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Geography matters: A model of production, pollution and health over Space and Time

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Keywords: Environmental Economics, Spatial dynamics, Pollution, Geography, Sustainable development, Optimal control, Spatial Economic Growth, Partial Differential Equations

MASTER'S THESIS

Geography matters: A model of production, pollution and health over Space and Time.

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Abstract

More and more contributions in Environmental Economics put forward the pressing necessity of incorporating realistic representation of ecological systems to adequately tackle the sustainable development question. In this regard, it seems primordial to extend dynamic optimization models into spatial settings, as the spatial dependency is embedded in a vast array of economic and environmental problems. However, standard macroeconomics models have for long restricted their attention to time. Recently some authors have tried to extend economic models to incorporate a spatial dimension in order to describe in a more realistic manner the world's economic and environmental problems. Although encouraging, these attempts have been restricted for the main part to an abstract, essentially featureless, approach of space in one dimension, without accounting for natural and geographical factors.

The aim of this master's thesis project is thus to build on the recent theoretical developments made in the spatial growth literature to study dynamic pollution management in two-dimensional stylized geographical settings, with appropriate and realistic descriptions of economic and environmental interacting systems. Focusing on the impact on health of pollution from industrial sources, we will take into account its transport and diffusion through air, soil or water to study the role of geography in the process of sustainable development and to explore the design of spatially dependent regulation in an optimal control problem. This work is thought as a combination of theoretical developments, numerical treatment and empirical applications, solving the model in typified illustrative examples as well as in the context of real world regions with polluting activities.

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1 Introduction

The relationship between economic growth and environmental pollution has been extensively studied in the academic literature and remains a pressing issue. While it is recognized that economic activities are an important determinant of environmental degradation, the design of policy interventions to reduce the accumulation of pollution and ensure, if possible, a sustainable economic development requires careful investigation. In this regard, recent contributions in environmental economics emphasize the crucial importance of incorporating realistic representation of ecological systems if one wants to properly account for the interconnection linking the economy and the environment. However, the standard macroeconomics models that have been used in the economic literature to study economic and environmental dynamics have restricted their attention to the temporal dimension, although the spatial dependency is embedded in a vast array of economic and environmental problems. This raises the need to extend dynamic optimization models into spatial settings, in order to describe in a more realistic manner the world's economic and environmental problems, including questions related to the geographic patterns of pollution exposure and their evolution through time, yet unaddressed. This approach is particularly relevant for a variety of environmental issues in which the transport phenomena are critically important, among which pollution management, because, in accordance with its physical characteristics, the stock of pollution in a specific location is strongly affected by the emissions generated in others. Although it spares several mathematical challenges, overlooking these transport effects amounts to ignoring the impact of one's actions on all agents located in different sites, thus potentially leading to inaccurate prospects for sustainability. The consideration of geographical aspects then yields a renewed and more complex picture of the sustainable development problem, allowing to reduce the regulatory inefficiencies that arise from ignoring the spatial interactions of production and pollution in the design of optimal policies.

Numerous contributions pertaining to the New Economic Geography literature have been dedicated to study the drivers of the organisation of economic activities across space and its consequences. More recently, a strand of literature on spatial economic growth models has additionally introduced capital accumulation and analyzed the effects of spatial diffusion (of production factors or technology) on economic growth in a continuous space structure (see among others Brito (2004), Zou and Camacho (2004), Camacho et al. (2008), Brock and Xepapadeas (2008), Capasso et al. (2010),Boucekkine et al. (2009, 2013a, 2018),Fabbri (2016)). Concurrently, some attempts have been made to bring together the macroeconomic geography literature and the sustainable development debate by examining the role of the spatial dimension in environmental and resource economics problems specifically. Notable examples can be found in Brock and Taylor (2010); Brock and Xepapadeas (2010); Brock et al. (2014), Camacho and Pérez-Barahona (2015), Desmet and Rossi-Hansberg (2015), La Torre et al. (2015), Marsiglio et al. (2019) or Augeraud-Véron et al. (2017, 2019b, 2021). The common characteristic of these problems with spatio-temporal dynamics is that the state equations are described by partial differential equations (PDEs). Accordingly, these contributions converge to show that technical difficulties quickly arise when considering an optimization problem over space and time (see Boucekkine et al. (2013b) for instance, discussing the emergence of ill-posed problems in this type of setting, and Augeraud-Véron et al. (2019a) for a review of new mathematical research lines associated to the specificities of spatially structured optimal control problems in Environmental Economics). It is undoubtedly for these reasons that most contributions consider a simplified and unique spatial dimension. This representation remains essentially abstract, taking away the relevance and the meaningfulness of the non-homogeneous and patterned nature of geography, in particular that of the natural environments in which economic systems are embedded. Therefore, this dissertation aims at contributing to these growing strands of literature by extending existing approaches to propose a benchmark framework to study a spatial economy with pollution transport in continuous time and two-dimensional space, with spatially variable

geographical and economic elements. More specifically, we will look into the role of geography in the process of sustainable development in a context of point-source production-induced pollution where environmental quality feeds back on economic opportunities through the health of the labor force.

The consideration of the labor production factor in spatial economic growth model is in reality extremely rare in comparison to physical capital. An interesting exception can be found in Marsiglio et al. (2019), in which the authors consider a simple macroeconomic model with a spatial dimension and show that neglecting the mutual relation of population and pollution on one hand, and overlooking spatial externalities on the other, together leads to overly optimistic prospects for sustainability. Although we will be interested in similar topics, the latter contribution is not the most closely related to ours. Our model partly builds on La Torre et al. (2015), which analyzes the spatio-temporal dynamics of pollution and capital in an economic growth model with economic and abatement activities, financed through environmental taxation. The authors develop and study both a Solow-type framework and a Ramsey-type problem. The analytical tools that will be used in the following however brings us closer to Boucekkine et al. (2009), which lays the foundations of the Spatial Ramsey model, and subsequently, Camacho and Pérez-Barahona (2015). The latter studies an optimal land use problem with environmental degradation as a spatial, local and global, externality, putting forward the emergence of relevant spatial patterns even in one dimension and the role of the abatement technology for the existence of a sustainable equilibrium. Similarly, Augeraud-Véron et al. (2017, 2019b, 2021) analyze the control of groundwater pollution from agricultural activities. Their model unfolds in a very comprehensive hydrogeological state equations system in 3D accounting in a rigorous and detailed way for soil heterogeneity. Their treatment is essentially analytical and more complex than the scope of this dissertation, but we share the underlying goal to highlight the sensitivity of this type of problems to geographical elements. Their contributions also has the originality to account for both advection and diffusion, while almost every related work only consider diffusion in the pollution transport phenomenon, as we will come back to when presenting our model, featuring both types of transport mechanisms.

Solution methods and tools for the qualitative analysis of the type of models developed in the following mostly rely on the application of the classical Pontryagin maximum principle to derive necessary conditions, and on mathematical results from the literature on the control of partial differential equations (one can again refer to Augeraud-Véron et al. (2019a) and Brock et al. (2014) for a presentation of relevant tools). The procedure generally leads to systems that cannot be solved analytically, except by making restrictive functional form assumptions. Thus, to illustrate the richness of the model and to consider applications to realistic geographical settings, we will rely on numerical simulations. Accurate and stable algorithms were developed in the course of this master's thesis project to solve and simulate the PDEs and the optimal control problems at hand in the most insightful and reliable way.
The objectives of studying the spatio-temporal dynamics of production, "healthy" human capital and pollution in such economic growth model are plural. Broadly, the underlying question is to determine in which cases the society we are modelling is able to deal with its pollution problem, given the significance of each source of spatial heterogeneity and the design of policy interventions Policy implications for territorial development and land settlement considerations are to be explored as well, in terms of the relative arrangement of agglomerations and industrial centers, for instance.

