with Proposition 1.

Proposition 7 hereafter is the counterpart of Proposition 6 in the case where some agents are not pure.

**Proposition 7:** If  $N = N^a = P^a \cup M^a$ ,  $P^a \neq \emptyset$ , then the possible terminal classes are:

- $[\emptyset, M^a] \cup [P^a, N^a]$
- $[\emptyset, N^a]$

*Proof.* In a terminal class, either all the pure agents say 'yes' for sure, or they 'no' for sure, or they have a probability  $p \notin \{0,1\}$  to say 'yes'. They may also alternate between a probability  $p = 1$  to say 'yes' and a probability  $p = 1$  to say 'no'. This makes four candidates to examine:  $[\emptyset, M^a]$ ,  $[P^a, N^a]$ ,  $[\emptyset, M^a] \cup [P^a, N^a]$  and  $[\emptyset, N^a]$ .

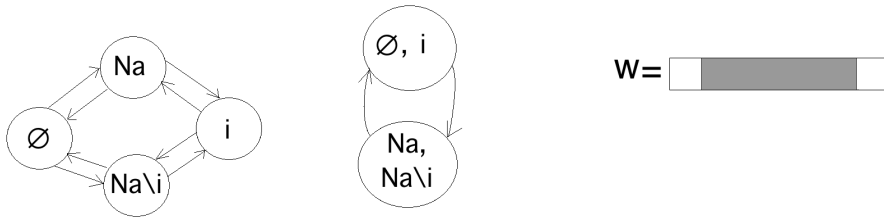
- $[\emptyset, M^a]$ : impossible because  $\emptyset \rightarrow N^a$  occurs with a non-zero probability.
- $[P^a, N^a]$  cannot be a terminal class since  $p_{a\pi}(1_{N^a}) = 1$  if and only if  $k + 1 > n^a$ , that is,  $k \geq n^a = n$ , which is impossible.
- $[\emptyset, M^a] \cup [P^a, N^a]$  can be a terminal class. For example with  $w_1 = w_n = 0$  and  $m^a = 1$  (so  $p^a = n^a - 1$ ).
- If  $w_1 \neq 0$  and  $w_n \neq 0$ , then  $[\emptyset, N^a]$  is a terminal class.

□

When all agents are pure,  $[\emptyset, M^a] \cup [P^a, N^a]$  collapses into  $\emptyset \rightarrow N$ , which is consistent with Proposition 6.2. The interesting point of this proposition is that when some agents are not pure, then  $[\emptyset, N^a]$  (that is, the class  $2^N$  when  $N^c$  collapses into  $\emptyset$ ) can be a terminal class, which was not the case under the hypothesis of Proposition 6.

### Illustration of the terminal class $[\emptyset, M^a] \cup [P^a, N^a]$

On the figure below,  $i$  is a mixed agent.



Proposition 8 is not a generalization of Proposition 7 since it imposes  $N^c \neq \emptyset$ . However we will see after the proof how these two propositions can be discussed jointly.

**Proposition 8:** Let  $N^c = P^c \cup M^c \neq \emptyset$  and  $N^a = P^a \cup M^a \neq \emptyset$ . We assume that at least one agent is not pure. Then possible terminal classes are boolean or unions of boolean lattices as listed below:

1.  $2^N$

2.  $[P^a, N^a \cup M^c]$
3.  $[P^c, N^c \cup M^a]$
4.  $[N^c, N]$

*Proof.* 1. Cycles and terminal states are characterized by a sequence of transition from states to states with probability 1 (where a terminal state can be seen as a cycle of length 1). If at least one agent is not pure, then for any  $S, T \subseteq 2^N$  we have  $\lambda_{S,T} \notin \{0,1\}$ .

2. When  $w_1 \neq 0$  and  $w_2 \neq 0$ , the terminal class is  $2^N$  (Proposition 2). For what follows, we can restrict our analysis to left or right-null vectors.

*Case 1:  $w$  left-null of order  $k$ .*

There are three subcases<sup>5</sup>:

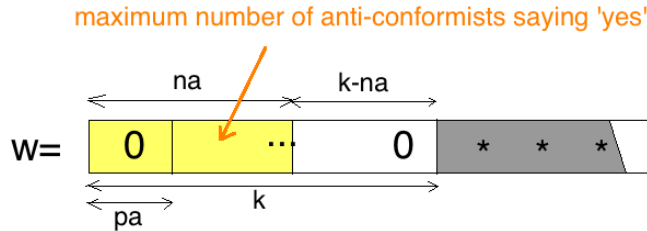
(a)  $k = n^a$ . Two sub-subcases:

- If  $M^c = \emptyset$ , then the terminal class is  $[P^a, N^a]$ .
- If  $M^c \neq \emptyset$ , then the terminal class is  $2^N$ .

(b)  $k < n^a$ . The terminal class is  $2^N$ .

(c)  $k > n^a$ . Two sub-subcases:

- If  $m^c \leq k - n^a$ , the terminal class is  $[P^a, N^a \cup M^c]$ .
- If  $m^c > k - n^a$ , Starting from a state with a maximum of  $k$  agents saying 'yes', there can be  $n^a$  anti-conformists who say 'yes' at the next state. If in addition to these  $n^a$  anti-conformists saying 'yes' there is more than  $k - n^a$  conformist agents saying 'yes' we have  $\sum_{i=1}^n w_i x_{(i)} \notin \{0,1\}$  at the next state. This situation is possible since  $m^c > k - n^a$ . Therefore the terminal class is  $2^N$ .



*Case 2:  $w$  right-null of order  $k$ .*

There are three subcases<sup>6</sup>:

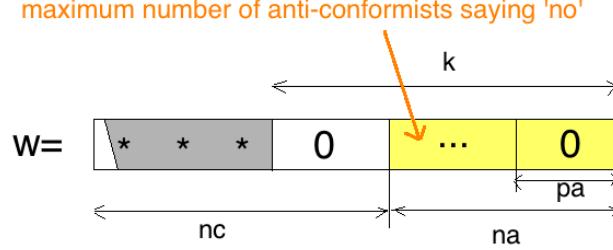
(a)  $k = n^a$ . Two sub-subcases:

- If  $M^c = \emptyset$ , then the terminal class is  $[N^c, N]$  (all the conformists+some anti-conformists)
- If  $M^c \neq \emptyset$ , then the terminal class is  $2^N$ .

<sup>5</sup>The first one can be grouped with the third one to make the proof shorter but we prefer this presentation that allows to exhibit some interesting special cases discussed in the paragraph after the proof.

<sup>6</sup>Same remark.

- (b)  $k < n^a$ . The terminal class is  $2^N$ .
- (c)  $k > n^a$ . Two sub-subcases:
- If  $m^c \leq k - n^a$ , the terminal class is  $[P^c, N^c \cup M^a]$ .
  - If  $m^a > k - n^a$ , the terminal class is  $2^N$ .



□

In particular when at least one agent is not pure, then the terminal classes are always booleans (see Proposition 12 in the heterogeneous framework). This will be true also in the heterogeneous framework. Anticipating on Theorem 3 presented in the section on heterogeneous weigh vector (this theorem is *a fortiori* true in the homogeneous framework) we can notice that two important terminal classes are covered by Propositions 7 and 8:  $[\emptyset, N^a]$  and  $[N^c, N]$  (see Proposition 13.1). Under the hypothesis of Proposition 8,  $[N^c, N]$  can be a terminal class. A necessary condition for  $[N^c, N]$  to be a terminal class is  $M^c = \emptyset$  (see Proposition 13.2). We can check that Proposition 8 is consistent with Proposition 7 since  $[N^c, N]$  collapses into  $[\emptyset, N^a]$  when  $N^c = \emptyset$ . But this is not permitted under the hypothesis of Proposition 8 since we impose  $N^c \neq \emptyset$ . However it is still possible under the hypothesis of Proposition 8 to have the terminal class  $[\emptyset, N^a]$ ; necessary conditions are  $M^c = \emptyset$  and  $P^a = \emptyset$ . If at least one anti-conformist agent is pure,  $[\emptyset, N^a]$  cannot be a terminal class under the hypothesis of Proposition 8 and necessary conditions for  $[\emptyset, N^a]$  to be a terminal class are  $N^c = \emptyset$  and at least one anti-conformist is not pure.

Finally, we notice that the sets  $[P^c, N^c \cup M^a]$  cannot collapse into  $[\emptyset, M^a]$  since a necessary condition for it would be that the weigh vector is left or right-null of order  $k \geq n^a = n$ , which would mean that all the entries of the weigh vector are 0. This is consistent with Lemma 2.1.

## 4 Terminal states and classes with GOWA: heterogeneous framework

In this section and all the propositions exposed, all agents decide according to a GOWA aggregation function.

In the previous section we assumed that all agents had the same weigh vector  $w$ . However it seems more natural to characterize an agent with a couple  $(\alpha^p, w^p)$  where  $p = a$  or  $c$ . In this section we introduce heterogeneity by taking two types of agents: conformists and anti-conformists, all having the same  $\alpha$  and the same  $w$  within a same type.

$k^a$  and  $l^a$  extend naturally the degrees of left and right-nullity respectively. We recall that  $k+1$  is the first non-zero index of  $w$  and  $n-l$  the last one.

Our study of the homogenous case before was worth being done before the case of two types of agents in order to see what changes in the results when we introduce some heterogeneity. The natural extension of this present master thesis is to examine more and more heterogeneous societies in order to see the regularities and changes each time we generalize the framework. We don't do it here but we can already appreciate the additional terminal classes that it is possible to obtain in the heterogeneous framework.

## 4.1 Adapting the propositions of the homogeneous framework to the heterogeneous one

In all this section, conformist agents have a weigh vector  $w^c$  and anti-conformist agents a weigh vector  $w^a$ . If  $w^c = w^a$  the results of the previous section apply.

Propositions 1 and 2 can be easily adapted to the case of two types of agents by requiring that restrictions on  $w$  now become restrictions on both  $w^a$  and  $w^c$ . Proposition 6 can be generalized in a straightforward manner too. We group all these generalizations under the name "Proposition 9".

Propositions 3, 4, 5, 7 and 8 cannot be transposed as easily. For example it is wrong that the terminal classes are either singletons,  $N^a \rightarrow N^c$  or  $2^N$ .

**Proposition 9** (Generalization under heterogeneity of Propositions 1, 2, 6 and Corollary 1): Assume that all agents are pure. Then:

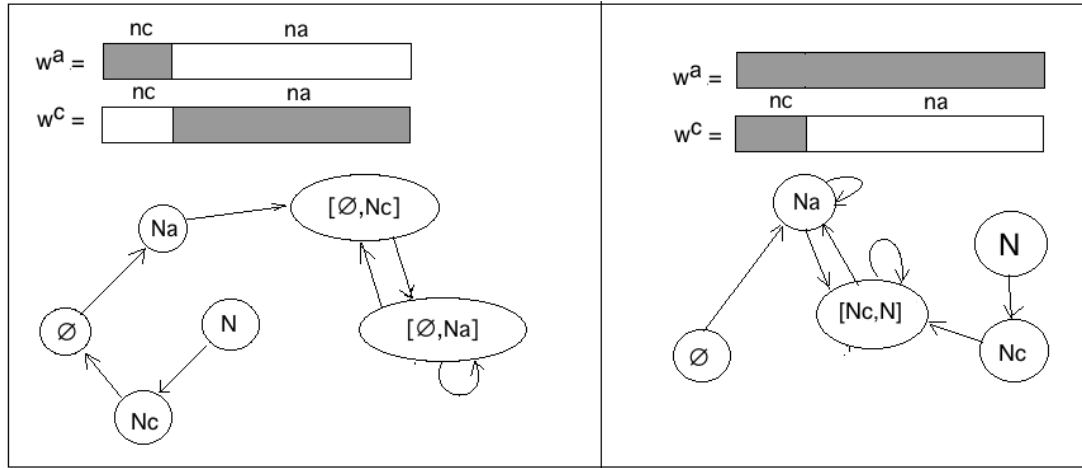
1.  $\lambda_{N,N^c} = 1$  and  $\lambda_{\emptyset,N^a} = 1$ .
2. If  $N^a \neq \emptyset$ , then  $\emptyset$  and  $N$  are never terminal states.
3. Assume that  $w^a$  and  $w^c$  are such that  $w_1^p \neq 0$  and  $w_n^p \neq 0$  (for both  $p = a$  or  $c$ ) and that  $N^c \neq \emptyset$  and  $N^a \neq \emptyset$ . Then the terminal class is  $2^N$ .
4. If  $N^c = \emptyset$ :
  - There are no terminal states.
  - If all agents are pure, the only terminal class is the cycle  $\emptyset \rightarrow N$ .

The next subsection examines a special case of heterogeneity: when the weigh vectors of the conformists and anti-conformists are left or right-null of order  $n^c$  or  $n^a$ . This will allow us to make conjectures and pursue our study of the terminal classes. We briefly introduce the problem of terminal classes with not pure agents in subsection 4.8.

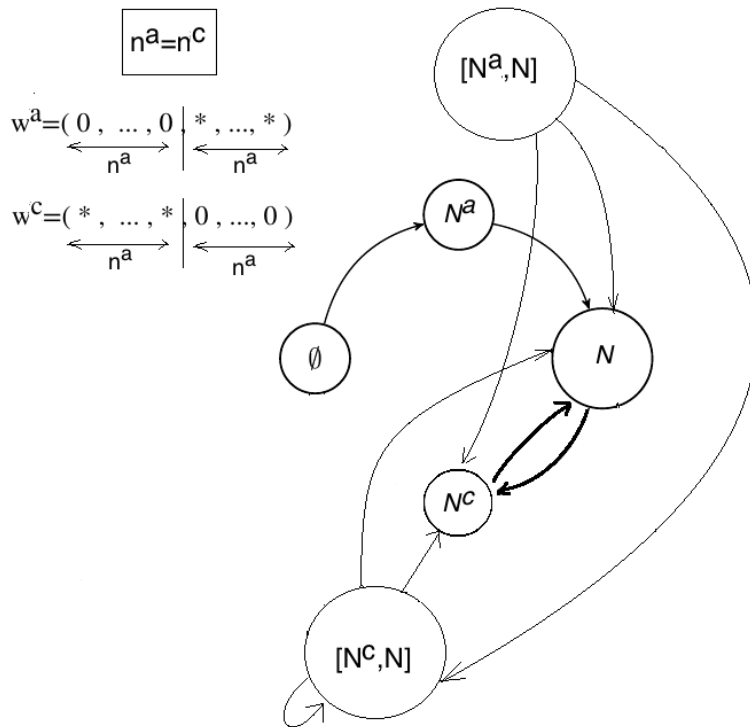
From the rest of the section, agents are assumed to be pure by default. We tell explicitly when this is not the case.

## 4.2 $(n^a, 0), (0, n^a), (n^c, 0)$ or $(0, n^c)$ weigh vectors

The two examples below, boxed, are two cases of societies where  $n^a > n^c$ . The left one is one of the examples given in one of the three tables further down the text. The right one is not.



For the rest of this subsection we only examine special cases for  $w^a$  and  $w^c$ : the  $(k, *)$  and the  $(*, l)$  weigh vectors<sup>7</sup> in order to formulate conjectures (the left one of the previous two examples is one of them). We will have to study three cases:  $n^a < n^c$ ,  $n^a = n^c$  and  $n^a > n^c$ . This makes overall  $16 + 4 + 16 = 36$  cases to examine. We show another example below.



<sup>7</sup>Though all computations are made with GOWA, building  $w^p$  ( $p = a$  or  $c$ ) as  $n^a$  or  $n^c$  left or right-null anticipates on non anonymous aggregation functions since agents screen other agents choices by choosing to filter the votes according to the number of 'conformists'/'anti-conformists'. This anticipates further studies, not presented in this master thesis, in which we will build aggregation functions that treats differently conformists and anti-conformists. We can expect GOWA to be an interesting special case of it.

### Terminal classes with $n^c < n^a$

$\downarrow w^c$ $w^a \rightarrow$	$(\underbrace{*, \dots, *}_{n^c}, \underbrace{0, \dots, 0}_{n^a})$	$(\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$	$(\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^c})$	$(\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^c})$
$(\underbrace{*, \dots, *}_{n^c}, \underbrace{0, \dots, 0}_{n^a})$	$N^c$	$N^c \rightarrow N$	$N^c \rightarrow N$	$[N^c, N]$
$(\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$	$[\emptyset, N^a] \cup [N^c, N]$	$2^N$	$2^N$	$2^N$
$(\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^c})$	$\emptyset \rightarrow N^a$	$[\emptyset, N^a]$	$N^a$	$\emptyset \rightarrow N^a$
$(\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^c})$	$2^N$	$2^N$	$[N^c, N] \cup [N^a, N]$	$2^N$

### Terminal classes with $n^c = n^a$

This table is a special case of the one below; actually we could make only two tables:  $n^c < n^a$  and  $n^c \geq n^a$ .

$\downarrow w^c$ $w^a \rightarrow$	$(\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^a})$	$(\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^a})$
$(\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^a})$	$N^c$	$N^c \rightarrow N$
$(\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^a})$	$\emptyset \rightarrow N^a$	$N^a$

### Terminal classes with $n^c > n^a$

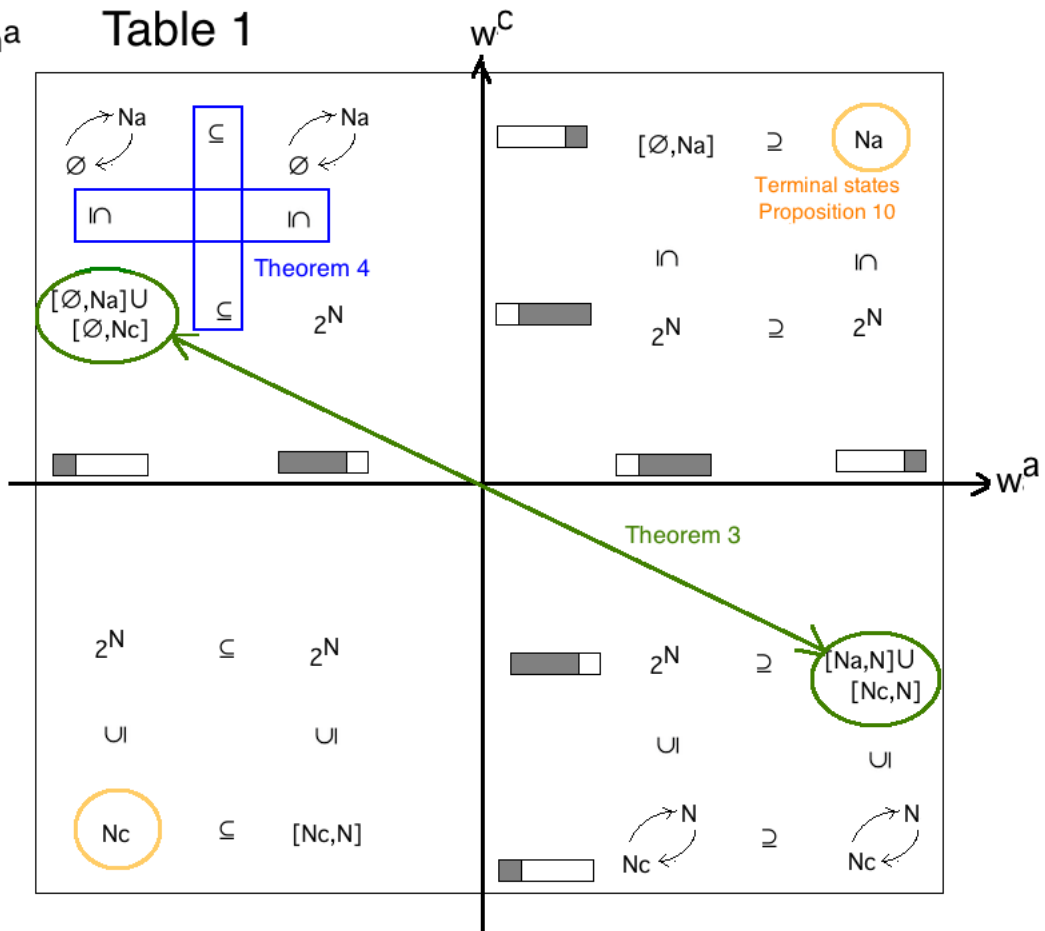
$\downarrow w^c$ $w^a \rightarrow$	$(\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^c})$	$(\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^c})$	$(\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$	$(\underbrace{*, \dots, *}_{n^c}, \underbrace{0, \dots, 0}_{n^a})$
$(\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^c})$	$N^c$	$[N^c, N]$	$N^c \rightarrow N$	$N^c$
$(\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^c})$	$\emptyset \rightarrow N^a$	$N^a$	$N^a$	$[\emptyset, N^a]$
$(\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$	$\emptyset \rightarrow N^a$	$N^a$	$N^a$	$[\emptyset, N^a]$
$(\underbrace{*, \dots, *}_{n^c}, \underbrace{0, \dots, 0}_{n^a})$	$N^c$	$[N^c, N]$	$N^c \rightarrow N$	$N^c$

### Summary of the terminal classes and conjectures

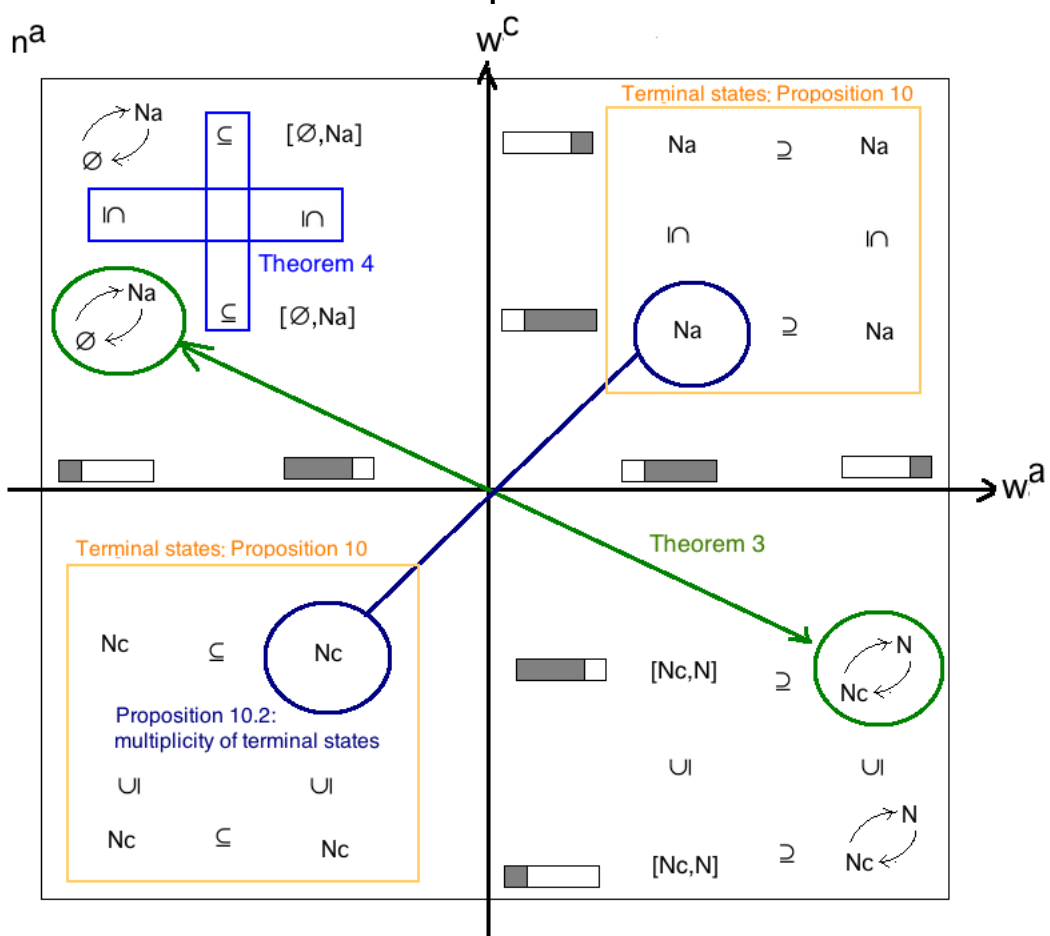
From these three tables, we build Table 1 (next page). It is done by splitting those 36 situations into two cases:  $n^c < n^a$  and  $n^c \geq n^a$ . Table 1 is made of the corresponding two sub-tables of double-entries in which we report the terminal class depending on  $w^a$  and  $w^c$ . This is one of most important picture of this master thesis since it is the one that permits to make conjectures of several key propositions: Proposition 10, Theorem 3 and Theorem 4.

$n^c < n^a$

Table 1



$n^c \cong n^a$





First we notice that we have more terminal classes than in the homogeneous framework. Studying the  $(k,*)$  and the  $(l,*)$  weigh vectors gives us an idea of the possible terminal classes. The general result will be Theorem 1 of section 4.4. As we will see, taking a more general form for  $w$  adds three additional terminal classes, which happen to be cycles:  $N^a \rightarrow N^c$ ,  $\emptyset \rightarrow N^a \rightarrow N^c$  and  $N^a \rightarrow N \rightarrow N^c$  (all particular all the booleans can be obtained with  $(k,*)$  and  $(l,*)$  weigh vectors). To get these additional cycles, we need 0 on the left and on the right of the weigh vectors (necessary conditions will be stated explicitly); that is why they don't appear in Table 1.

Then we notice that the two sub-tables that Table 1 is made of are very different from one another. This means that the balance between the number of anti-conformists and conformists is of a crucial importance to determine the terminal classes. Jumping from  $n^c \geq n^a$  and *vice-versa* changes radically all the terminal classes obtained. The only exceptions are the four "corners" in which the blank zones ('0' in the weigh vectors) are large enough so that the terminal classes are robust a reversal of the group of the largest cardinality.

There are clearly thresholds in the left/right-degrees of nullity of  $w^a$  and  $w^c$  with jumps from one type of terminal class to another. However overall, keeping  $n^a$  and  $n^c$  fixed, we see that there is some stability in the terminal class as we slightly change the weigh vectors. It is interesting in particular to notice that cycles can be quite robust (they are not degenerate cases of terminal classes). This stability is more "horizontal" (small modifications of  $w^a$ ) when  $n^c < n^a$  and more "vertical" (small modifications of  $w^c$ ) when  $n^c \geq n^a$ . If, for example, there are more anti-conformists than conformists, then large changes in  $w^a$  won't change the terminal class or, if it does, the new one in many cases is a subset or a supset of the former one (this is a rule of the thumb, neither a precise nor a general statement). Stated differently, the weigh vectors of the smallest group crucially determines the nature of the terminal classes. This has the flavor of the glove-market game in cooperative game theory.

Within a same table, we observe symmetries with respect to the origin of the axis (green circles). If, both  $w^a$  and  $w^c$ , we make the following changes: (1) A  $(k,*)$ -weigh vector is changed into a  $(*,k)$ -weigh vector (2) A  $(*,l)$ -weigh vector is changed into a  $(l,*)$ , then we need to make the following changes to get the new terminal class:  $\emptyset \leftrightarrow N$  and  $N^a \leftrightarrow N^c$ . More generally, we will see that if we make the following transformation:  $k \leftrightarrow l$  for both  $w^a$  and  $w^c$ , then each state is replaced with its complement in the new terminal class. This remarks holds for the all the *transitions* between states: replacing a  $(k,l)$ -weigh vector by a  $(l,k)$ -weigh vector (both for  $w^a$  and  $w^c$ ) gives a society whose reduced graphs of transitions are the same, up to the replacement of each set by its complement. This will be formalized by Lemma 3 (that states that the transition graphs are the same up to a change of each state for its complement) and Theorem 3 (that states that the terminal classes are the same up to a change of each state for its complement). Theorem 3 directly follows from Lemma 3.

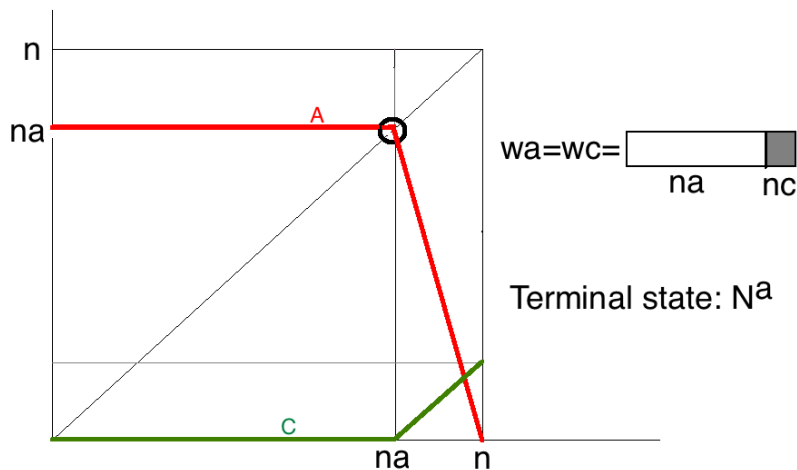
In Table 1 we can also localize the singletons, which allows us to conjecture Proposition 10. Finally, we observe inclusions of terminal classes in others (we boxed in light blue some inclusions). Those inclusions of terminal classes occur when the set of non-zero coefficients in the weigh vectors of the corresponding society are included in one another. This will be Theorem 4. We can also conjecture (not proved in this master thesis) that exchanging  $w^a$  and  $w^c$  gives new terminal classes that are either a subset or a supset of the previous ones.

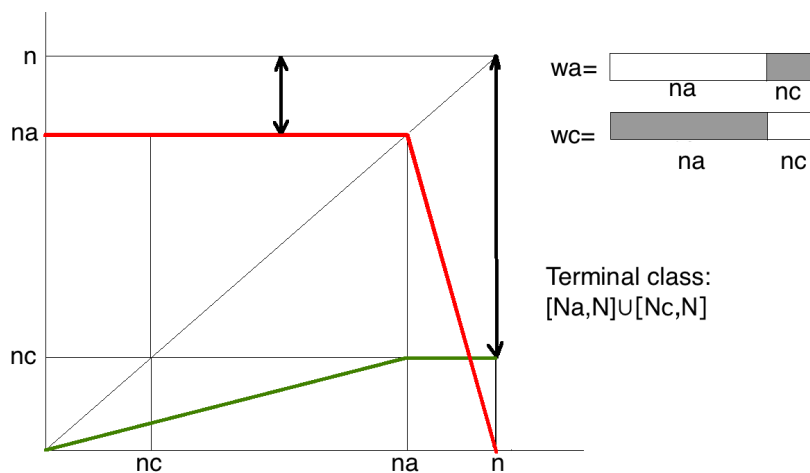
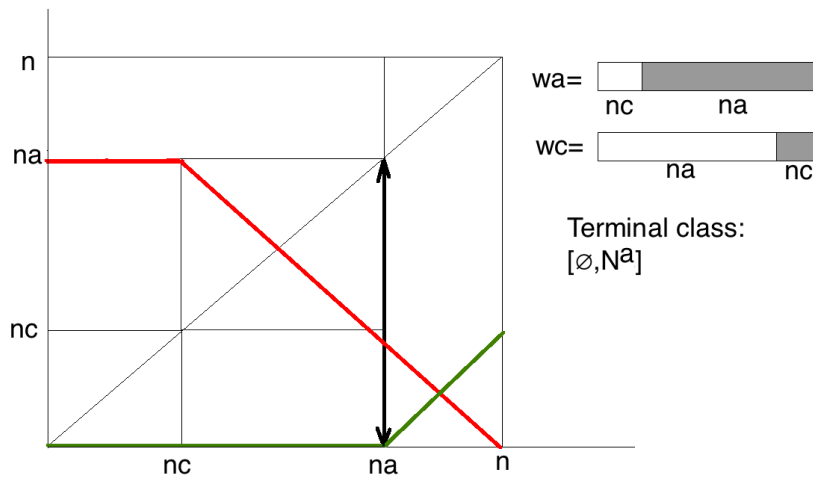
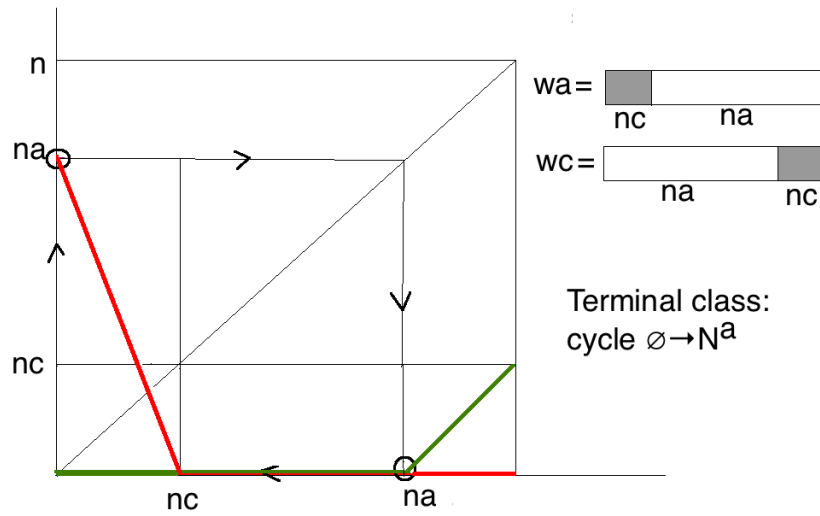
### 4.3 Graphical computation of the terminal classes and correspondances

The terminal classes can be computed visually and rapidly using continuous graphs. The number of people saying 'yes', that is,  $s$ , is represented on the x-axis. The red curve is a representation of the probability for an anti-conformist to say 'yes' at the next state. All matters is whether  $\sum_i w_i^a x_{(i)}$  takes the value  $-1$ ,  $1$  or something else. In the latter case we draw an oblique line. The green curve is a representation of the probability for a conformist to say 'yes' at the next step. Of course we should pay attention to the fact that here we look at cardinalities, not sets themselves, but for a heuristic purpose this is not an issue.

Then we can use correspondances, that we deduce entirely from the red and green curves. When, say the red line, is oblique then all anti-conformists, independently from each others, can say 'yes' or 'no'. We can see graphically, using the 45 degrees line, the fixed points (terminal states), cycles or "fixed zones" (booleans). We want to make a graph of a discrete phenomenon using continuous functions. Therefore we must pay attention that when we look at the correspondance of an interval  $[s_1, s_2]$ , the right boundary of the correspondance set is included in the set, while the left-boundary is not. For example in the first graph below, if there are  $n^a$  agents who say 'yes', then there will still be  $n^a$  agents who say 'yes' at the next step and they are all anti-conformists. Therefore  $N^a$  is a terminal state (and it is the only one here).

We notice in particular that all types of terminal classes can be quite robust to small changes of the weigh vectors. For example in the case of the cycle  $\emptyset \rightarrow N^a$ , we can increase  $l^a$ , increase  $k^c$  and still having the same cycle). The terminal states can be robust too, which will appears in particular in Proposition 10.





*Note 1:* These graphs can also be used to conjecture conditions on the weigh vectors to obtain a certain set of states as a terminal class.

*Note 2:* Similar graphs can still be used when there are not pure agents.

## 4.4 Theorem 1: list of possible terminal classes

For  $A$  and  $B$  two matrices of the same dimensions, we write  $A \leq B$  if each entry of  $A$  is lower or equal to each respective entry of  $B$ .

Proposition 10.1 generalizes proposition 4 to the heterogeneous framework. Proposition 10.2 is a special case of Theorem 2 that generalizes Corollary 2. Theorem 2 states that in order to have several terminal classes, we need (necessary condition)  $n^c > n^a$ .

**Proposition 10** (existence and uniqueness of terminal states with heterogeneous weigh vectors). Assume that  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$ .

1. There exists a terminal state if and only if the following two conditions are true :

- All agents are pure.
- Either one of these two conditions holds:
  - $w^a$  and  $w^c$  are left-null of order  $k^a$  and  $k^c$  respectively, with  $k^a \wedge k^c \geq n^a$ . (Case AC)
  - $w^a$  and  $w^c$  are right-null of order  $l^a$  and  $l^c$  respectively, with  $l^a \wedge l^c \geq n^a$ . (Case BD)

This terminal state is either  $N^a$  (Case AC) or  $N^c$  (Case BD).