The present dissertation is thus thought as a combination of theoretical analysis and computational methods, as well as empirical applications of the model to some real-word context, with their own demography and geography.

All in all, this work can therefore be thought as an illustration of why geography matters

for the sustainable development issue in environmental and resource economics, relying on a spatially structured economic model applicable to real geographies, and more generally, as an effort to develop tools to tailor socio-economic and environmental policies to the spatial context.

The dissertation is organized as follows. Section 2 presents the structure and the assumptions of the model. Section 3 characterizes the behavior of the model by means of numerical simulations, in a Solow-type framework. The role of the determinant factors of the model is explored and the application of the model in several realistic/real case studies is proposed. Section 4 then extends the setting to an optimal control problem and provides theoretical findings. Section 5 illustrates the spatio-temporal dynamics in the Ramsey-type model, picking up on the examples developed beforehand. Finally, Section 6 presents some concluding remarks. Proofs and technical details about the numerical scheme are provided in the appendix section.

2 A model of industrial pollution with spatial transport

This section presents the theoretical framework that was developed over the course of this research project. It builds on the developments made in the contemporaneous economic literature on spatial economic growth models in an effort to design a model that allows to represent a vast array of realistic problems and spatial configurations. In this regard, we will extend existing approaches to cases with two-dimensional spatial regions and spatially varying coefficients (for diffusivity, productivity, etc...). Except in the work of Augeraud-Véron et al. (2017, 2019a, 2021), these two aspects are usually discarded to avoid technical complications, but they are primordial to be able to represent meaningful spatial distribution patterns (for population, production and pollution) and to allow the application of the model to some real world contexts, which is also a further originality of our work in this literature.

Focusing on the impact on health of pollution from industrial sources, the objective is then to include into the economic model an explicit modeling of the spatio-temporal dynamics of the pollution transport process. Namely, as described in the following, we will use a spatial growth model in which production uses labor and generates revenues but also pollution, harming the human capital (i.e. health) and spreading over space, according to the hydrogeological properties of the environment at study and its spatial heterogeneity.

Consider a continuum of locations along a two-dimensional region $\mathcal{R} \subset \mathbb{R}^2$. We assume that $\mathcal{R}$ is a bounded and connected real set and is a quarrable domain of the xy-plane. For simplicity, we will assume that $\mathcal{R}$ is a rectangular region: $\mathcal{R} = [X_1, X_2] \times [Y_1, Y_2]$.
Each location has a given population density $h(x, y, t)$, where h is an integrable and strictly positive function such that $\iint_{\mathcal{R}} h(x, y, t)\, dx\, dy = 1$. Note that we allow h to be time-dependent for the sake of completeness, but we will not deal with population growth or migration in the present report and we leave these cases for future research.

There exists a unique consumption good which is produced using labor. The revenue from production is devoted to two different activities: consumption c and pollution abatement u. We denote by $F(\cdot)$ the production function, which is the same in each location and satisfies the following standard assumptions:

(A1) $F(\cdot)$ is non-negative, increasing, concave, its first and second derivatives exist and are non-negative and it verifies the Inada conditions, that is:
$F(\cdot) \in C^2, F(0) = 0, F'(\cdot) > 0, F''(\cdot) \leq 0, \lim_{s\to 0} F'(s) = +\infty, \lim_{s\to+\infty} F'(s) = 0.$

Local pollution comes directly from the production of the consumption good, which is distributed across the region $\mathcal{R}$ according to a given distribution $f(x, y, t)$, with $\iint_{\mathcal{R}} f(x, y, t)\, dx\, dy = 1$. The

inclusion of the functions h and f will allow to study the effect of spatial patterns in industrial and urban territorial organization in this type of model, in relation to their concentration for instance. In most of our simulations and because it is generally observed to be the case in reality, we will indeed think of the productive activity as clustered, that is, not uniformly distributed across space, which means that the model can feature persistent spatial heterogeneity as a consequence of the industrial distribution only. Making sense of the location of economic activity and, more specifically, of its agglomeration has been a popular strand of literature in Economic Geography, which has tried to rationalize the emergence of industrial clusters by cost savings and efficiency motives (Marshall, 1920) or economies of scale (Krugman, 1991). Note however that in this model we will consider industrial density as given and will not try to account for the potential forces behind existent patterns. Nevertheless we can point out that, while population density has received somewhat more attention, the spatial distribution of the productive activity has been ignored in almost every problem similar to ours in the literature[1]. It is generally assumed that each point in space can produce and does so with its own resources, which seems like an unlikely assumption, except in the specific case of an agricultural economy. In the present model, human capital enters the production function as an aggregate, where each point is weighted by population density, health and distance, taking into account that the location of the industry influences the size of the labor force it is able to attract but also the spatial distribution of pollution discharges (according to f). Typical situations to be represented in this model would be regions[2] where a major industrial sector can be identified, concentrating a significant part of revenues and employment within the area, and is known to be an important source of pollution through discharges or emissions.

We assume that each unit of production generates θ unit of pollution. This parameter thus can be thought as a measure of the environmental inefficiency of of the production process. Namely, we assume that pollution generation increases linearly with production at a constant rate θ. In turn, pollution feeds back on the economy by lowering the level of output through the effect on health (assimilated as a decrease in the human capital available for production or, in a close interpretation, as a decrease in labor productivity). Therefore, a local increase in output increases the wealth of the economy but also tends to rise the stock of pollution in the whole region because the contaminant travels across space.
The transport model we use, although it is inspired by the famous Gaussian plume atmospheric dispersion model, can encompass a variety of situations of air, water or soil pollution[3] ruled by a reaction–diffusion–advection equation. The latter describes the transport of a mass (or concentration) of pollutants particles being emitted punctually (in space and time) and transferred inside the physical system due to the combination of two processes: diffusion, driven by a quantity gradient (Fick's law), and advection, which refers to the movement of the material induced by the motion of the surrounding fluid (typically, a river flow, current, wind). Since the first contributions to the spatial growth literature (Brito (2004),Zou and Camacho (2004), Brock and Xepapadeas (2008, 2010)), such directional advective effects have almost always been discarded to focus on diffusion, which may be suited for economic growth model with capital mobility, migrations or technological diffusion, but less so for pollution models, as the transport of pollution in a real environment usually results from a combination of diffusion and advection.

[1] To my best knowledge, the only exception are Boucekkine et al. (2018) where spatial distributions for technology and population are introduced, and the inclusion of a static "proximity" externality with close interpretation in La Torre et al. (2015) and related contributions Aniţa et al. (2013, 2015), in the form of a kernel function.

[2] The exact spatial scale to be covered and its perimeter, whether it is determined by administrative boundaries, a whole water basin, or geographical specificities, is to be chosen accordingly to each problem's characteristics and depending on boundary conditions.