2. The only case where there are several terminal states is when  $n^c > n^a$  and  $w^a$  and  $w^c$  are both lower-null of order  $k \geq n^a$ . The terminal states in this case are  $N^a$  and  $N^c$ .

Before proving this result, let us notice that, just as in Proposition 4, the condition for the multiplicity of terminal states relies on the number of anti-conformists. This is a asymmetry between the roles played by the conformists and the anti-conformists.

*Proof.*

We can verify that the two conditions involved are sufficient.

For the necessary direction, we only provide a visual proof, using graphs of correspondances. A formal (and cumbersome) proof uses arguments of contraposition as we did in Propositions 3 and 4.

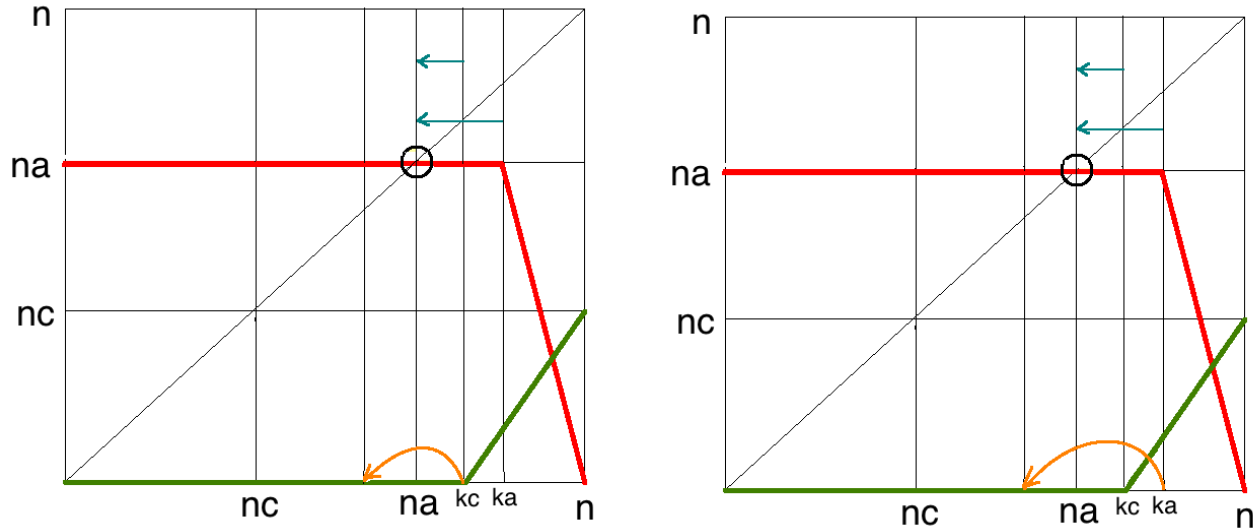
We use the same conventions as in section 4.3 for graphs. The horizontal arrows show until where we can push  $k^a$  and  $k^c$  to keep the terminal state (the limit of the arrow not being included). The orange arrows represent situations under which the necessary and sufficient conditions to get the corresponding terminal states is violated.

That all agents must be pure is obviously a necessary condition. We split the proof in four cases. We are going to show graphically that:

- If  $n^c < n^a$  one of these two statements must be true:
  - $w^a$  and  $w^c$  are left-null of order  $k^a$  and  $k^c$  respectively, with  $k^a \wedge k^c \geq n^a$ . (Statement A)
  - $w^a$  and  $w^c$  are right-null of order  $l^a$  and  $l^c$  respectively, with  $l^a \wedge l^c \geq n^a$ . (Statement B)

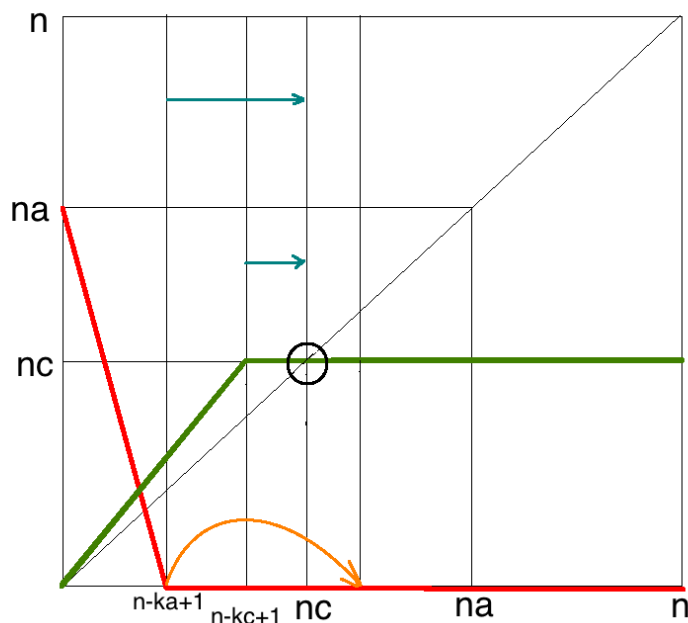
- If  $n^c \geq n^a$  one these two statements must be true:
  - $w^a$  and  $w^a$  are left-null of order  $k^a$  and  $k^c$  respectively with  $k^a \wedge k^c \geq n^a$ . (Statement C)
  - $w^a$  and  $w^a$  are right-null of order  $l^a$  and  $l^c$  respectively with  $l^a \wedge l^c \geq n^a$ . (Statement D)

**Agents pure + Statement A**



*Note:* If, for example  $k^c$  jumps below  $n^a$  (on the left),  $N^a$  cannot be a terminal state anymore because at a state where  $n^a$  people say 'yes', the cardinality of the next state  $T$  can be anything in  $[n^a, n]$ . If  $k^a$  jumps below  $n^a$  (on the right), then when  $n^a$  people say 'yes', the cardinality of the next state  $T$  can be anything in  $[0, n^a]$  and the terminal class is now  $[\emptyset, N^a]$ <sup>8</sup>.

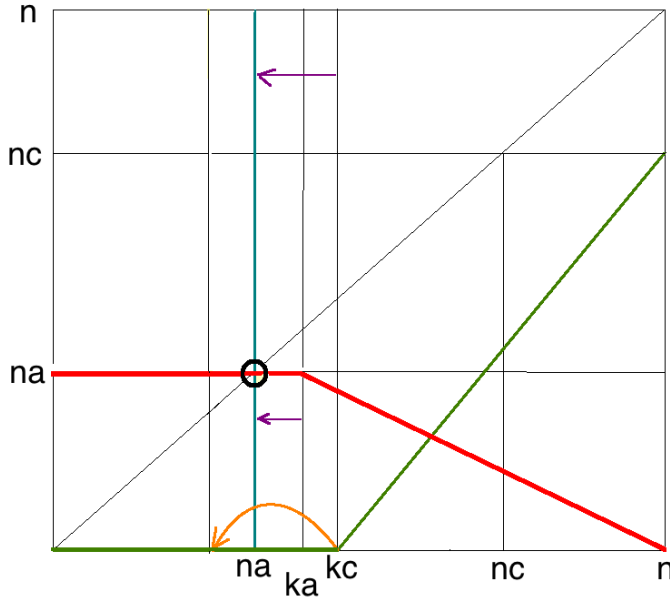
**Agents pure + Statement B**



<sup>8</sup>It is worth noticing that the states alternate between  $N^a$  and  $[\emptyset, N^a]$

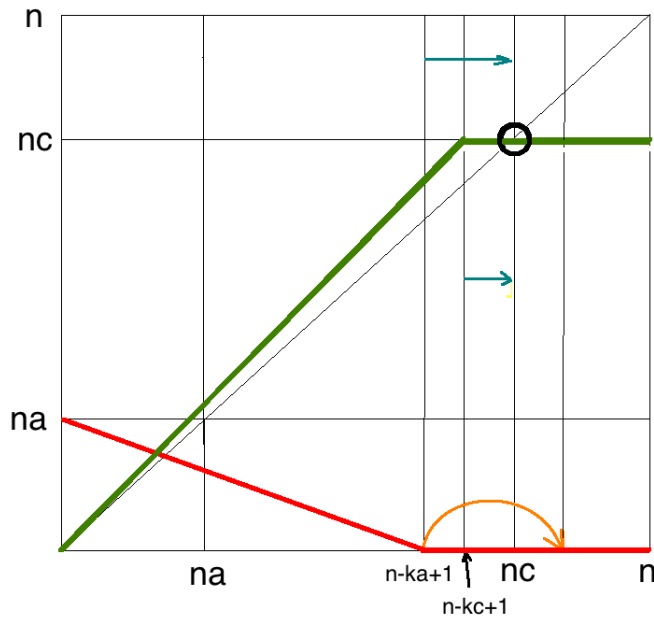
Note: If, for example, we push  $k^c$  strictly below  $n^a$ , then from  $n^c$  people saying 'yes', any number between  $[n^c, n]$  can be the cardinality of the next state and  $N^c$  cannot be a terminal state anymore.

**Agents pure + Statement C**



Note: If for example  $k^c$  jumps below  $n^a$ ,  $N^a$  would not be a terminal state anymore because from  $n^a$  people saying 'yes' we can have any cardinality in  $[n^a, n]$  for the next state. If  $k^a$  jumped below  $n^a$ , the terminal class would be  $[\emptyset, N^a]$ , as shown also in Table 1 that summarizes the terminal classes in the special cases of weigh vectors.

**Agents pure + Statement D**



Note: If, for example, we push  $k^a$  strictly below  $n^a$ , then from  $n^c$  saying 'yes', we can then have any number in  $[0, n^a]$  for the cardinality of the next state and  $N^c$  cannot be a terminal state anymore.

□

Theorem 1 hereafter that the terminal classes exposed in Propositions 10 and 11 are the only ones. This theorem is an adaptation of a Theorem from "*A model of influence based on aggregation functions*" (Theorem 2 in their paper) to the context of GOWA. Their theorem states as follows :

**Theorem "no.2"** [Grabisch, Rusinowska]: Consider an influence process  $\mathbf{B}$  based on mappings  $A: [0,1]^n \rightarrow [0,1]$  satisfying:

1.  $A(0, \dots, 0) = 0, A(1, \dots, 1) = 1$  (boundary conditions)
2. If  $\mathbf{x} \leq \mathbf{x}'$  then  $A(\mathbf{x}) \leq A(\mathbf{x}')$  (monotonicity)

Then terminal classes are:

1. either singletons  $\{S\}, S \in 2^N$
2. or cycles of nonempty sets  $\{S_1, \dots, S_k\}$  of length  $2 \leq k \leq C_n^{\lfloor n/2 \rfloor}$  (and therefore they are periodic of period  $k$ )<sup>9</sup>, with the condition that all sets are pairwise incomparable (by inclusion)
3. or collections  $\mathbf{C}$  of nonempty sets with the property that  $\mathbf{C} = \mathbf{C}_1 \cup \dots \cup \mathbf{C}_p$ , where each subcoalition  $\mathbf{C}_j$  is a Boolean lattice  $[S_j, S_j \cup K_j], S_j \neq \emptyset, S_j \cup K_j \neq N$  and at least one  $K_j$  is nonempty.

This theorem does not apply in our context since GOWA mappings do not verify boundary conditions and monotonicity. Theorem 1 in an analogous theorem in the GOWA framework.

**Theorem 1 (Generalization of Proposition 5 to the heterogeneous framework):** Assume that all agents are pure and that  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$ . There are 12 possible terminal classes which are:

1. Either one of the following singletons: (i)  $N^c$  (ii)  $N^a$ .
2. or one of the following cycles: (i)  $\emptyset \rightarrow N^a$  (ii)  $N^c \rightarrow N$  (iii)  $N^a \rightarrow N^c$  (iv)  $\emptyset \rightarrow N^a \rightarrow N^c$  (v)  $N^a \rightarrow N \rightarrow N^c$ .
3. or one of the following boolean sets or union of boolean sets:
  - $2^N$
  - $[\emptyset, N^a]$
  - $[N^c, N]$
  - $[\emptyset, N^a] \cup [\emptyset, N^c]$
  - $[N^a, N] \cup [N^c, N]$

*Proof.* 1. Agents of the same type must make the same choice. Therefore the only candidates for terminal states are  $\emptyset, N^a, N^c$  and  $N$ .

- $\emptyset$  cannot be a terminal state because the next state is  $N^a$  with probability 1.
- $N$  cannot be a terminal state because the next state is  $N^c$  with probability 1.

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<sup>9</sup>The maximum length of an antichain is stated by the Sperner's lemma. An elegant proof is proposed in *Supermath*, Pierre Bornsztein, problem R49, éditions Vuibert.

- $N^a$ : It is a terminal state for example with  $n^c \geq n^a$  and  $w^a = w^c = (\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^c})$ .
- $N^c$ : It is a terminal state for example with  $n^c \geq n^a$  and  $w^a = w^c = (\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^c})$ .

2. First we notice that :

- The transition  $N^a \rightarrow N$  occurs with probability 1 if and only if :  $\begin{cases} n^a \leq k^a & (1) \\ n^c \leq l^c & (2) \end{cases}$
- The transition  $N^c \rightarrow \emptyset$  occurs with probability 1 if and only if :  $\begin{cases} n^a \leq l^a & (3) \\ n^c \leq k^c & (4) \end{cases}$
- The transition  $N^a \rightarrow N^c$  occurs with probability 1 if and only if :  $\begin{cases} n^c \leq l^a & (5) \\ n^c \leq l^c & (6) \end{cases}$
- The transition  $N^c \rightarrow N^a$  occurs with probability 1 if and only if :  $\begin{cases} n^c \leq k^a & (7) \\ n^c \leq k^c & (8) \end{cases}$

Cycles must have by definition a probability 1 of transition between two states. All agents of the same type must make the same choice, each state of the cycle must include either all or neither of agents of each type. Therefore no other state than  $\emptyset$ ,  $N^a$ ,  $N^c$  and  $N$  can be part of a cycle.

Now we examine the candidates for cycles. Necessary and sufficient conditions were given in Proposition 11.

*Cycles of order 2:*

The fact that  $\lambda_{\emptyset, N^a} = 1$  and  $\lambda_{N, N^c} = 1$  makes only three cases to examine.

- $\emptyset \rightarrow N^a$  can be a cycle. For example with  $n^c \geq n^a$ ,  $w^a = (\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^c})$  and  $w^c = (\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$ .
- $N^c \rightarrow N$  can be a cycle. For example with  $n^c \geq n^a$ ,  $w^a = (\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$  and  $w^c = (\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^c})$ .
- $N^a \rightarrow N^c$  can be a cycle. For example with  $n^a = 3$  and  $n^c = 1$ ,  $w^a = (0, *, *, 0)$  and  $w^c = (0, *, *, 0)$ .

*Cycles of order 3:*

Since  $\emptyset \rightarrow N^a$  for sure for any  $w$  and  $N \rightarrow N^c$  for sure for any  $w$ , the candidates to examine are : (a)  $\emptyset \rightarrow N^a \rightarrow N^c$ , (b)  $N^a \rightarrow N \rightarrow N^c$

(a)  $\emptyset \rightarrow N^a \rightarrow N^c$  : (3)+(4)+(6) give  $n - l^a \leq n^c < k^c + 1 \leq n - l^c \leq n^a$ .<sup>10</sup> We can take for

<sup>10</sup>As mentionned in the corresponding case in Proposition 11, the constraint (5), which is redundant here.



example  $n^c = 2$ ,  $n^a = 7$ ,  $w^a = (\underbrace{*,*}_2, \underbrace{0, \dots, 0}_7)$  and  $w^c = (\underbrace{0, 0, 0}_3, \underbrace{*,*}_2, \underbrace{0, 0, 0, 0}_4)$ , which indeed gives one such cycle.

(b)  $N^a \rightarrow N \rightarrow N^c$ : (8)+(2)+(1) give  $n^c < k^c + 1 \leq n - l^c \leq n^a \leq k^a$ .<sup>11</sup> We can take for example  $n^c = 2$ ,  $n^a = 7$ ,  $w^a = (\underbrace{0, \dots, 0}_7, \underbrace{*,*}_2)$  and  $w^c = (\underbrace{0, 0, 0, 0}_4, \underbrace{*,*}_2, \underbrace{0, 0, 0}_3)$ , which indeed gives one such cycle<sup>12</sup>.

*Cycle of order 4:*  $\emptyset \rightarrow N^a \rightarrow N \rightarrow N^c$ : (3)+(4) give  $n - l^a \leq n^c < k^c + 1$  and (1)+(2) give  $n - l^c \leq n^a \leq k^a$ . We have:

$$k^c < n - l^c \leq n^a < k^a + 1 \leq n - l^a \leq k^c$$

where the last inequality comes from (3)+(4). We get  $k^c < k^c$ . Absurd. Therefore such a cycle does not exist.<sup>13</sup>

3. (a) Let's  $\mathbf{C}$  be a terminal class. Each set  $S_i \in \mathbf{C}$  can be written  $S_i = S_i^a \cup S_i^c$ . There are three cases. The first and the second one are not mutually excludable. Case 3 is by definition the negation of Case 1 and Case 2:

- Case 1: There exists  $S_i \in \mathbf{C}$  such that  $\emptyset \subset S_i^a \subset N^a$ . This implies that for all  $L \in [S_i^c, S_i^c \cup N^a]$ ,  $L \in \mathbf{C}$ .
- Case 2: There exists  $S_i \in \mathbf{C}$  such that  $\emptyset \subset S_i^c \subset N^c$ . This implies that for all  $L \in [S_i^a, S_i^a \cup N^c]$ ,  $L \in \mathbf{C}$ .
- Case 3: In all other cases, we get easily that  $\mathbf{C}$  is either a terminal state or a cycle.

We know thanks to Cases 1 and 2 that if  $\mathbf{C}$  is a terminal class different from terminal states and cycles, then it must be one of these booleans or unions of booleans:  $2^N$ ,  $[\emptyset, N^a]$ ,  $[N^c, N]$ ,  $[\emptyset, N^a] \cup [\emptyset, N^c]$ ,  $[N^a, N] \cup [N^c, N]$ ,  $[\emptyset, N^c]$  and  $[N^a, N]$ .

(b) Let's show that  $[N^a, N]$  cannot be a terminal class. If this was the case, by definition  $N \in \mathbf{C}$ . From Proposition 1,  $N^c \in \mathbf{C}$ , which contradicts  $[N^a, N]$  being a terminal class.

Let's show that  $[\emptyset, N^c]$  cannot be a terminal class. If this was the case, by definition  $\emptyset \in \mathbf{C}$ . From Proposition 1,  $N^a \in \mathbf{C}$ , which contradicts  $[\emptyset, N^c]$  being a terminal class.

Let's show that the five other cases enumerated are possible:

- $2^N$  is a terminal class for example with  $n^c < n^a$  and  $w^a = w^c = (\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$ .
- $[\emptyset, N^a]$  is a terminal class for example with  $n^c < n^a$  and  $w^a = (\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$   
and  $w^c = (\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^c})$ .

<sup>11</sup>We can notice that it implies the constraint (7), which is therefore useless here.

<sup>12</sup>Notice that we just switched  $k \leftrightarrow l$  both for  $w^a$  and for  $w^c$  in the example of the previous cycle. Of course we could choose something else but we wanted to point out again a manifestation of Theorem 3.

<sup>13</sup>Two inequalities prevent the equality to occur. Only one is sufficient of course, but this is worth being noticed. We don't know yet how to exploit this.

- $[N^c, N]$  is a terminal class for example with  $n^c < n^a$  and  $w^a = (\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^c})$   
and  $w^c = (\underbrace{*, \dots, *}_{n^c}, \underbrace{0, \dots, 0}_{n^a})$ .
- $[\emptyset, N^a] \cup [\emptyset, N^c]$  is a terminal class for example with  $n^c < n^a$ ,  $w^a = (\underbrace{*, \dots, *}_{n^c}, \underbrace{0, \dots, 0}_{n^a})$   
and  $w^c = (\underbrace{0, \dots, 0}_{n^c}, \underbrace{*, \dots, *}_{n^a})$ .
- $[N^a, N] \cup [N^c, N]$  is a terminal class for example with  $n^c < n^a$ ,  $w^a = (\underbrace{0, \dots, 0}_{n^a}, \underbrace{*, \dots, *}_{n^c})$   
and  $w^c = (\underbrace{*, \dots, *}_{n^a}, \underbrace{0, \dots, 0}_{n^a})$ .

□

In addition to this exhaustive list of terminal classes, Theorem 1 tells us two interesting things that relates to the behavior of anti-conformists. Not all possible terminal classes find their counterpart by changing  $N^a$  for  $N^c$ . For example  $[\emptyset, N^a]$  can be a terminal class but not  $\emptyset, N^c$ . The symmetry in the terminal classes obtained does not occur at the level  $N^a \leftrightarrow N^c$ , but rather at the level of set complementation ( $N^a \leftrightarrow N^c$  and  $\emptyset \leftrightarrow N$ ). This is formalised Lemma 3 and Theorem 3. We notice also that all the possible booleans except  $[\emptyset, N^a] \cup [N^c, N]$  and  $2^N$  are booleans of anti-conformists (in the sense that all the conformists either say 'yes' or 'no' for sure, altogether). We see also that if a boolean is a terminal class, then there must exist a state where some anti-conformists say 'yes' (therefore there exists also a state where they all say 'yes'). On the contrary the boolean  $[\emptyset, N^a]$  (all the conformists say 'no' for sure, once for all) can be a terminal class. There is no symmetry in the role played by the conformist agents and the anti-conformist agents. This asymmetry in Theorem 1 tells us an important thing on the structure of GOWA: anti-conformism is not *just* the contrary of conformism. It was tempting to believe that we would just need to adapt the results to include agents who do the contrary of the conformists; what we start seeing here is that anti-conformists agents play a central role in the dynamic of the society. We won't develop this idea in this master thesis, but we discuss this question in the concluding section.

**Corollary 3** (Incompatibility of sure transitions):

1.  $(N^a \rightarrow N \text{ with probability } 1) \Rightarrow \text{Not}(N^c \rightarrow \emptyset \text{ with probability } 1)$ .
2.  $(N^a \rightarrow \emptyset \text{ with probability } 1) \Rightarrow \text{Not}(N^c \rightarrow N \text{ with probability } 1)$ .

*Proof.* 1. This has been incidentally shown in the proof for the non-existence of the cycle  $\emptyset \rightarrow N^a \rightarrow N \rightarrow N^c$  in the previous proof, by showing that (1)+(2) was incompatible with (3)+(4).

2. Having both requires the following inequalities:  $\begin{cases} n - l^a \leq n^a & (9) \\ k^c + 1 > n^a & (10) \end{cases}$  and  $\begin{cases} k^a + 1 > n^c & (11) \\ n - l^c \leq n^c & (12) \end{cases}$ .

Using (9)+(10)+(12)+(11)<sup>14</sup>, we get  $n - l^a \leq n^a < k^c + 1 \leq n - l^c \leq n^c < k^a + 1$ , that is,  $n - l^a < k^a + 1$ . This contradicts  $n - l^a \geq k^a + 1$ .

□

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<sup>14</sup>This time, all the constraints are needed.

We now extend Proposition 10 to the other terminal classes (the first two points of Proposition 11 are a reformulation of Proposition 10) by giving necessary and sufficient conditions to get each of the possible cases listed by Theorem 1. Using graphs is useful to see graphically all the possible weigh vectors such that a certain set of states is a terminal class. These conjectures that can also be done with Table 1.

**Proposition 11:** If all agents are pure:

1.  $N^a$  is a terminal state if and only if  $n^a \leq k^a, k^c$ .
2.  $N^c$  is a terminal state if and only if  $n^a \leq l^a, l^c$ .<sup>15</sup>
3.  $[\emptyset, N^a]$  is a terminal class if and only if  $n^a \leq k^c$  and  $k^a < n^a < n - l^a$ .
4.  $[N^c, N]$  is a terminal class if and only if  $n^a \leq l^c$  and  $k^a < n^c < n - l^a$ .<sup>16</sup>
5.  $\emptyset \rightarrow N^a$  is a terminal class if and only if  $n - l^a \leq n^a \leq k^c$ .
6.  $N^c \rightarrow N$  is a terminal class if and only if  $n - l^c \leq n^c \leq k^a$ .<sup>17</sup>
7.  $[\emptyset, N^a] \cup [\emptyset, N^c]$  is a terminal class if and only if  $n^c \leq k^c \leq n^a \leq l^a$  and  $n^c < n^a$ .
8.  $[N^c, N] \cup [N^a, N]$  is a terminal class if and only if  $n^c < l^c < n^a \leq k^a$ .<sup>18</sup>
9.  $N^a \rightarrow N^c$  is a terminal class if and only if  $n^c \leq k^a, l^a, k^c, l^c \leq n^a$ .
10.  $\emptyset \rightarrow N^a \rightarrow N^c$  if and only if  $n^c \leq k^c < n - l^c \leq n^a \leq l^a$ .<sup>19</sup>
11.  $N^a \rightarrow N \rightarrow N^c$  if and only if  $n^c \leq k^c \leq n - l^c \leq n^a \leq k^a$ .<sup>20</sup>
12.  $2^N$  is a terminal class in all the other cases.

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<sup>15</sup>This can be obtained from Case 1 using the transformation rule  $k^p + 1 \leftrightarrow n - l^p, >\leftrightarrow \leq$  and  $n^a \leftrightarrow n^c$ . This remark is formalized by Theorem 3.

<sup>16</sup>This can be obtained from Case 3 using the transformation rule  $k^p + 1 \leftrightarrow n - l^p, >\leftrightarrow \leq$  and  $n^a \leftrightarrow n^c$ . This remark is formalized by Theorem 3.

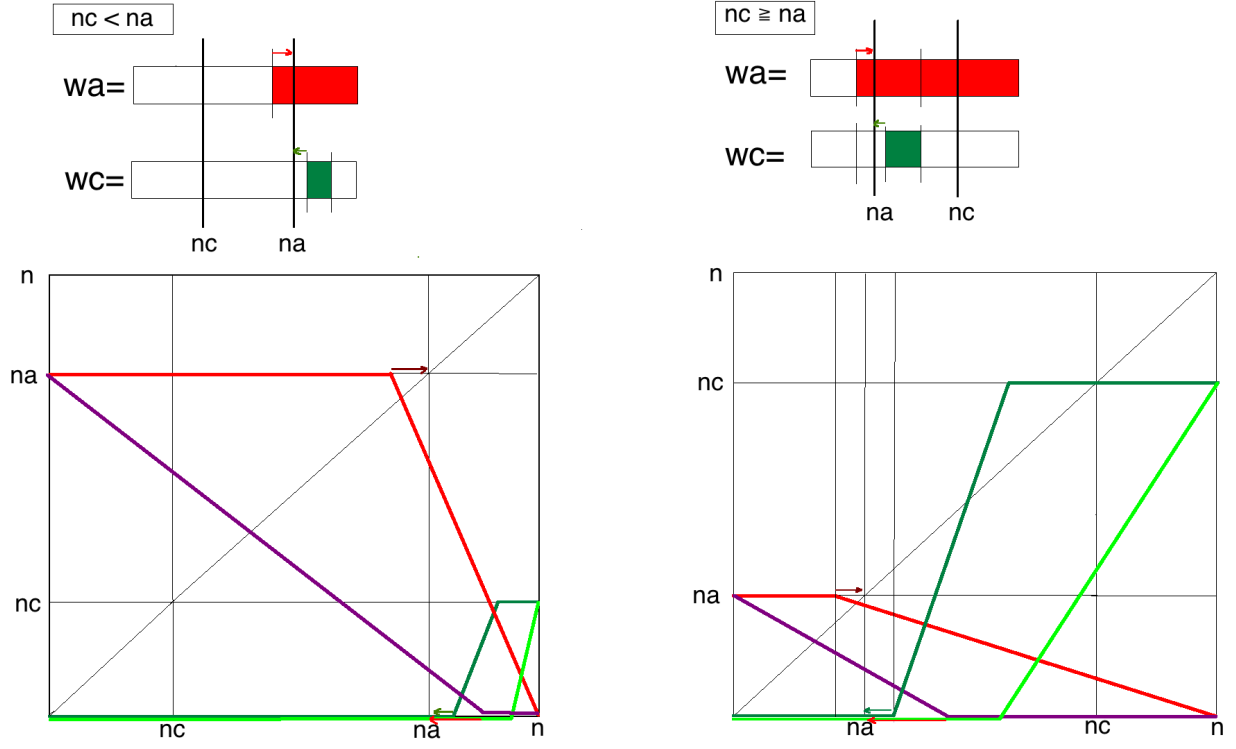
<sup>17</sup>This can be obtained from Case 5 using the transformation rule  $k^p + 1 \leftrightarrow n - l^p, >\leftrightarrow \leq$  and  $n^a \leftrightarrow n^c$ . This remark is formalized by Theorem 3.

<sup>18</sup>This can be obtained from Case 7 using the transformation rule  $k^p + 1 \leftrightarrow n - l^p, >\leftrightarrow \leq$  and  $n^a \leftrightarrow n^c$ . This remark is formalized by Theorem 3.

<sup>19</sup> $N^a \rightarrow N^c$  with probability 1 if and only if (i)  $n - l^a \leq n^a$  and (ii)  $n - l^c \leq n^a$ .  $N^c \rightarrow \emptyset$  with probability 1 if and only if (iii)  $n - l^a \leq n^c$  and (iv)  $k^c + 1 > n^c$ . We know also that (v)  $k^c + 1 \leq n - l^c$ , since the first non-zero coefficient of  $w^c$  is lower or equal than the last non-zero coefficient of  $w^c$ . (ii)-(iv)-(v) imply that  $n^c < n^a$  is a necessary condition. Since the inequality (iii)  $l^a \leq n^a$  implies (i)  $n - l^a \leq n^a$  (that we can also write  $n^c \leq l^a$ ) we get that the necessary inequality (i) is redundant. Redundancies of inequalities occur also in other cycles as we could show. We will point it out again when we will make the details of how to get the conditions for the cycles in the proof of Theorem 1. We don't know yet how to exploit this curious phenomenon.

<sup>20</sup>This can be obtained from Case 10 using the transformation rule  $k^p + 1 \leftrightarrow n - l^p, >\leftrightarrow \leq$  and  $n^a \leftrightarrow n^c$ . This remark is formalized by Theorem 3. In the previous case, the redundant inequality was one of the two conditions to get  $N^a \rightarrow N^c$  with probability 1. Therefore it is not surprising that this time, the redundant inequality is one of the two conditions to get  $N^c \rightarrow N^a$  with probability 1 (again, this is a manifestation of Theorem 3); this redundant inequality is indeed  $n^c \leq k^a$ .

*Proof.* The proofs are left to the reader and can be done in a similar fashion as in Proposition 3. We can do a graphical "proof" for Case 3 to show how useful it is to resort to graphs to formulate conjectures.



□

We can make some comments for the cycles involving both  $N^a$  only and  $N^c$  only (that is, a trend passes from the conformists only to the anti-conformists only, in a cycle. They are represented by Cases 9, 10 and 11). In these cases,  $w^a$  and  $w^c$  have their non-zero coefficients "in the middle". These cycle require (necessary condition) (1) societies with more anti-conformists than conformists (strictly in the case of cycles of order 3) (2) that agents decide not to weigh the outsiders, that is, their non-zero coefficients of the weigh vectors are "in the middle". In a sense whether agents are conformists or not, this is some kind of "conformism" (but we should use another word to describe this. We can think of "conservative"). In particular a necessary condition to get the cycle  $N^a \rightarrow N^c$  is that both weigh vectors must be lower-null of order  $n^c$ .

## 4.5 Theorem 2: on the multiplicity of terminal classes

Theorem 2 hereafter generalizes Corollary 2 that we met in the homogeneous framework. It is remarkable that the necessary and sufficient conditions only relies on  $w^c$ . The weigh vector  $w^a$  can be anything.

**Theorem 2:** Assume that all agents are pure. There exists two terminal classes if and only if  $n^a \leq k^c, n - l^c \leq n^c$ .

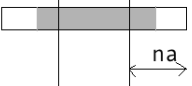
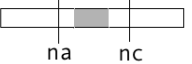

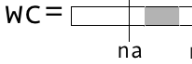
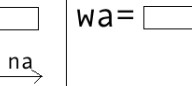
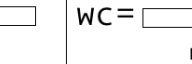
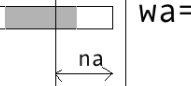
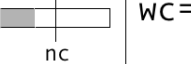
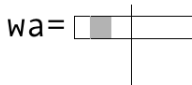
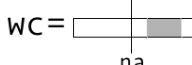
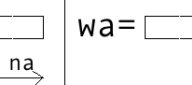
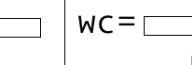
*Proof.* We know from Corollary 2.2 that there cannot be two cycles being terminal classes.

We know from proposition 10 that  $N^a$  and  $N^c$  can be terminal states together. The conditions stated are necessary.

We know by Theorem 1 that it remains to examine the booleans. If a terminal class includes neither  $N^a$  nor  $N^c$ , then it includes neither  $\emptyset$  nor  $N$ . But it is impossible that none of these four sets belongs to a given terminal class. Let us assume now that there are two terminal classes<sup>21</sup>. Let's show that if a terminal class includes both  $N^a$  and  $N^c$ , then there cannot be two terminal classes. Assume that  $N^a$  and  $N^c$  belong to  $\mathbf{C}_2$ . If  $K = K^c \cup K^a \in \mathbf{C}_1$  with  $\emptyset \subset K^a \subset N^a$  and  $\emptyset \subset K^c \subset N^c$  then  $\mathbf{C}_1 = 2^N$ . Absurd. If  $K = K^c \cup N^a \in \mathbf{C}_1$  with  $\emptyset \subset K^c \subset N^c$ , then  $N \in \mathbf{C}_1$ , which gives in turn  $N^c \in \mathbf{C}_1$ . Absurd. We just showed that if there are two terminal classes, then one contains  $N^a$ , the other contains  $N^c$ . The terminal class that includes  $N^a$  must be either a terminal state, or it must be one of these ones: (i)  $[\emptyset, N^a]$  or (ii)  $\emptyset \rightarrow N^a$ . By the same token we show that the terminal class containing  $N^c$  must be either  $N^c$ ,  $[N^c, N]$  or  $N^c \rightarrow N$ .

- By Corollary 3.2,  $\emptyset \rightarrow N^a$  and  $N^c \rightarrow N$  cannot both be terminal classes of the same society.
- $\emptyset \rightarrow N^a$  and  $[N^c, N]$  cannot be both terminal classes of the same economy, for we would have by Proposition 11:  $n^c < n - l^a \leq n^a \leq k^c < k^c + 1 \leq n - l^c < n^c$ .
- $[\emptyset, N^a]$  and  $N^c \rightarrow N$  cannot both be terminal classes of the same society, for we would have by Proposition 11:  $n - l^c \leq n^c \leq k^a < n^a < n - l^a$ . But  $n - l^c \geq k^c + 1 > k^c \geq n^a$  where the last inequality comes from 11.3. This would imply that  $n^a < n^a$ . Absurd.

Finally we are left with six possible couples. We put below a complete summary of all the societies where there are two terminal classes.

Case	1	2	3	4
Term. classes	$[\emptyset, N^a], [N^c, N]$	$[\emptyset, N^a], N^c$	$N^a, [N^c, N]$	$N^a, N^c$
Weight vectors	$w^a =$  $w^c =$ 	$w^a =$  $w^c =$ 	$w^a =$  $w^c =$ 	$w^a =$  $w^c =$ 
Condi-tions (Prop.11)	$ka+1 \leq na, nc < n-la$ $na \leq kc, lc$	$ka+1 \leq na < n-la,$ $na \leq kc, la, lc$	$ka+1 \leq nc \leq n-la$ $na \leq kc, ka, lc$	$na \leq ka, kc$ $na \leq la, lc$
Case	5	6		
Term. classes	$\emptyset \rightarrow N^a, N^c$	$N^a, N^c \rightarrow N$		
Weight vectors	$w^a =$  $w^c =$ 	$w^a =$  $w^c =$ 		
Condi-tions (Prop.11)	$n-la \leq na \leq kc, la, lc$	$n-lc \leq nc \leq ka$ $na \leq ka, kc$		

<sup>21</sup>With GOWA there can be at most two terminal classes.