[3] The initial formulation of the problem essentially refers to point-source pollution, because we consider the productive activity to be concentrated in the form of one or several industrial centers, as it is usually observed in reality. However, an application to agricultural non-point source pollutants could be envisioned in future research.

Hence, the dynamics of pollution at location (x, y) and at time t, $p(x, y, t)$, are given by the following second-order partial differential equation (PDE) of parabolic type:

$$\begin{aligned} p_t &= \theta F(H) f(x,y,t) + \nabla \cdot (D \nabla p) - \nabla \cdot (Vp) - \delta p - G(u) \quad [4] \\ &= \theta F(H) f(x,y,t) + \frac{\partial}{\partial x}\big[D p_x\big] + \frac{\partial}{\partial y}\big[D p_y\big] - \frac{\partial}{\partial x}\big[v_1 p\big] - \frac{\partial}{\partial y}\big[v_2 p\big] - \delta p - G(u) \\ &= \theta F(H) f(x,y,t) + D(p_{xx} + p_{yy}) + D_x p_x + D_y p_y - (v_1 p_x + v_2 p_y) - p(v_{1x} + v_{2y}) - \delta p - G(u) \end{aligned} \tag{1}$$

where $\delta = \delta(x, y, t)$ is the rate of natural decay, through absorption or chemical reactions for example, and G describes the abatement technology, which is assumed to satisfy the same conditions as $F(\cdot)$, gathered in (A1), with, in particular, $G'(\cdot) > 0, G''(\cdot) \leq 0$.

$D = D(x, y, t)$ is a local diffusion coefficient and the term $\nabla \cdot (D \nabla p)$ corresponds to the diffusion process, inducing pollution to flow from highly polluted areas to low p areas, that is according to a quantity or concentration gradient, hence the presence of second order partial derivatives. The advective effect appears in the term $\nabla \cdot (Vp)$, where $V = \begin{pmatrix} v_1(x,y,t) \\ v_2(x,y,t) \end{pmatrix}$ contains the coefficients describing the velocity field that the quantity is moving with in each direction. These terms together provide a net balance of the evolution of the pollution stock, at the (x, y) point in space, as a result of in and out flows.
Notice we consider spatio-temporally dependent diffusion and velocity coefficients, which is rarely the case in the economic literature but crucial from a geophysical point of view to account for inhomogeneities in the physical medium, whether it is air, soil or water, and thus to provide more flexibility for the model to describe various real world contexts. There exist various examples in which spatially dependent coefficients would be useful to provide an appropriate description, such as: regions exhibiting pressure variations (relevant for the dispersion of greenhouse gases), soil heterogeneity, hence varying diffusion potential according to soil types, or a multiple-aquifer system with cell-specific interconnections. Variable velocity coefficients are necessary to represent a hydrogeological network or merely a water stream that do not span the entire region, but also to introduce temporal variations in climatic conditions for example, affecting wind and river flows. We will also exploit these coefficients to depict physical barriers to pollution transport like mountain ranges or artificial buffer zones.

Finally, the term $\theta F(H) f(x, y, t)$ in Equation 1 corresponds to the production externality and is a reaction variable in the sense that it is itself a function of current pollution because the latter impairs the human capital H available for production, hence impacting the current production and emission levels. Notice also that it is multiplied by the density function $f(\cdot)$ to account for the in-homogeneous distribution of production across space (with $\iint_{\mathcal{R}} \theta F(H) f(x, y, t)\, dx\, dy = \theta F(H)$, the total emissions).
Thus, in this setting, new pollution only originates from production while in reality, especially in contexts of rapid urban development, population can be seen as a driver of pollution in itself, which we may have included in an additional source term as a function of distribution h. We choose however to focus the present study on the pollution dynamics associated to the industrial activity, as the effect of households' behavior would be limited anyway since we have excluded cases with migration or other population dynamics and it is not obvious how to model the link between one individual's health and emissions when she is assumed to be spatially immobile.

The production function writes $F(H) = BH^{\alpha}$ where $B = B(x, y, t)$ is the total factor productivity at point (x, y, t) and H is the total human capital used for production in the single industry,

[4] The symbol $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}^T$ represents the spatial gradient of a scalar function $f : \mathcal{R} \to \mathbb{R}$ and the symbol $\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$ represents the divergence of a vector function $f = [f_1, f_2] : \mathcal{R} \to \mathbb{R}^2$.

defined as:

$$H(t) = \iint_{\mathcal{R}} \phi(p(x,y,t), h(x,y,t), x, y)\, dx\, dy \tag{2}$$

Pollution reduces the level of output through the damage function ϕ, that represents the decrease in the productivity of labor due to pollution exposure at the location of workers. In our setting, the dependence of ϕ on h,x and y in addition to p is meant to represent the unequal importance of each location in the firm's labor force, not only because of their varying health but also in relation to the population density, availability and potentially other characteristics. One could think for example of discounting the contribution of populations that are located far away from the firm, as healthy as they might be, because they are more tired or less willing to work due to commuting.
We further assume ϕ to be a smooth function, twice differentiable with respect to p, and characterized by:

$$\textbf{(A2)} \quad \begin{cases} \phi_p(\cdot) < 0, \phi_h(\cdot) > 0, \\ \phi(p,h,x,y) = h(x,y,t)L(x,y) \text{ when } p(x,y,t) = 0, \\ \lim\limits_{p \to +\infty} \phi(p,h,x,y) = 0. \end{cases}$$

where $L(x,y)$ can be included as a discount factor, inversely proportional to the distance of point (x,y) to the production area.

Two elements are worth commenting in this formulation. First, the impact of local pollution on local health is contemporaneous, which is a simplifying assumption that obviously fits some types of pollution better than others. Namely, the health capital at one point in space is not instantaneously affected by the pollution generated or accumulated in other points in space but it depends immediately of the pollution stock prevailing locally and contemporaneously; in other words, there is no distinction between the time of exposure and the time of damage. This means that, while we do account for delays in the pollution stock accumulation dynamics in an explicit and precise way in accordance to the transport process, we discard cases in which impacts on health only arise after a certain duration of exposure to pollution. The opposite would require associating an additional PDE for health spatio-temporal dynamics to the initial problem or including a delay in the model's equation, which yields mixed-type equations (functional differential equations) that are much more complex to deal with, as shown by the extensive delayed optimal control literature.
Second, we can notice that we deal with a PDE that is perturbed by an integral term, in the form of H as defined in Equation 2, which brings our problem and its treatment closer to La Torre et al. (2015), Aniţa et al. (2013, 2015) and Camacho and Pérez-Barahona (2015). This type of integral terms is generally presented as a "non-local" term in the sense that it linked the emission flows at each location to one another, outside of considerations related to the transport process. For instance, Camacho and Pérez-Barahona (2015) study the impact of global versus local pollution by having both aggregate and local pollution entering the damage function as arguments. In La Torre et al. (2015), production has a disproportionate direct impact on neighbouring spatial locations (a proximity, static externality) in addition to affecting future pollution for all locations through diffusion. Our setting however slightly differs because, as each location does not engaged in its own production as we have previously discussed, there is no local pollution externality per se. But the local stock of pollution does impact every location's pollution flows, firstly because pollution travels across space over time but also because the local health capital individually affects the labor force available for the whole region, which is aggregated in H, so that it impacts the production level, hence the amount of newly emitted pollution. This second channel is what is embodied by the integral term. In terms of derivations, the presence of the integral will mainly prevent us from obtaining closed-form analytical solutions in a linear

framework as in Boucekkine et al. (2013a, 2018, 2019), explicit solutions being generally not available without this simplified structure anyway.