We can check that in all cases we must have  $n^a < n^c$ :

1. Case 1:  $n^a \leq k^c < k^c + 1 \leq n^c$ .
2. Case 2:  $n^a \leq k^c < k^c + 1 \leq n - l^c \leq n - n^a = n^c$ .
3. Case 3:  $n^a \leq k^a < k^a + 1 \leq n^c$ .
4. Case 4:  $n^a \leq k^a < k^a + 1 \leq n - l^a \leq n - n^a = n^c$ .
5. Case 5:  $n^a \leq k^c < k^c + 1 \leq n - l^c \leq n^c$
6. Case 6:  $n^a \leq k^c < k^c + 1 \leq n - l^c \leq n^c$

Moreover the  $C_3^2 + 3 = 6$  cases considered span all the possible configurations for  $w^a$ . In all cases we have  $n^a \leq k^c, n - l^c \leq n^c$ . This necessary condition can easily be shown to be sufficient.  $\square$

In particular we generalized Corollary 2: there cannot be several terminal classes if  $n^c \leq n^a$ <sup>22</sup>. Actually we can show easily in the homogeneous framework that a necessary and sufficient condition for the existence of two terminal classes in the homogeneous framework is  $k \leq n^a, n - l \leq n^c$ . That  $n^a < n^c$  is a necessary condition means that we need sufficient stability for the society for two terminal classes to exist. In particular we must not have too many anti-anticonformists.

As a rule of the thumb, we can remember that there are two terminal classes if the coefficients of  $w^c$  are "centered in the middle".  $w^a$  can be any vector, it will not determine the multiplicity of terminal classes but only their nature.

We notice that if agents are not *statu quo*-biased, in the sense that what action designates 'yes' and which one designates 'no' is only a matter of convention, in other words that  $w_i = w_{n-i+1}$  for all agents, then we cannot have two terminal classes of different dynamics (one singleton and one boolean, or one singleton and one cycle), as described by Cases 2, 3, 5 and 6.

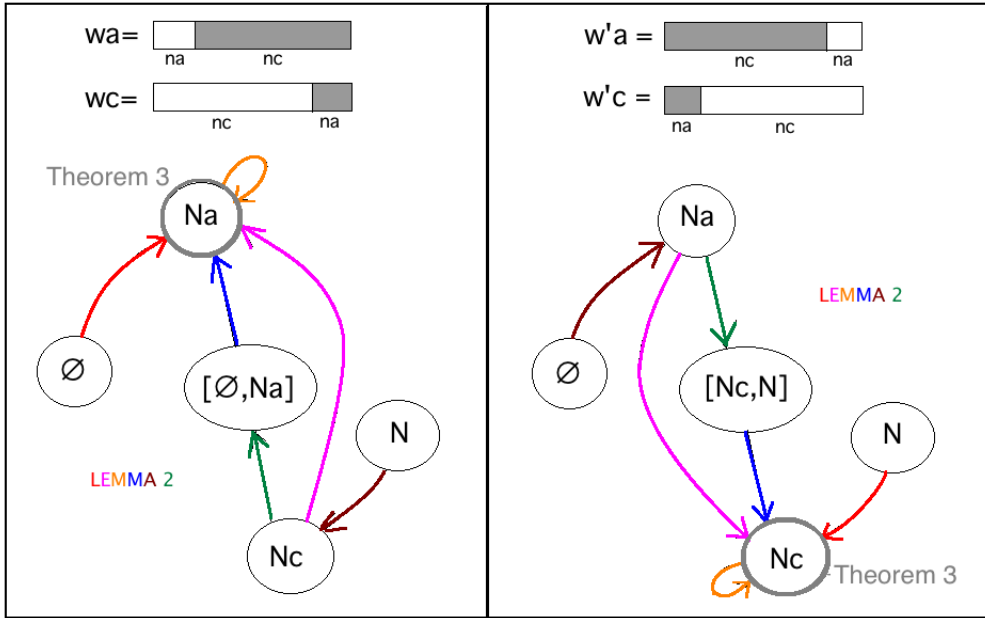
## 4.6 Theorem 3: on the symmetry structure of GOWA

In this subsection we present two theorems in which we compare the terminal classes of two societies where  $n^a$  and  $n^c$  remain fixed, but  $w^a$  and  $w^c$  are modified. In Theorem 3, we switch the coefficients of the weigh vectors as indicated on Table 1.

The figure hereafter illustrates Lemma 3 and Theorem 3 (with pure agents in this figure but this is not required for these propositions). A glance at Table 1 shows how the terminal classes are transformed one into another when we do the transformation  $k \leftrightarrow l$  on  $w^a$  and on  $w^c$  (we say that  $w^a$  is switched into a vector topologically equivalent to  $w'^a$ ).

---

<sup>22</sup>We can notice that this is the second time in this master thesis that a condition is not (either " $n^a > n^c$ " or " $n^a \leq n^c$ "). The first time was precisely in Corollary 2. This shows that the case  $n^a = n^c$  is not only a subcase of the case  $n^c \leq n^a$  as we could be tempted to think by observing the Table 1. Actually there are really three subcases to describe the dynamism of GOWA:  $n^a < n^c, n^a = n^c$  and  $n^a > n^c$ . Some situations are splitted into  $n^a \geq n^c$  and  $n^a < n^c$ . Other situations are splitted into  $n^a > n^c$  and  $n^a \leq n^c$ .



Notation:  $w \sim w'$  if and only if  $k = k'$  and  $l = l'$ .

Two such weigh vectors are said to be *topologically equivalent*. Two topologically equivalent vectors are drawn in the same fashion on our pictures with gray and blank zones. We can define an equivalent class of topologically equivalent weigh vectors (Fact 6).

Fact 6:  $\sim$  is reflexive, symmetric and transitive.

We introduce now the notation that tells " $w'$  is topologically equivalent to the switched weigh vector of  $w$ ":

Notation:  $wRw'$  if and only if  $k = l'$  and  $l = k'$ .

Fact 7:  $wRw'$  and  $w'Rw'' \Rightarrow w \sim w''$ .

In what follows we need to sets of weigh vectors such that the society verifies a certain property, permutation matrices and other properties of terminal classes. We introduce also the following notations to make the statements and proofs lighter:

**Notations:**

- The set of terminal classes of  $q := (w^a, w^c, n^a, n^c)$  is denoted  $\bigcup_j T_j(q)$ , where each  $T_j$  is a terminal class. We denote by  $L_q$  the set of indexes  $j$  (in the context of GOWA,  $j = 1$  or  $2$ ).
- We write  $E(S, T) := \{(w^a, w^c) \mid \lambda_{S, T} \neq 0\}$ .
- $C(K)$  is the terminal class that  $K$  belongs to. If  $K$  does not belong to any terminal class, we write  $C(K) = \emptyset$ . If there is an ambiguity of the society considered, we write  $C^q(K)$ .

- We define the binary matrix  $\Theta$  of  $\Lambda$  by:  $(\Theta_{S,T}) = 1$  if  $\lambda_{S,T} > 0$  and 0 otherwise. It is the matrix of the reduced graph of the stochastic process defined by the society  $q$  (we can put an upper-script  $q$  on  $\Theta$  if a misunderstanding is possible).
- Let  $\Sigma$  be the permutation matrix defined on  $2^N \times 2^N$  by  $(\Sigma_{S,T}) := \begin{cases} 1 & \text{if } T = \tilde{S} \\ 0 & \text{if } T \neq \tilde{S} \end{cases}$ .

*Fact 8:*  $\Sigma^2 = Id$ .

The following notion helps disentangling the different cases to treat in Lemma 3.  $Int(G(S))$  is a subset of  $G(S)$  in which we find all the sets  $T$  of  $G(S)$  such that if all agents of one group say unanimously 'yes' or 'no', then they could not have said something else. Whether  $T$  is in  $Int(G(S))$  or not is important to recover whether  $\sum_i w_i x_{(i)}$  is equal to  $-1$ ,  $1$  or is between strictly between the two (for  $w = w^a$  or  $w^c$ ). If for example all conformists say 'yes', it does not mean in general that  $\sum_i w_i^c x_{(i)} = 1$ . It could be that  $\sum_i w_i^c x_{(i)} \notin \{-1, 1\}$  and that by chance all the conformists say 'yes'. But if  $T$  is taken in  $Int(G(S))$  then we can infer from the unanimity of agents of one group in their decision that they *had* to take this decision.

**Definition:** We define the interior of  $G(S)$  as follows:

$$Int(G(S)) = \left\{ T = T^a \cup T^c \in G(S) \mid \begin{cases} T^a = \emptyset \Rightarrow p_{a^\pi}(1_S) = 0 \\ T^c = \emptyset \Rightarrow p_{c^\pi}(1_S) = 0 \\ T^a = N^a \Rightarrow p_{a^\pi}(1_S) = 1 \\ T^c = N^c \Rightarrow p_{c^\pi}(1_S) = 1 \end{cases} \right\}$$

We define the boundary of  $G(S)$  to be  $Fr(G(S)) := G(S) \setminus Int(G(S))$ .

The following Lemma 3, from which follows Theorem 3 on the terminal classes, is a proposition on the symmetries among the graphs of societies whose weigh vectors have been applied the transformation  $k \leftrightarrow l$ . Once this lemma on the reduced graphs is proved, it is straightforward to deduce Theorem 3.

**Lemma 3:**  $E(\tilde{S}, \tilde{T}) = \{(w'^a, w'^c) \mid (w^a, w^c) \in E(S, T), w^a R w'^a, w^c R w'^c\}$ .

which can be equivalently stated as follows:

**Lemma 3** (other formulation): Assume that  $w^a R w'^a$  and  $w^c R w'^c$ . Let  $q := (w^a, w^c, n^a, n^c)$  and  $q' := (w'^a, w'^c, n^a, n^c)$ . Then:  $\Theta^{q'} = \Theta^q \Sigma$ .

*Proof.* (Lemma 3)

1. Case 1: All agents are pure.

Let  $S, T \subseteq 2^N$ . We have  $s' = n - s$ .

For  $T = T^a \cup T^c$ , we only treat the case  $\emptyset \subset T^a \subset N^a$  and  $\emptyset \subset T^c \subset N^c$ . This allows us

to write  $\lambda_{S,T}^q \neq 0 \Leftrightarrow \begin{cases} p_a(1_S) \notin \{0, 1\} \\ p_c(1_S) \notin \{0, 1\} \end{cases}$ . The proof of the other subcases are similar. In

particular we need to treat separately  $T \in \{\emptyset, N^a, N^c, N\}$ . We extend a bit the proof in Case 2 to show how the other cases can be treated.



We want to show that for all  $q' = (n^a, n^c, w'^a, w'^c)$  such that  $w^a R w'^a$  and  $w^c R w'^c$  we have  $\lambda_{S,T}^q \neq 0 \Leftrightarrow \lambda_{\tilde{S},\tilde{T}}^{q'} \neq 0$ .

$$\begin{aligned} \lambda_{S,T}^q \neq 0 &\Leftrightarrow \begin{cases} p_a(1_S) \notin \{0,1\} \\ p_c(1_S) \notin \{0,1\} \end{cases} \Leftrightarrow \begin{cases} \sum_i w_i^a x_{(i)} \notin \{-1,1\} \\ \sum_i w_i^c x_{(i)} \notin \{-1,1\} \end{cases} \Leftrightarrow \begin{cases} k^a + 1 \leq s < n - l^a \\ k^c + 1 \leq s < n - l^c \end{cases} \\ &\Leftrightarrow \begin{cases} n - (n - l^a) = l^a = k'^a < n - s \leq n - (k^a + 1) = n - l^a - 1 \\ n - (n - l^c) = l^c = k'^c < \underbrace{n - s}_{s'} \leq n - (k^c + 1) = n - l^c - 1 \end{cases} \\ &\Leftrightarrow \begin{cases} k'^a + 1 \leq s' < n - l^a \\ k'^c + 1 \leq s' < n - l^c \end{cases} \Leftrightarrow \lambda_{\tilde{S},\tilde{T}}^{q'} \neq 0 \end{aligned}$$

2. Case 2:  $S = (\underbrace{P^a \cup M^a}_{S^a}) \cup (\underbrace{P^c \cup M^c}_{S^c})$ .

We only give brief outlines of a non exhaustive list of subcases. All the other subcases work the same way.

- The definition of  $Int(G(S))$  imposes to treat separately  $T \in \{\emptyset, N^a, N^c, N\}$ , since the negation of " $T = \emptyset \Rightarrow \dots$  and  $T = N^a \Rightarrow \dots$  and  $\dots$ " in the definition of  $Int(G(S))$  does not translate into the negation each of these statements. We can check that Lemma 3 is verified in these cases.

- $T = T^a \cup T^c$  with  $\emptyset \subset T^a \subset N^a$  and  $\emptyset \subset T^c \subset N^c$ .

The proof goes as before by noticing that under these conditions  $\lambda_{S,T}^q \neq 0 \Leftrightarrow \begin{cases} p_{a^\pi}(1_S) \notin \{0,1\} \\ p_{c^\pi}(1_S) \notin \{0,1\} \end{cases}$  and making similar computations to Case 1.

- $T \in Int(G(S)) \cap \{T = T^c \mid \emptyset \subset T^c \subset N^c\}$ . We have  $\lambda_{S,T}^q \neq 0 \Leftrightarrow \begin{cases} p_{a^\pi}(1_S) = 0 \\ p_{c^\pi}(1_S) \notin \{0,1\} \end{cases}$ , the computations after being analogous to Case 1.
- $T \in Fr(G(S)) \cap \{T = T^c \mid \emptyset \subset T^c \subset N^c\}$ . We have  $p_{a^\pi}(1_S) \notin \{0,1\}$  and  $p_{c^\pi}(1_S) \notin \{0,1\}$  and the computations after being analogous to Case 1.
- $T \in Fr(G(S)) \cap \{T = N^a \cup T^c \mid \emptyset \subset T^c \subset N^c\}$ . As the previous case we have  $p_{a^\pi}(1_S) \notin \{0,1\}$  and  $p_{c^\pi}(1_S) \notin \{0,1\}$  and the computations after being analogous to Case 1.

□

Theorem 3 is a direct consequence of Lemma 3:

**Theorem 3:** Let  $q := (n^a, n^c, w^a, w^c)$  and  $q' := (n^a, n^c, w'^a, w'^c)$  where  $w'^a R w^a$  and  $w'^c R w^c$ . Then  $\bigcup_j T_j(q') = \{\tilde{K} \mid K \in \bigcup_j T_j(q)\}$ .

*Proof.* By Lemma 3, the reduced graphs of the societies  $q$  and  $q'$  (in which the states have been replaced by their complements) are the same. Therefore the terminal classes are the same.

□

In this master thesis, we only study the terminal classes, in other words only the reduced graphs of the matrices of transition are of interest for us (we are not interested in the probabilities of transition, they are either 0, 1 or between the two; we don't need more precisions). However, let us mention briefly that we can give a necessary and sufficient condition for the probabilities of transition to be the same. We call it called Lemma 3.bis. The proof is given in Appendix 2, in which we provide also a discussion on the converse. The interest of presenting Lemma 3.bis in our framework is to show that in order to get identical transition probabilities in the new graph (under conditions of the absolute values of the degrees of conformism), we need to *switch* all the indexes of the weigh vectors, not *translate* the coefficients leftwards or rightwards, which tells us that the structure of GOWA is about symmetries, not translations.

**Lemma 3.bis:** If  $w'_i = w_{n-i+1} \forall i \in [1, n]$  then  $\Lambda^{q'} = \Lambda^q \Sigma$ .

Lemma 3 is more adapted to prove Theorem 2 than Lemma 3.bis. It is not a straightforward consequence of Lemma 3.bis (and therefore Theorem 2 is not either) because Lemma 3 takes all the weigh vectors within a same equivalence class and all the possible degrees of (anti)-conformism. To prove Theorem 3, an approach with equivalent classes is unavoidable. Using Lemma 3.bis would not allow us to escape this deep study since we need to generalize its specific framework, where we assumed that the weigh vectors and degrees of (anti)-conformism have a special form.

## 4.7 Theorem 4: on the inclusion of terminal classes

We provide a last theorem that formalized the remark that we did when looking at the Table 1: when we increase the order of left or right-nullity of some weigh vectors, the new terminal classes are included in the old ones. The proof of Theorem 4 relies on Lemma 4; this lemma is not tied to GOWA and applies to more general situations.

*Notation:* We write  $(a, b) \leq (c, d)$  if  $a \leq c$  and  $b \leq d$ .

**Theorem 4:** Let  $q = (n^a, n^c, w^a, w^c)$  and  $q' = (n^a, n^c, w'^a, w'^c)$ . Assume that the left and right-orders of nullity of  $w^a$  and  $w^c$  are respectively  $(k^a, p^a)$  and  $(k^c, p^c)$ . The left and right-orders of nullity of  $w'^a$  and  $w'^c$  are respectively  $(k'^a, p'^a)$  and  $(k'^c, p'^c)$ . If  $(k^a, l^a) \leq (k'^a, l'^a)$  and  $(k^c, l^c) \leq (k'^c, l'^c)$  then  $T_j(q') \subseteq T_j(q)$  for any  $j \in L_q$ .

This theorem can of course be proved in a cumbersome manner from Proposition 11. The proof presented here however aims at being more simple. Moreover, a disjunction of cases is not satisfying for our curiosity. Disjunctions of cases are always empirical in the sense that they establish a proof without shedding any light on the underlying mechanisms at play.

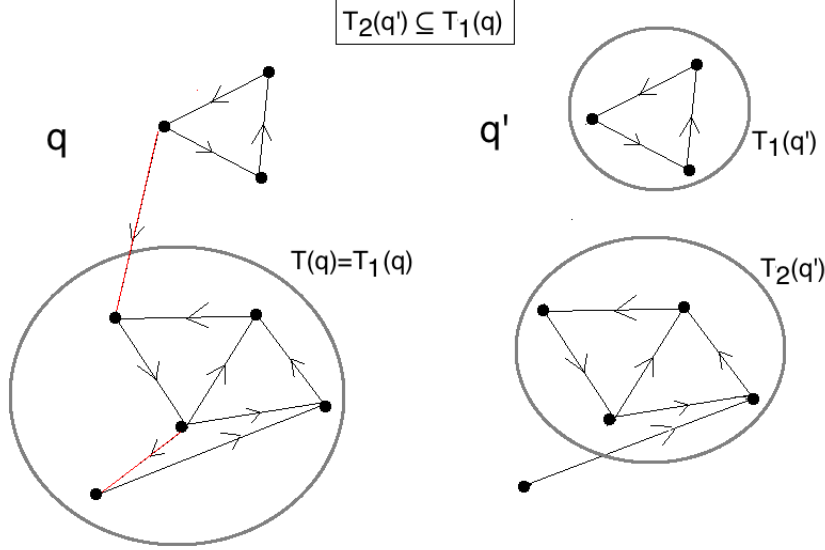
**Lemma 4:** Let  $q$  and  $q'$  be such that  $\Theta(q') \leq \Theta(q)$ . Let  $K \in T_j(q')$ . If  $K \in T_{j'}(q)$  then  $C^{q'}(K) \subseteq C^q(K)$ , which can also be written  $T_{j'}(q') \subseteq T_j(q)$ .

*Proof.* (Lemma 4) Assume that  $K, L \in C_{q'}(K)$ . By definition of a terminal class there exists  $K_1, \dots, K_p \in C_{q'}(K)$  such that  $\Theta(q')_{K, K_1} = 1, \Theta(q')_{K_1, K_2} = 1, \dots, \Theta_{K_p, L}(q') = 1$  (or, equivalently  $(\Theta^{p+1})_{K, L} > 0$ ). Since  $\Theta(q') \leq \Theta(q)$ , we have  $\Theta(q)_{K, K_1} = 1, \Theta(q)_{K_1, K_2} = 1, \dots, \Theta(q)_{K_p, L} = 1$ , that

is,  $L \in C_q(K)$ .

□

**Illustration of Lemma 4:**



The difficulty of Theorem 4 is that removing some links can potentially create multiple terminal classes. This is the case, for example, with  $n^a = 1$ ,  $n^c = 2$ ,  $w^a = w^c = (0, *, *)$ . In this case there is one terminal class ( $N^a$ ). With  $n^a = 1$ ,  $n^c = 2$ ,  $w^a = w^c = (0, *, 0)$ , there are now two terminal classes ( $N^a$  and  $N^c$ ).

*Proof.* (Theorem 4) We only treat two subcases:

1.  $T = T^a \cup T^c$  where  $\emptyset \subset T^a \subset N^a$  and  $\emptyset \subset T^c \subset N^c$ .

Assume that  $\begin{cases} \lambda_{S,T}^{q'} \neq 0 \\ (k^a, l^a) \leq (k'^a, l'^a) \\ (k^c, l^c) \leq (k'^c, l'^c) \end{cases}$ . Given the form of  $T$  we have  $\begin{cases} p_{a^\pi}^{q'}(S) \notin \{0,1\} \\ p_{c^\pi}^{q'}(S) \notin \{0,1\} \end{cases}$  which

can be rewritten  $\begin{cases} \sum_i w_i^{a'} x(i) \notin \{-1,1\} \\ \sum_i w_i^{c'} x(i) \notin \{-1,1\} \end{cases}$  that is, equivalently:  $\begin{cases} k'^a + 1 \leq s < n - l'^a \\ k'^c + 1 \leq s < n - l'^c \end{cases}$ .

Since  $(k^a, l^a) \leq (k'^a, l'^a)$  and  $(k^c, l^c) \leq (k'^c, l'^c)$  we have:  $\begin{cases} k^a + 1 \leq k'^a + 1 \leq s < n - l'^a \leq n - l^a \\ k^c + 1 < k'^c + 1 \leq s < n - l'^c < n - l^c \end{cases}$ .

This implies  $\lambda_{S,T}^q \neq 0$ .

2.  $T = N^a \cup T^c \in \text{Int}(G(S))$  where  $\emptyset \subset T^c \subset N^c$ .

Assume that  $\begin{cases} \lambda_{S,T}^{q'} \neq 0 \\ (k^a, l^a) \leq (k'^a, l'^a) \\ (k^c, l^c) \leq (k'^c, l'^c) \end{cases}$ . Given the form of  $T$  we have  $\begin{cases} p_{a^\pi}^{q'}(S) = 1 \\ p_{c^\pi}^{q'}(S) \notin \{0,1\} \end{cases}$  which

can be rewritten  $\begin{cases} \sum_i w_i^{a'} x(i) = -1 \\ \sum_i w_i^{c'} x(i) \notin \{-1,1\} \end{cases}$  that is, equivalently:  $\begin{cases} s < k'^a + 1 \\ k'^c + 1 \leq s < n - l'^c \end{cases}$ . Since

$(k^a, l^a) \leq (k'^a, l'^a)$  and  $(k^c, l^c) \leq (k'^c, l'^c)$  we have:  $\begin{cases} s < k'^a + 1 \leq n - l'^a \leq n - l^a \\ k^c + 1 \leq k'^c + 1 \leq s < n - l'^c \leq n - l^c \end{cases}$ .

The anti-conformists have a non-zero probability to say 'yes'<sup>23</sup> and the conformists can say

<sup>23</sup> $\lambda \neq 0$  can be  $\lambda \notin \{0,1\}$  or  $\lambda = 1$ . We don't know and we don't need more precisions. But we know from Table 1 that increasing a left-or right order of nullity can make a transition become certain.

'yes' but not for sure. Therefore  $\lambda_{S,T}^q \neq 0$ .

The other subcases can be treated similarly in the spirit of the proof of Theorem 3. The implications obtained are true for any  $S, T \subseteq 2^N$ , therefore  $\Theta(q') \leq \Theta(q)$ . We apply Lemma 4 to get Theorem 4.  $\square$

## 4.8 Introduction to not pure agents in the heterogeneous framework

**Proposition 12** (with heterogeneous weigh vectors): If at least one agent is not pure, then terminal classes are a union of boolean lattices.

*Proof.* 1. Let  $S$  be the current state. We express  $N$  as the union of conformist and anti-conformist agents, each of these subsets being the union of pure and not pure agents :  $N = \underbrace{(M^a \cup P^a)}_{N^a} \cup \underbrace{(M^c \cup P^c)}_{N^c}$ . There are eight cases:

- (a) *Case 1:* At the state  $S$  we have  $\sum_{i=1}^n w_i^c x(i) = -1$  and  $\sum_{i=1}^n w_i^a x(i) = -1$ . Then the set of directly reachable sets is  $[P^a, N^a] \cup [\emptyset, M^c]$ .
- (b) *Case 2:* At the state  $S$  we have  $\sum_{i=1}^n w_i^c x(i) = 1$  and  $\sum_{i=1}^n w_i^a x(i) = -1$ . Then the set of directly reachable sets is  $[\emptyset, M^a] \cup [P^c, N^c]$ .
- (c) *Case 3:* At the state  $S$  we have  $\sum_{i=1}^n w_i^c x(i) = -1$  and  $\sum_{i=1}^n w_i^a x(i) = 1$ . Then the set of directly reachable sets is  $[\emptyset, M^a] \cup [\emptyset, M^c]$ .
- (d) *Case 4:* At the state  $S$  we have  $\sum_{i=1}^n w_i^c x(i) = 1$  and  $\sum_{i=1}^n w_i^a x(i) = 1$ . Then the set of directly reachable sets is  $[\emptyset, M^a] \cup [P^c, N^c]$ .
- (e) *Case 5:* At the state  $S$  we have  $\sum_{i=1}^n w_i^a x(i) = 1$  and  $\sum_{i=1}^n w_i^c x(i) \notin \{-1, 1\}$ . Then the set of directly reachable sets is  $[\emptyset, N^c] \cup [\emptyset, M^a]$ .
- (f) *Case 6:* At the state  $S$  we have  $\sum_{i=1}^n w_i^a x(i) = -1$  and  $\sum_{i=1}^n w_i^c x(i) \notin \{-1, 1\}$ . Then the set of directly reachable sets is  $[\emptyset, N^c] \cup [P^a, N^a]$ .
- (g) *Case 7:* At the state  $S$  we have  $\sum_{i=1}^n w_i^c x(i) = 1$  and  $\sum_{i=1}^n w_i^a x(i) \notin \{-1, 1\}$ . Then the set of directly reachable sets is  $[\emptyset, N^a] \cup [P^c, N^c]$ .
- (h) *Case 8:* At the state  $S$  we have  $\sum_{i=1}^n w_i^c x(i) = -1$  and  $\sum_{i=1}^n w_i^a x(i) \notin \{-1, 1\}$ . Then the set of directly reachable sets is  $[\emptyset, N^a] \cup [\emptyset, M^c]$ .

Any state  $S$  can only reach a boolean, therefore a final class can only be a union of booleans.  $\square$

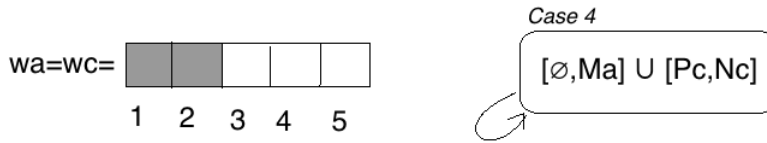
The proof of the previous proposition leaves us with a list of possible booleans. If we want to get all the possible terminal classes, we need to examine which transitions are compatible with each others within a terminal class.

**Transition from  $S$  to  $T$ :**

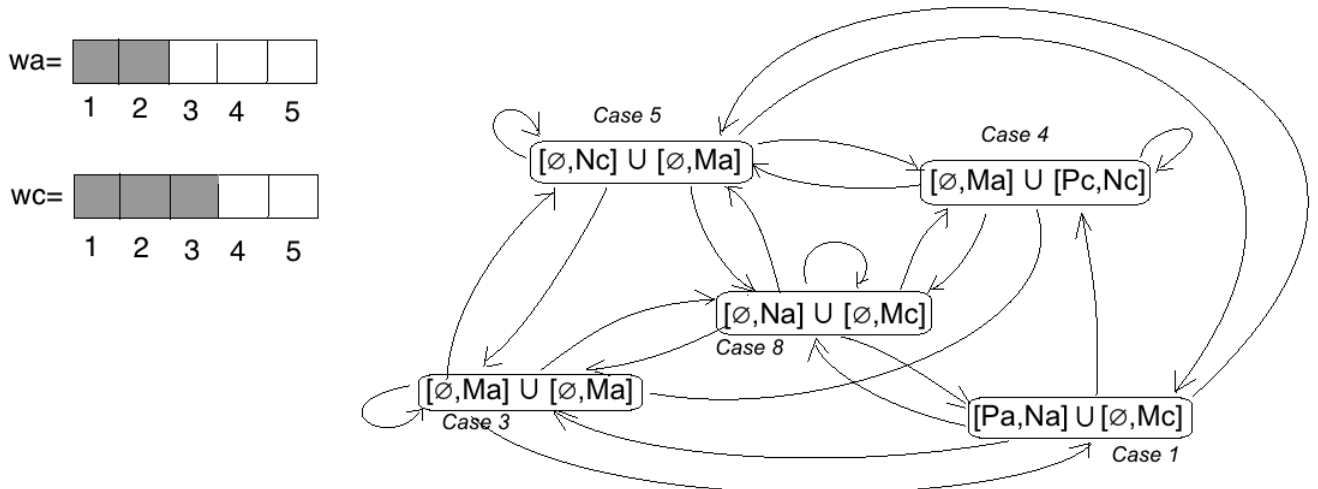
	Necessary and sufficient conditions of $S$	Property of $T$
Case 1	$s < k^c + 1$ and $s < k^a + 1$	$t \in [p^a, n^a + m^c]$
Case 2	$s \geq n - l^c$ and $s < k^a + 1$	$t \in [p^c, n^c + m^a]$
Case 3	$s < k^c + 1$ and $s \geq n - l^a$	$t \in [0, m^a + m^c]$
Case 4	$s \geq n - l^c$ and $s \geq n - l^a$	$t \in [p^c, m^a + n^c]$
Case 5	$s \geq n - l^a$ and $k^c + 1 \leq s < n - l^c$	$t \in [0, m^a + n^c]$
Case 6	$s < k^a + 1$ and $k^c + 1 \leq s < n - l^c$	$t \in [p^a, n]$
Case 7	$s \geq n - l^c$ and $k^a + 1 \leq s < n - l^a$	$t \in [p^c, n]$
Case 8	$s < k^c + 1$ and $k^a + 1 \leq s < n - l^a$	$t \in [0, n^a + m^c]$

The set of possible terminal classes clearly depends on  $k^a, k^c, l^c$  and  $l^a$ . Let us expose two examples, the second one being a slight modification of the first one to appreciate the change.

**Example 1:** We take  $n^a = 2, n^c = 3, k^a = 0, l^a = 3, k^c = 0$  and  $l^c = 3$ . We take  $p^a = 1, m^a = 2 - 3 = 1, p^c = 2, m^c = 3 - 2 = 1$ . We represent below the terminal class as a union of booleans with the range of cardinality of people saying 'yes'.



**Example 2:** We take  $n^a = 2, n^c = 3, k^a = 0, l^a = 3, k^c = 0$  and  $l^c = 2$ . We take  $p^a = 1, m^a = 2 - 3 = 1, p^c = 2, m^c = 3 - 2 = 1$ . We represent below the terminal class as a union of booleans with the range of cardinality of people saying 'yes'.



The list of possible cases, that are unions of booleans if at least one agent is not pure, is very likely to be robust to the limit case where all agents are pure ; in this case terminal classes are not always boolean sets but they are limit cases of the possible classes that a future theorem will

hopefully list.

The generalization of Proposition 8 to the heterogeneous framework is not given in this master thesis but we give a weaker proposition hereafter, that tells everything we need to know to derive interesting qualitative results. The first part of it states that among the terminal classes characterized when all agents are pure (Theorem), let aside  $2^N$ , only the booleans of anti-conformists remain. The second part states that, apart from  $2^N$ , these terminal classes can only be obtained when all conformists are pure. In other words, we can allow some anti-conformist agents to be not pure and still get the same terminal classes<sup>24</sup>, but we cannot allow the conformist agents to be not pure. We discuss the implications of this result in the concluding section.

**Proposition 13:** Assume that at least one agent is not pure and that  $N^a \neq \emptyset$  and  $N^c \neq \emptyset$ .

1. Among the terminal classes listed by Theorem 1, only three of them can be terminal classes by choosing appropriate weight vectors:
  - $2^N$
  - $[\emptyset, N^a]$
  - $[N^c, N]$
2. The last two classes listed can be terminal classes only if (necessary condition) all conformist agents are pure.

*Proof.* Singletons and cycles obviously cannot be terminal classes anymore.  $[\emptyset, N^a] \cup [\emptyset, N^c]$  cannot be a terminal class; assume indeed without loss of generality that there exists a not pure conformist agent. That some anti-conformists say 'yes' does not prevent the not pure conformist agent from saying 'yes'. By the same token,  $[N^a, N] \cup [N^c, N]$  cannot be a terminal class. We can easily find examples of societies where the classes listed are terminal classes. It is obvious that all conformist agents being pure is a necessary condition for  $[\emptyset, N^a]$  and  $[N^c, N]$  to be terminal classes.  $\square$

## 5 Application: the history of fashion in the light of anti-conformism

### 5.1 Discussion on the dynamism of fashion

This master thesis provides an extensive analysis of the terminal classes and states. We can wonder whether it can explain stylised facts in the industry of fashion, in particular its intrinsic dynamism. If fashion was only a matter of imitation, there would be no trends. How come that fashion needs to change, how come that some trends are copied by some groups and given up by others ? Is following the fashion trends a conformist or an anti-conformist behavior ? Does this depend on the epoch ? Can some trends be embedded once for all in the moeurs (terminal state) or disappear and reappears regularly (cycles) ? Can we reconstitute from the evolution of the fashion trends, by looking at the terminal class that are the most likely to explain these evolutions, whether there are

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<sup>24</sup>We don't prove here whether  $w^a$  and  $w^c$  can still be the same as in the case of all agents being pure. It could be *a priori* that we can still get  $[\emptyset, N^a]$  and  $[\emptyset, N^c]$  to be terminal classes when we allow some anti-conformists at the cost of modifications of the weight vectors. We don't examine this issue here.

more anti-conformists in the society than conformists, their weigh vectors and if this has changed over time ?

People are not intrinsically conformists or anti-conformists, it depends on the object of fashion. Some trends are definitely adopted by the whole society (blue jean); in this sense there are no anti-conformists (maybe because they choose to be anti-conformist for something else). Other trends follow booms and bursts, and booms and burst again, etc... (maybe this remark can resuscitate theories of cyclical financial crisis. In this case, uncertainty would be at the root of those phenomena and anti-conformism would be another word to talk about mixed strategies).

Fashion has already been adressed by Bikhchandani and al. (1992). Their model in its spirit resembles Banerjee (1992). Our model takes a different turn since there is no sequential choices and some agents are anti-conformists while others are conformists. However, our analysis allows to explain dynamical phenomena. To the best of our knowledge, it is the first time that a process of imitation is generated that can explain various dynamical phenomena like stable and persistent consensus, large fluctuations or booms and bursts.

Our analysis on anti-conformism is particularly relevant with surnames. Some (anti-conformist) parents want to give their son a surname which is not so popular. But they may not be the only ones to have this idea (maybe because there is a best-seller book where the main character has this name); on top of that the surname they have chosen can be copied by later (conformist) parents. This in turn can make the surname popular in the subsequent years, before disappearing again and reappearing (cycle  $N^a \rightarrow N^c \rightarrow \emptyset$ ). We can argue that GOWA only focuses on a binary choice, which does not seem to fit the choice of surnames at first glance. However we can argue that parents consider candidates surnames sequentially and provide a 'yes' or 'no' answer to each of them. Of course things are more complex since they only make one choice (they don't chose a subset of surnames). In other words when a surname becomes popular, the others tend to become mechanically less popular. This incidentally raises the question of how to generalize GOWA to more than two outcomes and whether the behaviour of terminal classes will be similar.

## 5.2 An example of experiment on the field

We propose in this subsection an experiment that could test our results on the terminal classes with anti-conformism.