We assume that the initial distribution $p(x, y, 0) = p_0(x, y)$ is a known continuous function satisfying

(A3) $p_0 \in L^\infty(\mathcal{R})$[5], $p_0(x, y) \geq 0$ *almost everywhere* in $\mathcal{R}$.

We also assume that there is no pollution flow through the boundary of $\mathcal{R}$, i.e. no pollution flow along the normal to the boundaries of the space, which describes the Neumann boundary conditions. Such "no-flux" boundaries are standard in the literature and used initially to describe perfect insulation in thermodynamics. This of course lacks realism and there is no real situation in which a region would have no pollution flowing in nor flowing out except by being surrounded by regions with the exact same pollution level at their borders at any time. However, making a priori assumptions on the level of pollution prevailing at boundaries (Cauchy or Dirichlet boundaries) would somehow contaminate the study of the region's problem as it would induce additional flows. Note also that we are free to choose a spatial region with a large enough extent around the industrial cluster so that the remote areas will generally be relatively unpolluted and there is no spurious pollution accumulation at borders. So we can say that Neumann boundary conditions seem like the most adequate and flexible choice in our setting. The specification of the boundary conditions of course affects the solution of the PDE and results are presented for Neumann boundaries in the following, but it would be possible to modify them to accommodate specific problems[6], by considering more complex boundary conditions or even dividing the boundaries into different segments, with different boundary conditions.

Finally, adding initial and boundary conditions, we can write the problem as:

$$\mathcal{P} \begin{cases} p_t = \theta B H(t)^\alpha f(x, y, t) + \frac{\partial}{\partial x}[D p_x] + \frac{\partial}{\partial y}[D p_y] - \frac{\partial}{\partial x}[v_1 p] - \frac{\partial}{\partial y}[v_2 p] - \delta p - G(u) \\ H(t) = \iint_{\mathcal{R}} \phi(p(x, y, t), h(x, y, t), x, y)\, dx\, dy \\ p(x, y, 0) = p_0(x, y) \qquad , (x, y) \in \mathbb{R}^2, \\ \lim\limits_{x \to \partial R_X} p_x = \lim\limits_{y \to \partial R_Y} p_y = 0 \qquad , \forall t \in [0, +\infty) \end{cases} \tag{3}$$

in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$.
In the last expressions, ∂R denotes the boundaries of the (square) region. In particular, $\partial R_X = \{(x, y) \in \mathcal{R} \text{ s.t. } x \in \{X_1, X_2\}\}$ and $\partial R_Y = \{(x, y) \in \mathcal{R} \text{ s.t. } y \in \{Y_1, Y_2\}\}$.

The standard PDE literature can be extended to equations with variable coefficients (see for instance de Frutos and Martín-Herrán (2019)) and nonlocal integral terms such as H (Aniţa and Capasso (2009),Veliov (2008)) to provide us with an existence result for problem $\mathcal{P}$.

If the set of previously stated hypotheses (A1), (A2) and (A3) holds, as well as the following additional assumptions:

- B and δ are smooth non-negative functions and they have finite upper-bounds,
- D is continuously differentiable and there exist two constants $0 < d_m \leq d_M$ such that $d_m \leq D(x, y, t) \leq d_M$ for all $(x, y) \in \mathcal{R}$,
- V is a smooth convective field and the mean velocity of the flow is bounded, in $L^\infty(\mathcal{R})$,
- $\phi \in L^\infty(\mathcal{R} \times \mathcal{R} \times (0, \infty))$ and $\phi(p, h, x, y) \geq 0$ a.e.$(x, y, t) \in \mathcal{R} \times \mathcal{R} \times (0, \infty)$,

[5] $L^\infty(\mathcal{R})$ denotes the Lebesgue ∞-space and consists of all measurable functions $f : \mathcal{R} \to \mathbb{R}$ that are "essentially bounded", i.e. there exists a positive number $0 < M < \infty$ such that $|f(x, y)| \leq M$ for almost every $(x, y) \in \mathcal{R}$, i.e. except on a set of measure zero.

[6] It is for example trickier to construct a "large enough" space structure in cases with a strong advective current dragging pollution away quickly in one direction.

then, for each choice of controls (c, u), there exists a unique global continuous, non-negative and bounded, solution $p^{c,u}$ to problem $\mathcal{P}$.
Additionnally, the conditions imposed on the production function, satisfied by standard production functions exhibiting diminishing returns, guarantee the existence of a nontrivial stable equilibrium and the convergence to this steady state, which would no longer be true with a production function that is not globally concave [7]. The steady state, for each given couple (c, u), is defined as a smooth time-invariant function $\overline{p}^{c,u} \in C^2(\mathcal{R})$ solving the following ordinary differential equation:

$$0 = \theta B(x,y)H^{\alpha}f(x,y) + \frac{\partial}{\partial x}\left[D(x,y)p_x(x,y)\right] + \frac{\partial}{\partial y}\left[D(x,y)p_y(x,y)\right] - \frac{\partial}{\partial x}\left[v_1(x,y)p(x,y)\right] - \frac{\partial}{\partial y}\left[v_2(x,y)p(x,y)\right] - \delta(x,y)p(x,y) - G(u(x,y)) \quad (4)$$

Moreover, a *nontrivial* equilibrium refers to a steady state that is not merely characterized by a generalized collapse of the economy in the long-term, due to ever growing pollution levels driving health, hence production, to zero everywhere in the region.

3 Spatio-temporal dynamics of pollution and health

We first analyze a "Solow-Type" model, assuming the amount of resources devoted to pollution abatement activities $u(x, y, t)$ is given exogenously and more specifically as a given fraction of total revenue, to ensure budget balance. We perform numerical simulations to visualize the long-run behavior of the solutions to P. This exercise essentially aims at demonstrating the model's flexibility and relevance in various contexts and at illustrating a range of important dynamics that would not emerge in a model without any space dimension, or even in models with one spatial dimension only, which will be more apparent when we move to realistic geographical settings at the end of this section.
To do so, because we cannot obtain an analytical solution, we use an adapted Crank-Nicolson finite difference method to provide a numerical approximation to the solution of the PDE. Details on this numerical scheme are provided in Appendix A. It is then applied to a spatial grid, representing a discretized version of the region at study, where each mesh contains data on local diffusivity, $D(x, y)$, velocity, $v_1(x, y), v_2(x, y)$, population density, $h(x, y)$, industrial density, $f(x, y)$, and initial pollution conditions $p_0(x, y)$. Throughout this section, although we will attempt to represent real geographies, the choice of values for the variables cited above and for the other parameters of the model will be elected primarily to provide clear and insightful illustrations. The diffusion and advection parameters in particular do not correspond to an actual physical calibration.