A survey can be conducted in primary shools by giving the parents a list of questions provided in Appendix 2. We don't discuss here the usual issues of surveys (anchoring for example can be argued to be an issue the word "imitate" appears in the survey since imitation is often unconscious and regarded as a negative action, but there is room to improve this form), neither whether we should target parents whose children are very young or the elementary schools, where children are older. This is indeed an important thing to think of since we ask the question "do you think that the surname has become more popular in the subsequent years ?". At first sight, we can argue that it is good to target parents whose children have a large spectrum of ages (let's say from a few months to 10 years old<sup>25</sup>). Indeed we can argue that the "bell shape" of popularity of surnames (instead of two waves), goes in favor either of an imitation of other parents whose

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<sup>25</sup>But not too large since there is always an issue of statistical power.

child is of very young age, or of an imitation on a spectrum of different ages (which blurries the two waves, anti-conformists then conformists). We can argue in favor of this, because most of the imitation phenomena are unconscious, but the survey can accomodate this missing answer. We can also wonder whether asking the parents the surname of their child in order to check whether their personal impressions about its popularity is relevant.

Then the experimenter can make a simulation, based on the answers of this survey. He would take a subset of, let's say 26 surnames A to Z (an appropriate number can be designed properly and discussed), to see which weigh vectors can generate the shapes observed in the popularity of surnames (easy to find on the internet). Indeed, we can, thanks to the survey, estimate directly the proportion of anti-conformist parents (from the questions "did you want to give him a rare surname ?" and "was it a cultural/religious/familial tradition or tribute ?"). From these answers we can define an index of conformism that we normalize then to 0.9 or  $-0.9$  (or any other number between  $-1$  and  $1$  strictly). In the algorithm, the decision would be to consider the letters A to Z sequentially, provide a 'yes'-'no' answer to each of them, and then redo the job on the subset obtained until only one surname is left. Of course, we need to ensure that we indeed get one surname in the end (not zero, which can *a priori* be the case), but such sophistications can be easily designed.

Finally, now that we have an estimation of the proportion of anti-conformist parents and  $w^a$  and  $w^c$ , we make the heroic assumption that parents are a good sample of the whole population and we look at the history of fashion. From a few products or behavior (blue jean, chewing gum, this or that brand... or anything we can gets statistics of) we look at whether the parameters obtain from the study on surnames can explain the dynamism of fashion. We can even make a robustness check: we can start from the fashion and see whether the parameters obtained are similar and whether it can generate the popularity phenomena in surnames.

## 6 Summary and discussion of the results

In this master thesis, we generalized the notion of ordered weighted averages in an anonymous social network to anti-conformism. Each agent makes a 'yes'-'no' decision at each stage of influence based on the decisions of all agents. Agents are intrinsically either conformist or anti-conformist. We introduced a new aggregator to generalize the ordered weighted average aggregator to the context of anti-conformism. An agent is defined with two things: a weigh vector, just as in OWA, and with an index of (anti)-conformism. Influence is entirely anonymous in all this master thesis. We introduced the notion of impurity, akin the trembling-hand in game theory. We built our analysis of anti-conformism starting from what we called the "homogeneous framework" in which all agents, conformists and anti-conformists, have the same weigh vector<sup>26</sup>. We provided a complete analysis of the terminal states, classes and their unicity or not. Then we examined the "heterogeneous framework" in which conformist and anti-conformist agents have a different weigh vector. We did again the analysis of the terminal states, classes and their unicity or not.

We have shown that in the homogeneous framework there can only four terminal classes:  $N^a$ ,  $N^c$ , the cycle  $N^a \rightarrow N^c$  and  $2^N$  (Proposition 5), from which we deduce straightfully that if there

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<sup>26</sup>We allowed the coefficient of (anti)-conformism  $\alpha$ , which also captures the degree of trembling-hand, to vary among and accross groups.



are two terminal classes, then they are the two terminal states  $N^a$  and  $N^c$ . A necessary condition for the multiplicity of terminal classes (terminal states in the homogeneous framework) is that we have strictly more conformist agents than anti-conformist agents (Corollary 2). We have shown that the necessary condition  $n^c > n^a$  to have two terminal classes is still true in the heterogeneous framework (Theorem 2).

The terminal states obtained are shown to be consistent with OWA, in which  $\emptyset$  and  $N$  are always terminal states. We noticed that we have, in the homogeneous framework, a dynamism that we did not have with OWA, thanks to the cycles  $2^N$  and  $N^a \rightarrow N^c$ . Though elementary, this is an important result since it tells us that it is possible to generate dynamical situations in anonymous influence frameworks by introducing anti-conformism.

We have shown that in the heterogeneous framework and all agents being pure, the range of possible terminal classes is larger: there are, with respect to the homogeneous framework, two additional cycles (of order 3) and several booleans (Theorem 1). These booleans are remarkably booleans of anti-conformists, except for the obvious case  $2^N$  and the rather peculiar case  $[\emptyset, N^a] \cup [\emptyset, N^c]$ . We provided necessary and sufficient conditions to get each one of the twelve possible terminal classes under heterogeneity (Proposition 11). Theorem 1 is probably the key point of this master thesis. It makes the list of all the possible terminal classes and it enlightens the asymmetry between conformism and anti-conformism. It provides incidentally an interesting result on the incompatibility of some sure transitions (Corollary 3). In particular the 4-cycle  $\emptyset \rightarrow N^a \rightarrow N \rightarrow N^c$  is impossible with GOWA. We present briefly in Appendix 3 a semi-anonymous aggregator (PEWA) that allows this cycle to be a terminal class.

We first conjectured Theorem 3 and Theorem 4 with Table 1 in section 4.2 that summarizes the thirty-six terminal classes spread into two cases:  $n^c < n^a$  and  $n^c \geq n^a$ . In this table appears the symmetries expressed by Lemma 3, which state that replacing a  $(k,l)$ -weigh vector by a  $(l,k)$ -weigh vector (both for  $w^a$  and  $w^c$ ) gives a society whose reduced graphs of transitions are the same, up to the replacement of each set by its complement. We proved Lemma 3 and we directly obtained Theorem 3 that states that under this transformation, the terminal class(es) is (are) replaced by its (their) complement. Lemma 3.bis, though a bit aside from our focus since it addresses the values probabilities of transitions, is proved in Appendix 2 and enlightens that the symmetries in GOWA are not a matter of translations of coefficients, but rather about switching  $w_i$  and  $w_{n-i+1}$ . Theorem 4 states that when the set of non-zero coefficients in the weigh vectors shrinks, that is, when their orders of left and right-nullity increase, the new terminal class(es) is (are) included in the previous one(s).

That a necessary condition to have several terminal classes is to have more conformists than anti-conformists has an intuitive explanation: *anti-conformism brings instability to the process*; if we want to have two terminal classes, enough stability is needed. Therefore we must not have too many anti-conformist agents. We have shown that under this necessary condition, there are two terminal classes if and only if the coefficients of  $w^c$  are "centered in the middle" (we gave the precise mathematical conditions in the proof of Theorem 2). Conformist agents must stay conservative and not weigh opinions that are shared by too few agents (the coefficients of  $w^c$  are "in the middle").  $w^a$  can be *any* vector, there will be two terminal classes in all cases under these conditions. This means that *under the assumption that there are more conformists than anti-conformists and the assumption that the conformists are conservative, the behavior of the anti-conformists plays no*

*role in the existence of two terminal classes.* Anti-conformist agents can have any weight vector without any "harm" to the stability of the society (the actual terminal class, which can be thought of as being the persistence of traditions or any kind of customs).

The closer  $n^a$  and  $n^c$  become to each others, the more "difficult" it is to meet the constraints on  $w^c$  and therefore the more "difficult" it is to have two terminal classes. What matters is the number of anti-conformists relatively to the conformists, not their "behavior" (that is their weigh vector  $w^a$  plays no role). This result can be expressed in terms of fuzzy quantifiers (not done in this master thesis): if the non-zero coefficients of  $w^c$  are between  $a\%$  of  $n$  and  $b\%$  of  $n$  then when  $n^c$  becomes large enough relatively to  $n^a$ , there are two terminal classes.

Interesting analogies can probably be made with physics in which there exists systems, for example in thermodynamics, in which there are barriers of potential. A system with  $n^c \leq n^a$  and  $n^a$  and  $n^c$  close to each others has the flavor of metastability: an exogenous change in the inclination of a few agents can change the terminal class and possibly the dynamics of the society. Indeed the terminal class that the society is trapped for example in  $[\emptyset, N^a]$ , the current state being  $S$ , then the terminal class can possibly become  $N^c$  if  $k^c - s$  conformist agents exogenously decide to say 'yes'. It becomes  $N^c$  for sure if  $n^c - s$  conformist agents exogenously decide to say 'yes' (see Theorem 2). That the number of anti-conformists relatively to conformists is of a crucial importance to determine the terminal classes is something that we already noticed by looking at Table 1 that summarizes several cases of  $w^a$  and  $w^c$  (not exhaustively): two sub-tables that Table 1 is made of are very different from one another. This means that *the balance between the number of anti-conformists and conformists is of a crucial importance to determine the terminal classes.* For this reason the case  $n^a = n^c$  is a particular one that is not always a mere sub-case of  $n^a \geq n^c$  and  $n^a \leq n^c$ . Some propositions are splitted into  $n^a \geq n^c$  and  $n^a < n^c$ . Other ones are splitted into  $n^a > n^c$  and  $n^a \leq n^c$ . This analysis of the importance of size of each group can be pursued thanks to Table 1, which has also something to teach us about the stability of the societies considered. We see in it that the weigh vectors of the smallest group crucially determines the nature of the terminal classes. We drew an analogy with the glove-market game in cooperative game theory without examining this question in details.

The analysis on the multiplicity of terminal classes can be recasted in the a cultural framework to explain whether some trends or products will have success in some countries/communities/... and not in others. The same society (as defined with a relative number of conformists with respect to anti-conformists, their weigh vectors, that is, if they weigh or not the outsiders, if they are more inclined to weigh the 'yes' or the 'no' for a given topic - in this case, 'yes' and 'no' are not arbitray, in particular  $k$  and  $l$  can be different from each other -, etc...), with the same individuals of the same characteristics, could have different customs (be "trapped" in a different terminal class). In particular, some cross-sectional econometric studies which would try to recover dynamics of preferences (it can be any kind of preference, either political or relative to a global fashion) from individual characteristics can be highly problematic because of the potential multiplicity of terminal classes. The *actual* situation matters to determine preferences at equilibrium. Put it differently, the equilibrium of a given situation, or the outcome of a given variable that the econometrician wants to study, is not merely pinned down by the individual parameters. The same society can have different customs, tastes, behavior either qualitatively or dynamically (for example think of the two terminal classes  $N^c$  and  $\emptyset \rightarrow N^a$ ). A given society can be "trapped" in one terminal class but it could have been another one had History been different and it might not

be possible to infer individual characteristics from historical events. In particular our discussion of the History of fashion in the light of anti-conformism can be difficult to lead because of the possible multiplicity of terminal classes.

Tools shall be developed to recover weigh vectors from dynamical systems. The previous paragraph tells that multiplicity of terminal classes can be an issue. The econometrician will have to find a way to recover the true weigh vectors using some kinds of instrumental variables on which we have time series data. A last remark is of a practice kind: in econometric models where we try to recover the weigh vectors from the dynamic of a situation, it can be that  $w^a$  and  $w^c$  are entirely filled with non-zero coefficients<sup>27</sup>, but the largest may be localized in some places. We can use threshold to identify the largest coefficients are re-normalize them to 1 after.

## 7 Conclusion: applications and future research

This study of GOWA, and more generally of anti-conformism is likely to have a large number of empirical applications. This lack of deep examination of anti-conformism in the literature can probably be explained by the more difficult treatment than positive influence and the fact that in economic situations, agents try to coordinate, or they don't have the choice to do something else. This is the case for exemple with the impossibility to beat the market in finance. But there are many sociological and economic situations which are likely to be fitted by anti-conformism. We already mentionned in section 5 that our model can explain dynamical phenomenon like fashion trends, for example the popularity of surnames. To the best of our knowledge, it is the first time that a process of imitation is generated that can explain various dynamical phenomena like stable and persistent consensus, large fluctuations or booms and bursts. There are many economic situations where anti-conformism plays a role. For example the choice of a firm to go compatible or not with other firms is typically a problem of anticonformism. The firm may even choose with some probability to go compatibler or not, in which case it is akin a not pure agent. This situation is particularly well-described with GOWA<sup>28</sup>.

Though we did not formalize this idea, anti-conformism can be seen as a mixed-strategy, either in a complete or in an incomplete information framework. For example, there exists models of coordination of expectation in which there are two types of consumers (or traders): the informed ones and the non-informed ones, in which we recover the total demand as an composition of the "informed demand" and the "non-informed demand". In particular anti-conformism is likely to find applications in risky situations. Risk-adverse agents who have to choose between 'yes' and 'no' do not want to choose one of them with probability 1, even if there is an anonymous 'yes' or 'no'. If we think of Banerjee's model of herd behavior (1992), this makes sense. In his model, agents play sequentially<sup>29</sup> and wrong cascades can occur. Though it can be rational to follow the crowd, some anti-conformist agents may want to play a mixed-strategy: either following the crowd

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<sup>27</sup>But maybe not in all cases. If some econometric methods are found to recover the weigh vectors from dynamical situations of influence, it has to introduce the constraints  $w_i \geq 0$  for all  $i$  and  $\sum_i w_i x_{(i)}$ . The constraint might be binding and return zero coefficients. But if this is the case, it remains to think whether it would make sense or not to build an alternative GOWA model where negative coefficients in the weigh vectors are allowed.

<sup>28</sup>Not by PEWA. See Appendix 3.

<sup>29</sup>With GOWA agents don't play sequentially, but another question to examine is whether the dynamics of our model are robust to agents playing sequentially, and in any order.

or not. This is particularly true under bounded-rationality. Agents may not be able to know what is rational, for example because they lack information or if they don't have the time or the capacity to compute the probabilities that this or that action is the good one. They may play according to rules of the thumb like counting how many people said 'yes' rather than computing bayesian probabilities<sup>30</sup>. In Banerjee's model, they could very well rely on their own signal if they know that wrong cascades can occur and don't have the time or the possibility to compute the probabilities for each signal to be the good one. In this case they are anti-conformists if, when they receive a signal that does not match the choice of the herd they follow or not with some probability. The dynamic of the society will depend on the proportion of anti-conformists. For example we saw that  $2^N$  is often a terminal class when there are more anti-conformist agents than conformist agents. In this case, in real situations, we can expect large fluctuations in the decision process. Since a consensus is embodied by a terminal state, a consensus may never be reached and it is possible that cascades never occur.

We did not extend much on the meaning of allowing not pure agents, neither did we provide an extensive analysis of it. Examining societies with not pure agents, however, is definitely the direction that should take a future research on anti-conformism. We have given however two important results. The first one states that when at least one agent is not pure, then the terminal classes are always booleans (Proposition 8 in the homogeneous framework and Proposition 12 in the heterogeneous framework). The second one states that among the terminal classes possible in the case where everyone is pure (as listed in Theorem 1), only two of them can also be terminal classes when some agents are not pure (Proposition 13):  $[\emptyset, N^a]$  and  $[N^c, N]$ , that is, booleans of anti-conformists in which the conformists either say 'yes' or 'no' altogether (and therefore must be pure). This means that under some conditions, whether we impose or not that anti-conformists are pure does not change the terminal classes. Put it differently, we only care of whether conformists are pure altogether. But assuming that an agent is not pure, the sign of its  $\alpha$  plays no role to determine the terminal class(es). In other words the anti-conformists... can very well be not pure conformists; it won't change the terminal classes<sup>31</sup>. In this sense, *anti-conformism and impurity are similar notions*. In some particular equilibria, being anti-conformist or tossing a coin is the same thing. This link is the reason why it was really interesting in our definition of GOWA to introduce not pure agents, rather than allowing only either  $\alpha = 1$  or  $-1$  (if we had only these values, we would not be able to interpret anti-conformism as a mixed strategy). The interest was not only to introduce a measure of probability that quantifies the transitions between states. We discovered during this research that this tool is strongly connected with anti-conformism and this is one direction that this research could now take.

A future research will have to examine rigorously how to recast the context of trembling into the context of mixed strategies (where an agent would choose  $\alpha = 1$  or  $-1$  with a certain probability) in order to offer a new perspective to anti-conformism. We shall prove a result not presented here: even by allowing some agents to be not pure, booleans of only conformist agents never occur; if a boolean is a terminal class, then there must exist a state where some anti-conformists say

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<sup>30</sup>Chandrasekhar et al. in "Testing models of social learning on networks: evidence from a lab experiment in the field" show in laboratory that people tend to behave according to the DeGroot model than to Bayesian updating. This is consistent also with Anderson and Holt "Information cascades in the Laboratory" in which the authors show that counting is the most salient bias to explain departure from Bayesian updating

<sup>31</sup>Of course in this later case the terminal classes cannot be  $[\emptyset, N^a]$  or  $[N^c, N]$ , but it would be  $[\emptyset, M]$  and  $[P^c, N]$ , where  $P^c$  is the set of pure conformists and  $M$  the set of not pure agents+the anti-conformists

'yes' (therefore there exists also a state where they all say 'yes'; in other words this context of uncertainty and mixed strategies accomodates suddern peaks in agents choices). A future work shall study the strong link between anti-conformism and not-purity, that is, in another framework, between anti-conformism and mixed strategies. It shall also examine whether allowing some correlation in the choices of agents (by modifying a bit the GOWA function) can accentuate the booms and bursts in macroeconomic and financial situations.

This work carries a strong message: despite the apparent symmetry between the definition of conformism and anti-conformism in the definition of GOWA, *anti-conformism is not just the contrary of conformism; anti-conformists and conformists have different behaviours in nature*. The underlying reason for this is that agents using the GOWA aggregation function are "blind" in the sense that they don't look at the type of people saying 'yes' or 'no': what matters for them is only the number of people saying 'yes'. In particular anti-conformists weigh negatively their own decision. There exists other aggregation functions that overstep this issue, like PEWA presented briefly in Appendix 3. The properties of this aggregation function shall be examined too in a future work.

## 1 Appendix 1. On Lemma 3.bis

### 1.1 Proof of Lemma 3.bis

**Statement:** If  $w'_i = w_{n-i+1} \forall i \in [1, n]$ , then  $\Lambda^{q'} = \Lambda^q \Sigma$ .

*Proof.* We decompose  $T = T^a \cup T^c$  where  $t^p := |T^p|$  for  $p = a$  or  $c$ . Let:

- $w^a := (w_1^a, \dots, w_n^a)$  and  $w^c := (w_1^c, \dots, w_n^c)$
- $w'^a := (w_n^a, \dots, w_1^a)$  and  $w'^c := (w_n^c, \dots, w_1^c)$
- $q := (w^a, w^c, n^a, n^c)$  and  $q' := (w'^a, w'^c, n^a, n^c)$

We treat the case  $s \geq 1$ ; the case  $s = 0$  can be checked aside. The cardinality of  $\tilde{S} := N \setminus S$  is  $s' = n - s$ . For all  $S, T \subseteq 2^N$  we have:

$$p_{\alpha^p}^q = \frac{1}{2} \left( 1 + \alpha^p \left( \sum_{i=1}^s w_i - \sum_{i=s+1}^n w_i \right) \right) = \frac{1}{2} \left( 1 + \alpha^p \left( \underbrace{2 \sum_{i=1}^s w_i^p - 1}_{:=K_s} \right) \right)$$

for the overscript of  $\alpha$  being  $p = a$  or  $c$ . We write  $\alpha = \alpha_c$ . Then:

$$\begin{aligned} \lambda_{S,T}^q &= \left( \frac{1}{2} (1 + \alpha^c K_s^{w^c}) \right)^{t^c} \left( 1 - \left( \frac{1}{2} (1 + \alpha^c K_s^{w^c}) \right) \right)^{n^c - t^c} \\ &\quad \times \left( \frac{1}{2} (1 + \alpha^a K_s^{w^a}) \right)^{t^a} \left( 1 - \left( \frac{1}{2} (1 + \alpha^a K_s^{w^a}) \right) \right)^{n^a - t^a} . \\ &= \left( \frac{1}{2} (1 + \alpha^c K_s^{w^c}) \right)^{t^c} \left( \frac{1}{2} (1 - \alpha^c K_s^{w^c}) \right)^{n^c - t^c} \left( \frac{1}{2} (1 + \alpha^a K_s^{w^a}) \right)^{t^a} \left( \frac{1}{2} (1 - \alpha^a K_s^{w^a}) \right)^{n^a - t^a} \end{aligned}$$

We redo the computations with  $q'$ ,  $\tilde{T} := \tilde{T}^a \cup \tilde{T}^c$  and  $t^p := |\tilde{T}^p|$ .

$$\begin{aligned} p_{\alpha^p}^{q'} &= \frac{1}{2} \left( 1 + \alpha^p \left( \sum_{i=1}^{s'} w_i'^p - \sum_{i=s'+1}^n w_i'^p \right) \right) = \frac{1}{2} \left( 1 + \alpha^p \left( 1 - 2 \sum_{i=s'+1}^n w_i'^p \right) \right) \\ &= \frac{1}{2} \left( 1 + \alpha^p \left( 1 - 2 \sum_{i=n-s+1}^n w_{n-i+1}^p \right) \right) = \frac{1}{2} \left( 1 - \alpha^p \left( \underbrace{2 \sum_{i=1}^s w_i^p - 1}_{K_s^{w^p}} \right) \right) \end{aligned}$$

for  $p = a$  and  $c$ .

We get with similar computations as before:

$$\begin{aligned} \lambda_{\tilde{S}, \tilde{T}}^{q'} &= \left( \frac{1}{2} (1 - \alpha^c K_s^{w^c}) \right)^{t^c} \left( 1 - \left( \frac{1}{2} (1 + \alpha^c K_s^{w^c}) \right) \right)^{n^c - t^c} \\ &\quad \times \left( \frac{1}{2} (1 - \alpha^a K_s^{w^a}) \right)^{t^a} \left( 1 - \left( \frac{1}{2} (1 + \alpha^a K_s^{w^a}) \right) \right)^{n^a - t^a} \\ &= \left( \frac{1}{2} (1 - \alpha^c K_s^{w^c}) \right)^{t^c} \left( \frac{1}{2} (1 + \alpha^c K_s^{w^c}) \right)^{n^c - t^c} \left( \frac{1}{2} (1 - \alpha^a K_s^{w^a}) \right)^{t^a} \left( \frac{1}{2} (1 + \alpha^a K_s^{w^a}) \right)^{n^a - t^a} \end{aligned}$$

Since  $t^c = n^c - t^c$  and  $t^a = n^a - t^a$ , we get  $\lambda_{\tilde{S}, \tilde{T}}^{q'} = \lambda_{\tilde{S}, \tilde{T}}^{q'}$ . This can be written in a matricial form by multiplying  $\Lambda^q$  on the right by a permutation matrix which switches each state with its complementary; such a permutation matrix is indeed  $\Sigma$ . □

*Variant proof:* We can also use the following shortcuts. This is what we used implicitly at the cost of a slightly longer proof in order to bypass heavier notations.

If  $K_s^w$  is the expression  $2 \sum_{i=1}^s w_i - 1$ , and if  $w_k = w'_{n-k}$  for all  $k \in [0, n]$  then  $K_s^w = -K_{s'}^{w'}$ . Indeed,  $K_s^w = 2 \sum_{i=1}^s w_i - 1$  and  $K_{s'}^{w'} = 2 \sum_{i=1}^{s'} w'_i - 1 = \sum_{i=1}^{n-s} w'_i - \sum_{i=n-s+1}^n w'_i = -K_s^w$ .

## 1.2 Discussion on the converse

We define the operation  $\nabla$  on a weigh vector  $w$  as follows:  $w^\nabla$  is such that  $w_i^\nabla = w_{n-i+1}$ . To make notations lighter, let's denote  $\Lambda^q$  by  $\Lambda$  and  $\Lambda^{q'}$  by  $\Lambda'$ . Lemma 3.bis is true if  $\Lambda' = \Lambda \Sigma$  implies  $w' = w^\nabla$ . We conjecture that the converse is wrong in general but that it is possible to design conditions under which the converse is true.

First we can notice that in Lemma 3.bis, we don't precise the value of the matrix  $\Lambda$ . If the value of this matrix is given, we conjecture that we cannot recover the agent's inclinations if we don't know the coefficients of  $w$ . If if know them, then we can. To prove (or invalid) this conjecture, we can take inspiration from the work by Grabisch and Rusinowska in "*Iterating influence between*

*players in a social network*" (Proposition 1); we can conjecture that this proposition still holds with anti-conformism and it will be probably an important lemma. Now we need something a bit different: can we reconstitute the coefficients of  $w$  from the graphs of transition ? The problem of the converse of Lemma 3, though seemingly different, is probably related and maybe equivalent. We want to know if from  $\Lambda' = \Lambda\Sigma$  we can recover a unique  $w$  such that  $w' = w^\nabla$ . It might be the case that this problem is equivalent to the possibility of recovering the weigh vectors from the graphs of transitions of a GOWA process (with the probabilities associated).

We can easily show that the converse of Lemma 3.bis holds if and only if a set of  $n - 1$  polynomial expressions hold in all cases: **(1)** A polynomial expression in  $w_1$  equals *the same polynomial expression* in  $w'_n$  **(2)** A polynomial expression in  $w_1, w_2$  equals the same one in  $w'_n, w'_{n-1}, \dots$ , **(n-1)** A polynomial expression in  $w_1, w_2, \dots, w_{n-1}$  equals the same one in  $w'_n, w'_{n-1}, \dots, w'_2$ . We must have in addition to this  $\sum_i w_i = \sum_i w'_{n-i+1} = 1$  (C) with all the coefficients being positive. At first glance we could think that it is possible to define recursively  $w$  and  $w'$  such that there exists at least one  $i \in [1, n - 1]$  such that  $w_i \neq w'_{n-i+1}$ . This is not so easy however; this depends on the shape (injectivity or not on  $[0,1]$ ) of all the polynomes involved.

We conjecture that if the reverse of Lemma 3.bis does not hold, then the set of all couples of possible weigh vectors  $w$  and  $w'$  on the simplex of  $\mathbb{R}^n$  define closed, continuous and convex sets (two sets, or even more if one of the polynoms involved have more than one inflexion point and if condition (C) can still be respected). In other words, if it is not possible to recover a unique  $w$  from a GOWA process, then the weigh vectors  $w$  and  $w'$  that verify  $\Lambda' = \Lambda\Sigma$  are either equal (this is always possible since it is the sufficient condition of Lemma 3.bis) or generically different from one another<sup>32</sup>: there is no continuum of "slightly different weigh vectors  $w$  and  $w^\nabla$ " such that the equality holds). From  $\Lambda' = \Lambda\Sigma$ , we can always choose  $w = w'$  (this is Lemma 3.bis). But if the converse of Lemma 3.bis is wrong, then we can choose  $w$  in a closed, convex and continuous set and have several possibilities for  $w'$ , in a finite number. One of the possibilities is of course  $w' = w^\nabla$ . The number of other possibilities depends in particular, but not only, on the number of inflexion points on subsets of  $[0,1]$  of the polynoms involved. We conjecture that under some conditions on  $w$  (but not on  $\alpha^a$  and  $\alpha^c$ ), we can ensure that the only possibility will be  $w' = w^\nabla$ .

## 2 Appendix 2. An exemple of survey

Hello, i am doing a statistical and anonymous survey on the choice of surnames by the parents. The questions below are the only ones i am interested in but you can, instead of answering the questions or as a complement, explain me your choice in a few lines. All your answers will be treated anonymously and at a statistical level. Thank you very much for answering this survey.

What is your child's surname ?

How old is your child ?

Do you think that this surname was rather (very) popular or rather (very) rare when you decided to choose this surname ?

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<sup>32</sup>Strictly different from one another if we change slightly  $\alpha^a$  or  $\alpha^c$ .

Was it on purpose ? (yes or no)

Do you think that this surname is today (or starts being) more or less popular than at the moment of your choice ?

Before choosing this surname, have you met or seen (even if they did not know you) parents who gave their child the same surname ? (yes or no)

If yes (and even if you did not intend to imitate them), how old was their child at that moment ?

Was this surname a culturel, religious or family tradition or tribute ? (yes or no, no more precisions)

Have you voluntarily chosen the surname of a famous person or a fictious character ? (yes or no)

### 3 Appendix 3. A brief presentation of PEWA

GOWA is a "blind" aggregator: agents weigh the conformists and the anti-conformists in the same way. Only the number of people saying 'yes' matters for them. The aggregator PEWA hereafter changes this perspective. In this new framework, agents weigh differently conformists and anti-conformists. Conformist agents take the opposite of what anti-conformist do in order to deduce what action is conformist from what anti-conformists do. Anticonformist weigh positively the choices of the other anti-conformist agents and negatively the choices of conformist agents. The symmetry between the two groups is perfect. For this reason it would be abusive to call it conformism or anti-conformism. Rather, we should call it "homophily". More generally, it is very likely that any kind of non-homophile anti-conformism carries some asymmetry. Of course, all the analysis done with GOWA would worth being done with PEWA.

In the definition below, the minuscule letter  $a$  (resp.  $c$ ) designate the weigh vector of the anti-conformists (resp the conformists). The capital letter  $A$  (resp.  $C$ ) designates the weigh given by an agent of a group to choices of the anti-conformists (resp. the conformists).

**Definitions** (simple PEWA) : We say that a  $n$ -place aggregation function  $P$  is a simple personalized weighted average  $S$ =SPEWA with two weigh vectors  $w^a = (w_A^a, w_C^a)$  and  $w^c = (w_A^c, w_C^c)$ , i.e  $0 \leq w_A^a, w_C^a, w_A^c, w_C^c \leq 1$ ,  $w_A^a + w_C^a = 1$  and  $w_A^c + w_C^c = 1$  if :

$$P^a(x) = \alpha^a \left( \frac{w_C^a}{n^c} \sum_{i \in N^c} x_i - \frac{w_A^a}{n^a} \sum_{j \in N^a} x_j \right)$$

$$P^c(x) = \alpha^c \left( \frac{w_C^c}{n^c} \sum_{i \in N^c} x_i - \frac{w_A^c}{n^a} \sum_{j \in N^a} x_j \right)$$

for all  $x \in \{-1,1\}^n$ .  $\alpha \in [-1,1]$  is the coefficient of conformism of the agent. If  $\alpha > 0$ , the agent is said to be *conformist* and we write it  $\alpha^c$ . If  $\alpha < 0$ , the agent is said to be *anti-conformist* and



we write it  $\alpha^c$ .

$\alpha = 1$  and  $\alpha = -1$  are the *pure degrees of conformism*. An agent with a pure degree of (anti)-conformism is *pure*.

Let  $p_i(1_S) = \frac{1}{2}(1 + P(1_S))$  be the probability for agent  $i$  to say 'yes' at the next state.

We can represent our aggregation model by a time-homogeneous Markov chain with transition matrix of dimension  $2^N \times 2^N$ :  $\Lambda = (\lambda_{S,T})_{S,T \in \mathcal{N}}$ . By definition of PEWA, the probabilities of saying "yes" are independent among agents and only depend on the current state:

$$\lambda_{S,T} = \prod_{i \in T} p_i(1_S) \prod_{i \notin T} (1 - p_i(1_S)).$$

In other words, agents weigh differently conformists and anti-conformists. Conformist agents take the opposite of what anti-conformists do in order to deduce what action is conformist from what anti-conformists do. Anticonformist weigh positively the choices of the other anti-conformist agents and negatively the choices of conformist agents.

### Examples:

- If the weigh vector  $w^c$  for conformists is such that  $w_A^c = w_C^c = \frac{1}{2}$ , the weigh vector  $w^a$  for anti-conformists is such that  $w_A^a = \frac{2}{3}$  and  $w_C^a = \frac{1}{3}$ , if  $n^a = 2$  and  $n^c = 6$  and if the two anti-conformists and two conformists say 'yes' and four conformists say 'no', then  $P(x)$  for pure conformists is:

$$S_c(x) = 1 \times \left( \frac{1/2}{6} \times (2 - 4) - \frac{1/2}{2} \times 2 \right) = -\frac{2}{3}$$

which gives  $p_c(x) = \frac{1}{6}$ .

$P(x)$  for pure anti-conformists is:

$$S_a(x) = -1 \times \left( \frac{1/3}{6} \times (2 - 4) - \frac{2/3}{2} \times 2 \right) = \frac{7}{9}$$

which gives  $p_a(x) = \frac{8}{9}$ .

- $n^a = 5$  and  $n^c = 5$ ,  $w_C^c = w_A^c = w_C^a = w_A^a = \frac{1}{2}$ , all agents are pure and at state  $S$  all conformists say 'yes' and all anti-conformists say 'no', we have:

$$S_c(x) = 1 \times \left( \frac{1/2}{5} \times 5 - \frac{1/2}{5} \times (-5) \right) = 1$$

so that  $p_c(x) = 1$

$$S_a(x) = -1 \times \left( \frac{1/2}{5} \times 5 - \frac{1/2}{5} \times (-5) \right) = -1$$

so that  $p_a(x) = 0$

- If  $n^a = 2$  and  $n^c = 4$ ,  $w_C^c = w_C^a = \frac{2}{3}$  and  $w_A^c = w_A^a = \frac{1}{3}$  for  $p = a$  and  $c$ , one anti-conformist says 'yes' and the other says 'no', and one conformist says 'no' and the three other conformists say 'yes', then :

$$S_c(x) = 1 \times \left( \frac{2/3}{4} \times (3 - 1) - \frac{1/3}{2} (1 - 1) \right) = \frac{1}{3}$$

so that  $p_c(x) = \frac{2}{3}$ . For the anti-conformists:

$$S_a(x) = -1 \left( \frac{2/3}{4} \times (3-1) - \frac{1/3}{2}(1-1) \right) = -\frac{1}{3}$$

so that  $p_c(x) = \frac{1}{3}$

- The cycle  $\emptyset \rightarrow N^a \rightarrow N \rightarrow N^c$ , that we could not obtain with GOWA (Theorem 1) can be obtained with PEWA with  $w_A^a = w_A^c = 0$ ,  $w_C^a = w_C^c = 1$ . In this case, the conformists follow the anti-conformists and the anti-conformists run away from the conformists.

The generalized PEWA works in the same spirit, but the agents use weighted ordered averages within each group of conformism in their decision process.

**Definitions** (generalized PEWA<sup>33</sup>): We say that a  $n$ -place aggregation function  $P$  is a simple personalized weighted average  $P$ =PEWA with two weigh vectors  $w^p = (w_A^p, w_C^p)$  ( $p = a$  or  $c$ ), where  $w_A^p = (w_{A,1}^p, \dots, w_{A,n^a}^p)$  and  $w_C^p = (w_{C,1}^p, \dots, w_{C,n^c}^p)$ ,  $0 \leq w_{q,i}^p \leq 1$  ( $p, q = a$  or  $c$ ) for  $i = 1, \dots, n^a, n^a + 1, \dots, n$ ,  $\sum_{i \in N} w_i^p = 1$  if:

$$P^a(x) = \alpha^a \left( \sum_{i \in N^c} w_{C,i}^a x_{C,(i)} - \sum_{j \in N^a} w_{A,i}^a x_{A,(i)} \right),$$

$$P^c(x) = \alpha^c \left( \sum_{i \in N^c} w_{C,i}^c x_{C,(i)} - \sum_{j \in N^a} w_{A,i}^c x_{A,(i)} \right),$$

for all  $x = (x_A, x_C) \in \{-1, 1\}^n$  the inclinations of the anti-conformist and conformist agents, and:

- $\alpha \in [-1, 1]$  is the coefficient of conformism of the agent.
- $x_{A,(1)} \geq \dots \geq x_{A,(n^a)}$  and  $x_{C,(1)} \geq \dots \geq x_{C,(n^c)}$  are the ordered components of  $x_A$  and  $x_C$ .

If  $\alpha > 0$ , the agent is said to be *conformist*; in this case we note  $p = c$ . If  $\alpha < 0$ , the agent is said to be *anti-conformist*; in this case we note  $p = a$ .

$\alpha = 1$  and  $\alpha = -1$  are the *pure degrees of conformism*. An agent with a pure degree of (anti)-conformism is called *pure*.

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<sup>33</sup>We refer to it as "PEWA".

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