3.1 The benchmark case

In order to give a detailed picture of the transient and long-time behavior of the model, we will start by studying its application to a simple benchmark set-up with homogeneous diffusion, no advection, featuring one industrial area centered around a point (x_f, y_f) such that

$$f(x,y,t) = e^{-\frac{(x-x_f)^2+(y-y_f)^2}{2\sigma_f^2}} \quad \forall t,$$

[7] Capasso et al. (2010) for instance studies convergence in a similar, spatially structured, environment with a convex-concave "S-shaped" production function, which is thought to offer a better description of production processes in developing countries, and they find that it leads to the emergence of a saddle point behavior, resulting in potential poverty traps.

Table 1: Parameter values and functional assumptions for the benchmark case

$\mathcal{R} = [X_1, X_2] \times [Y_1, Y_2]$	$[-4, 4] \times [-4, 4]$	B	10
(x_f, y_f)	$(0, 0)$	α	0.65
(x_h, y_h)	$(1, -1)$	θ	0.02
$\overline{D}$	0.06	$u(x, y, t)$	0
p_l	0	δ	0.025
p_h	5		
$\phi(p(x, y, t), h(x, y, t)) = h(x, y, t)e^{-\gamma p(x,y,t)}$		γ	0.5
$f(x, y, t) = e^{-\frac{(x-x_f)^2+(y-y_f)^2}{2\sigma_f^2}}$		σ_f	0.3
$h(x, y, t) = e^{-\frac{(x-x_h)^2+(y-y_h)^2}{2\sigma_h^2}}$		σ_h	0.5

and, similarly, one residential area with a Gaussian distribution around a point (x_h, y_h), such

$$h(x, y, t) = e^{-\frac{(x-x_h)^2+(y-y_h)^2}{2\sigma_h^2}} \quad \forall t.$$

Of course, (x_f, y_f) and (x_h, y_h) do not necessarily coincide. Both industrial and population distributions are normalised to amount to $H(t) = 100$ over the region only to allow better readability. The region is taken to be large enough to avoid unrealistic boundary effects. The production function is a Cobb-Douglas production function $F(H) = BH^{\alpha}$ with labor share $\alpha = 0.65$, in consistence with the growth literature. Following the climate change economics literature, we select an exponential damage function: $\phi(p(x, y, t), h(x, y, t)) = h(x, y, t)e^{-\gamma p(x,y,t)}$. Again, the parameters of the previous functions as well as the diffusion coefficient $D(x, y, t) = \overline{D}$ $\forall (x, y, t)$ are chosen for convergence speed and visualization purposes. Parameter values and initial conditions for this section's numerical simulations are gathered in Table 1.

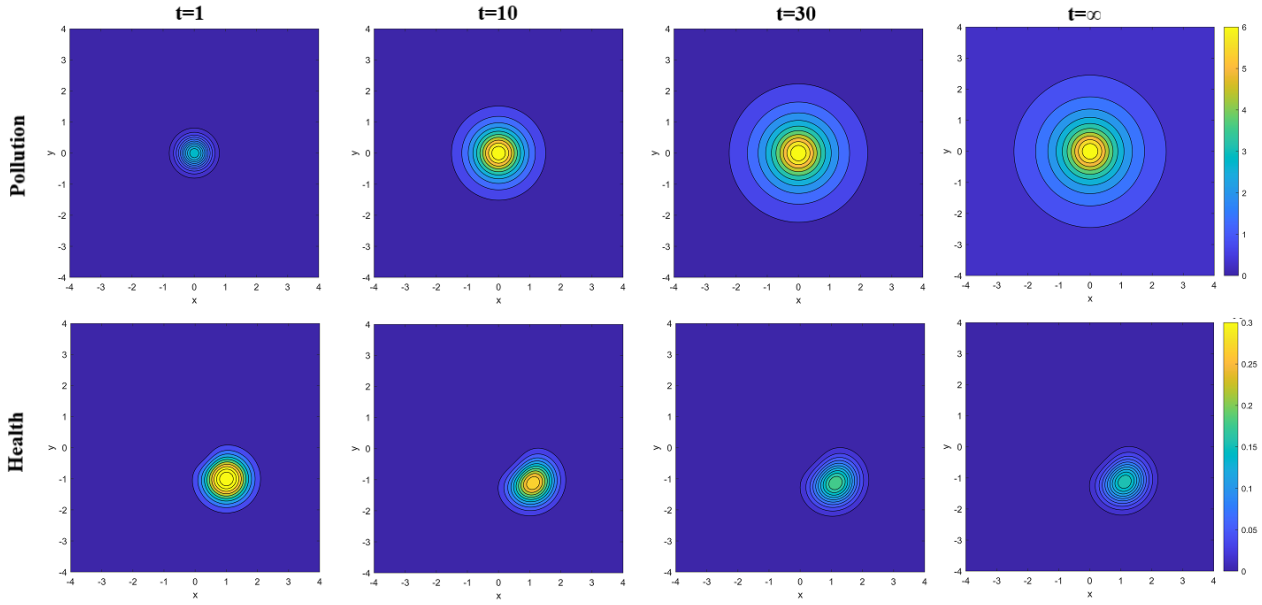
For the time being, consumption and abatement activities are exogenously given and we additionally assume that they are constant over time. We will present results for the case without any abatement activities, i.e. $u(x, y, t) = 0$, before looking at the role of this variable in isolation in a dedicated subsection.

We will consider several possibilities for the initial distribution of pollution: either low initial pollution, taking $p_0(x, y)_L = p_L = 0$ $\forall (x, y)$, for simplicity, or high initial pollution $p_0(x, y)_H$ with either homogeneous or non-homogeneous distribution. We will typically take $p_0(x, y)_H$ such that the initial health capital level $H(t = 0)$ is higher than the steady state level, to study convergence from above. We indeed recall that the model is built in such a way that, to some extent, the presence of pollution only matters if it is located in a populated area, thus the starting point in terms of health is the most determinant for the evolution of the system.

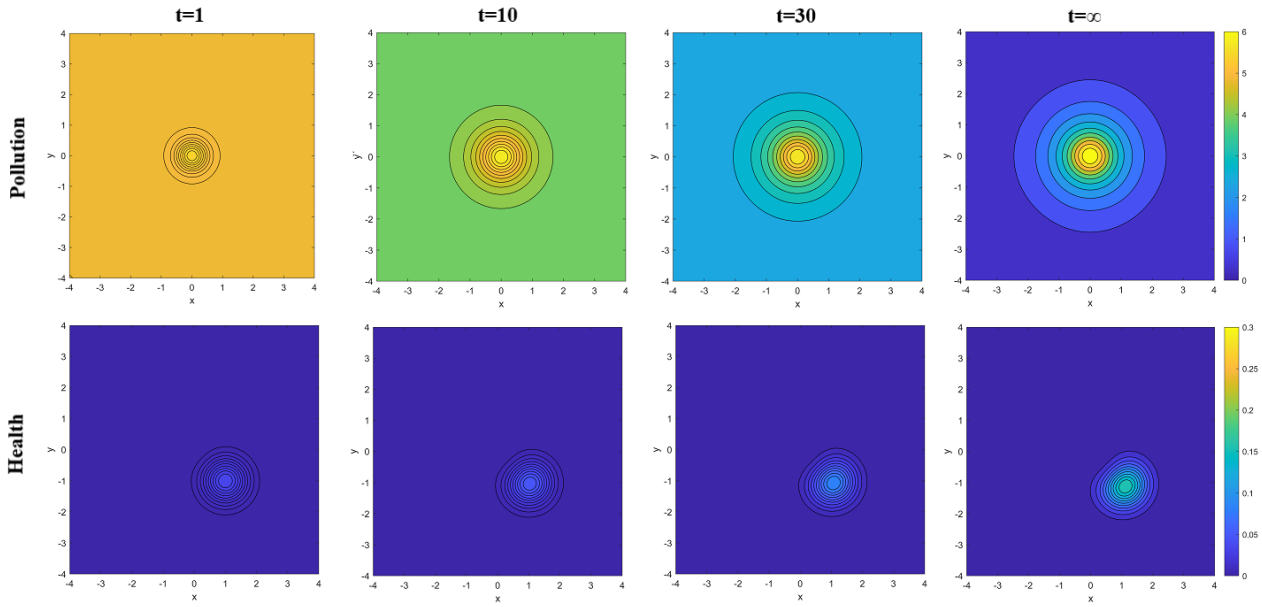
3.1.1 Evolution of pollution and health

The spatio-temporal evolution of pollution and health in the benchmark case is illustrated in Figure 1, for zero initial pollution and for widespread high initial pollution. Figure 2 shows the evolution of aggregate variables over time, that is, total health capital,$H(t)$, total production $F(H(t))$, and total pollution stock $P(t) = \sum_{\mathcal{R}} p(x, y, t)$.

Looking at the evolution of pollution in the upper panel of Figure 1, a first observation is that the spatial extent of the polluted area is bounded at all times, and the scope of the problem



(a) $p_0(x, y) = p_L = 0$



(b) $p_0(x, y) = p_H = 5$

Figure 1: Numerical solutions to $\mathcal{P}$ in the benchmark geographical setting with homogeneous diffusion (a) for zero initial pollution; (b) for high initial pollution.

therefore depends on the time horizon that one is willing to consider. This is actually a crucial feature of the type of problems studied in this paper, and these figures are helpful in showing how analyzing such problems in finite time, as it is often the case in the related literature, amounts to selecting which spatial territories are of interest, and which are ignored. One could argue however that significant changes tend to happen over a short time period after the starting point, while spatial evolution gets much slower as time elapses, and this relates to the driving force of the diffusion convergence mechanism: the speed (and the direction) of diffusion is proportional to the magnitude of the gradient, that is, the difference in pollution levels across space. The extent of the steady-state pollution stock, around the pollution source, is then determined

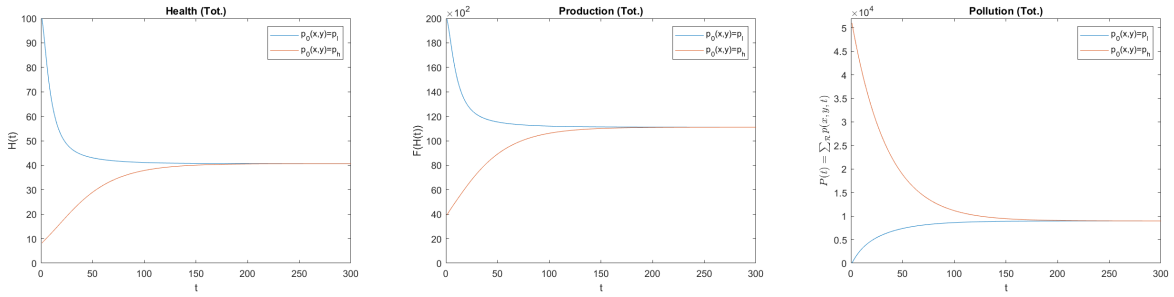


Figure 2: Evolution over time of aggregate healthy human capital, total production and total pollution stock in the benchmark case and for two levels of (homogeneous) initial pollution, high ($p_h = 5$) and low ($p_l = 0$).

by the relative size of the diffusivity coefficient D multiplying this gradient and the rate of natural decay δ, but also by the spatial concentration and the emission intensity of production, and of course by the interaction of health and pollution that modifies the evolution of production.

Comparing the evolution of the region for two levels of (homogeneous) initial pollution, high or low, we can see that in either case pollution first have dynamics then stabilizes at the same positive long run level.The concavity assumptions and other technical conditions we have previously imposed indeed ensure that the economy does not face take-off problems but instead encounters what we may call a sustainable development, although the health impacts of pollution remain significant. Note that even though the initial conditions obviously completely modify the transient dynamics faced by each location, the steady-state outcomes end up being identical, in aggregate and at each spatial point, because we assume no irreversibility in pollution impacts. Consequently, initial disparities disappear with time like in standard growth models. Starting with high, widespread, pollution levels sets the economy on an ascending path with growing production and health because health, hence production, is initially low enough not to offset the natural decay of the pollutant. Of course, the accumulated production over any time period still never catches up with the case with zero initial pollution.
In terms of health, we can see in the two corresponding panels of Figure 1 that the whole population is affected by pollution in the long-term, although it may have not been the case if the industrial hub and the residential area had been located further away from each other. As expected, the population living closer to the production sites are the most impacted by pollution, in magnitude but also starting sooner in time, which could bear important consequences in terms of inequalities in more stylized geographical settings. In any case, whether for pollution or health, although diffusion acts to smooth the spatial differences, the outcomes are, by construction, non-homogeneous across space, first because we assume the diffusive force not to be infinite, but more importantly because we have relaxed the unlikely assumption that production and population are distributed uniformly across space.

3.1.2 Non-monotonic convergence

One may notice in Figure 2 that the evolution of health towards the steady-state in the case $p_0(x, y) = p_H$ does not exhibit a strictly concave profile during the transitory period, which hints that the diffusion process is not as straightforward as it seems, even in a simple case with homogeneous diffusivity. More complex patterns may actually emerge in only slightly different scenarios. This can have important consequences for the stability of the consumption path over time for instance. Thus, we will quickly present some examples of more irregular transition paths to try to give additional insights on the role of initial conditions and on the mechanisms behind diffusion and convergence.
The initial level of pollution and its spatial distribution dictates the initial health, hence the

initial production and the level of emissions, but also, and perhaps more importantly, it affects the speed of diffusion: the level of pollution prevailing in the residential area not only determines the size of the pollution peak in the industrial area, through its damage on the health capital, but also the gradient of pollution between the two regions, therefore the rate of diffusion. To see this, we will look at a case with spatially heterogeneous initial conditions.
Figure 3 plots the evolution of health when the initial pollution level is set at 2. When this is the case in every location across the region, the path differs from the $p_0(x, y) = 5$ case, because the economy starts off at a higher production level which then backfires and causes a slight decrease in health when the newly emitted pollution reaches the populated area, with some delay corresponding to the diffusion process. The other curve shows the evolution of health when $p_0(x, y) = 2$ in the residential area only, that is, only in locations where $h(x, y, t) \geq 0.005$, and it is zero otherwise. We can observe that the urban area being initially more polluted somehow provides it some "protection": although production starts at a consequent level, this initial pollution reduces the pollution gradient between the residential center and its surroundings, hence dampening the diffusion of the additional pollution towards the area. This allows health to improve at first, as a result of the natural elimination of the pollutant and its diffusion from the residential area towards the initially less polluted neighbouring regions, before converging to the steady-state level as the industrial pollution spreads.
Therefore, there can be several mechanisms at stake in this simple diffusion process, and which of these dominates depends on the geographical setting and its characteristics.

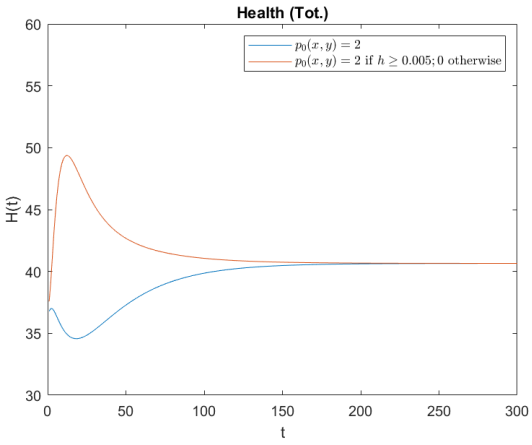


Figure 3: Comparing the evolution of the total health capital stock with homogeneous initial pollution versus spatially heterogeneous initial conditions.

3.1.3 Influence of different factors on the results

To give a more comprehensive overview of the model, we will now go through several important parameters and discuss their individual impact on the results. Note that for simplicity, we always consider the initial situation to be one with zero pollution, i.e. $p_0(x, y) = p_l = 0$. Unless otherwise specified, every other parameter is held equal, according to the values in Table 1.

- **Distance**:
  The distance between the industrial center and the residential area is a fundamental element of our setting as it determines both the amount and the timing of damage. However, it is simple to see that, in the benchmark case we have been studying, the further away the better, as distance unequivocally reduces exposure to pollution. This is true even after a very long time as a fraction of the pollution stock naturally vanishes while it diffuses away, so that distance indeed offers lower pollution in the long-run and not only an extra

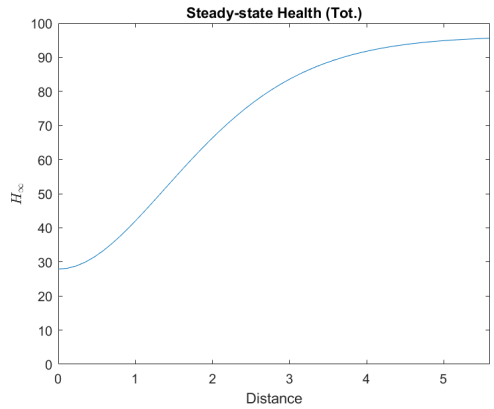
time delay. To make the problem more interesting, let us now introduce a discount factor representing the lesser availability of workers that are located further away from the firm, representing for instance the health or monetary costs of commuting. Namely, we rewrite the function

$$\phi(p(x,y,t), h(x,y,t), x, y) = h(x,y,t)e^{-\gamma p(x,y,t)}e^{-\xi l(x,y)}$$

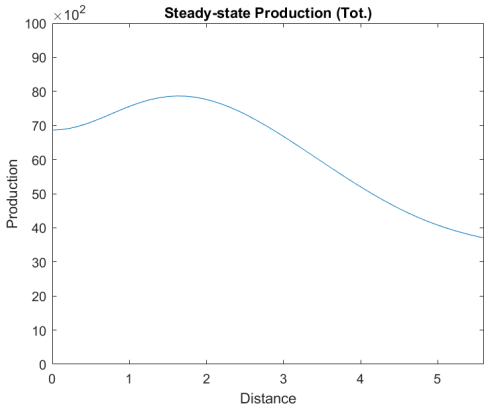
where $l(x,y)$ stands for the Euclidean distance of point (x,y) to the production area, such that $l(x,y) = \sqrt{(x-x_f)^2 + (y-y_f)^2}$, or more precisely to the center of the production area as an approximation. The relative size of coefficients γ and ξ thus determines the strength of each effect, respectively favoring or penalizing the distance of the residential area to the industry, and defines an optimal spatial positioning of the two areas of interest in terms of production maximization.
Keeping everything else equal, we perform the following exercise: we simulate the model while varying the position of the center of the residential area (x_h, y_h), and in each run we record the steady-state outcomes associated to the distance between (x_h, y_h) and fixed point (x_f, y_f). Then, we can describe long-term health and production as a function of distance, as plotted in Figure 4 illustrating this commuting problem with $\gamma = 1$ and $\xi = 2$. The optimal distance, here 1.62 ($x_h^* = y_h^* = 1.15$), of course heavily depends on these parameter values but we could argue that in an economy with low standards of living such that people cannot easily move across space to work, so $\xi > \gamma$, it can be optimal (in terms of total production) to locate an industry relatively close to populated areas although exposure to pollution and associated health damage may be high.

- **Diffusivity**:
Figure 5 plots the steady-state level of the total health capital as a function of parameter values in two situations: relatively close industrial and residential areas, with $(x_h, y_h) = (1, -1)$ (and still $(x_f, y_f) = (0, 0)$), or on the other hand a rather remote residential area, with $(x_h, y_h) = (3, -3)$. It can be counter-intuitive to observe that, in the first of these two cases, the steady-state health level increases with the diffusivity coefficient D (Figure 5a), as a larger D should lead to more locations being hit by pollution after the same amount of time. But more importantly, it implies stronger convergence forces: a larger D is associated with lower variance in the distribution of the pollution stock prevailing in each location at steady-state, and, because we have assumed an exponential damage function ϕ which is decreasing and convex, the fact that pollution is more spread out and less concentrated has a positive impact on health. In particular, it counteracts the local, disproportionate, accumulation of pollution right at the emission sources. Note the diffusivity coefficient D can also be interpreted as controlling the delay between production and the feedback effect of pollution on production, through health. However, with more distance separating the industrial center from the densely populated area, a higher diffusivity coefficient does not really act on the timing of feedback but rather on the magnitude of the effect, because pollution now reaches more locations and hence affects more people in the long run than it would have had with lower diffusivity. Besides, notice in Figure 5a that the health advantage brought by the distance to emission sources vanishes as D gets very large. Essentially, we can say that the diffusivity coefficient D determines the relevance of the spatial dimension for the model's long-term outcomes.

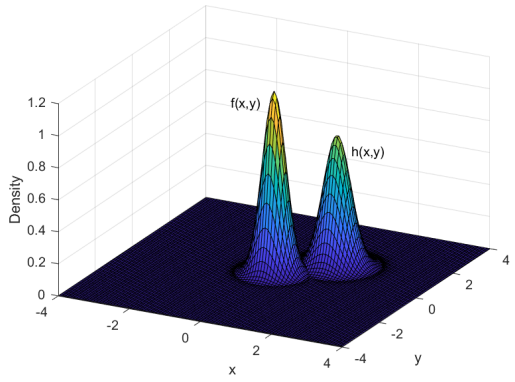
- **Natural pollution decay**:
The natural pollution decay δ also affects both the transient dynamics and the steady-state outcomes, although this time irrespectively of the spatial configuration (see Figure 5b). This parameter corresponds to nature's self-cleaning capacity and it determines how persistent pollution is across time, so it has a direct impact on the quantity of pollution that is spread across space at steady-state, but it also controls the share of emissions that eventually diffuses away and the perimeter in which it does so before vanishing. In other



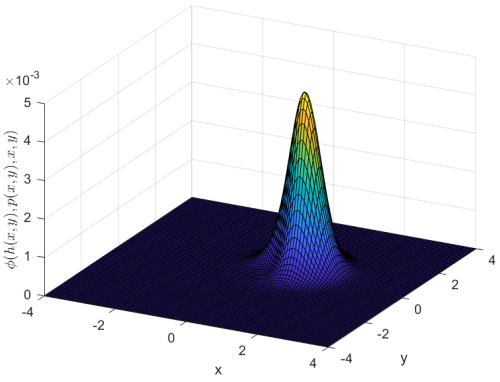
(a) Total health at steady-state as a function of distance.



(b) Total production at steady-state as a function of distance



(c) Optimal configuration for production maximization.



(d) Spatial distribution of the healthy human capital in the optimal configuration.

Figure 4: Steady-state outcomes when distance matters. Distance here corresponds to the Euclidean distance between (x_f, y_f) and (x_h, y_h), with fixed $(x_f, y_f) = (0, 0)$ while $x_h = y_h$ varies from 0 to 4.

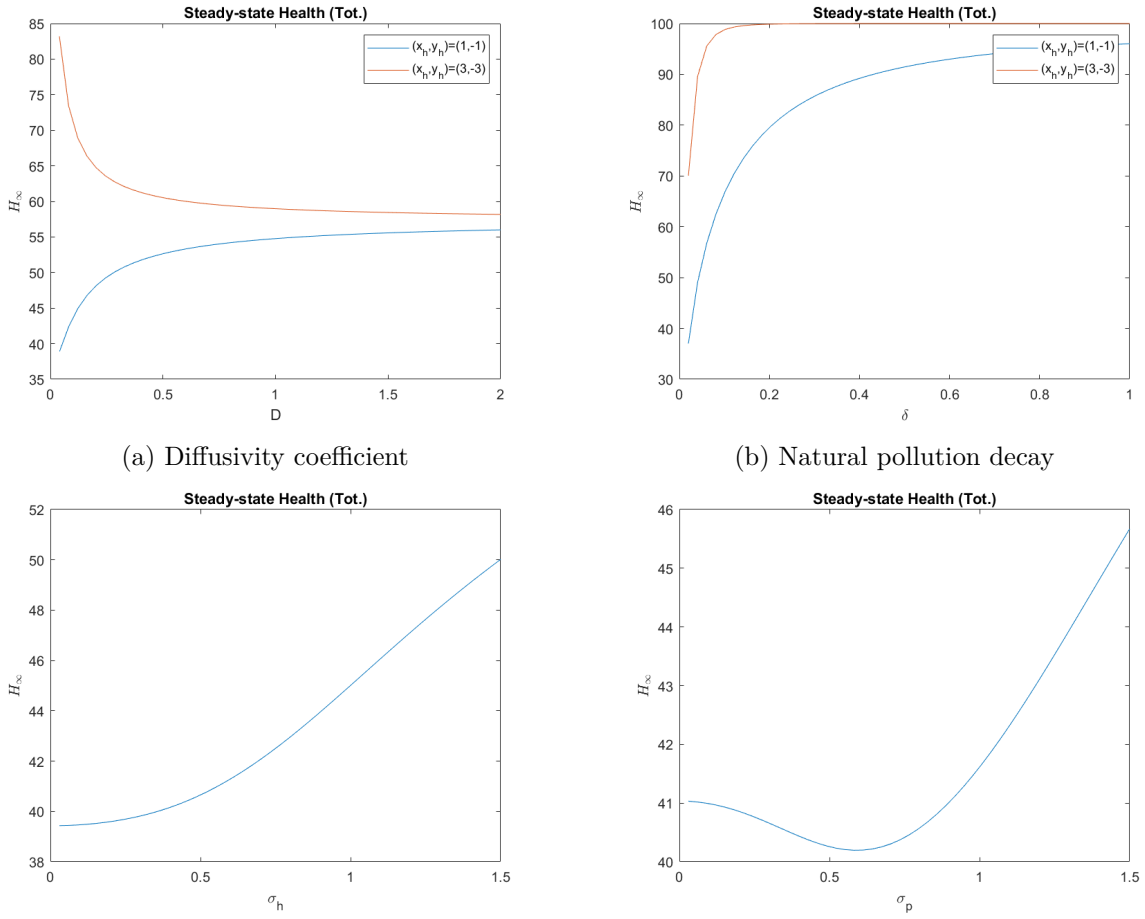
words, a higher δ somehow weakens the link between production and health, as less of the emitted pollution ends up reaching the population, thus dampening the accumulation process and yielding more favourable outcomes.

- **Production process**:
  In the formulation of the model, the labor factor productivity B and the environmental inefficiency θ are uncorrelated, so that a technological progress improving the efficiency of labor is associated with higher pollution and lower health in the long-term, because it allows to increase production, hence the quantity of emissions, using the same human capital. An increase in θ of course also yields a lower steady-state health level.

- **Agglomeration**:
  The concentration of the industrial activity and of population is controlled respectively by the standard deviation parameters σ_p and σ_h. We otherwise do not change the spatial setting compared to the benchmark case, meaning that we are still in a case where the production center and the urban center are close enough for the most densely populated area to be exposed to some pollution in steady-state. In this regard, a higher σ_h, that is, a larger spread of the population distribution in each direction, puts more people out of range of the pollution externality or at least further away from its source. A fraction of the residential area is of course more at risk, as it is extended in the direction of the

industrial area, but the overall effect of σ_h on equilibrium health remains positive (Figure 5c). The size of this effect remains however limited in comparison to the impact of other parameters.

As shown in Figure 5d, the role of industrial concentration is somewhat different. Starting from a very concentrated industrial area, an increase in σ_p is detrimental for long-term health because it increases the number of exposed people in steady-state, by reaching further in space. However, if σ_p is large enough for every location with $h(x, y, t) > 0$ to be affected in any case, then the more spread out the distribution of production is, the more spread out are the sources of pollution and consequently, the better is the health capital at steady-state. On a side note, industrial concentration also have an important role in the transient dynamics as it makes the pollution gradient between producing area and non-producing area more or less significant.



(a) Diffusivity coefficient

(b) Natural pollution decay

(c) Spatial spread of population density

(d) Spatial spread of the industrial distribution

Figure 5: Total health at steady-state as a function of parameter values.

- **Advection**:
  In a simple case where there are only two areas of interest, the role of advection is rather straightforward. Let us imagine a velocity field, representing wind for example, along the segment connecting (x_f, y_f) to (x_h, y_h). If polluting is transported from (x_f, y_f) to (x_h, y_h), taking for instance $v_1(x, y) = 0.3$ and $v_2(x, y) = -0.3$ for (x, y) on a straight 3 meshes wide stream joining these two points, steady-state health decreases from 41 (benchmark case) to 35. Recall that if initial pollution is set to zero, the advective field in this case goes in the same direction as diffusion, therefore amplifying the transport of pollution towards the residential area. Unsurprisingly, if wind instead blows pollution away from the city (take

$v_1(x,y) = -0.3$ and $v_2(x,y) = 0.3$ over the same segment), the long-term health level is brought up to 51. Advection does not otherwise affect the shape of the transition paths of the model.

- **Abatement**:
The discussion has been limited to production and pollution outcomes until now, because we have not introduced any sense of utility from consumption or else in the model, so that choosing a priori an abatement policy $u(x,y,t)$ is an arbitrary task, subject to resource waste. This last comment refers to cases where, taking $u(x,y,t)$ as given, we would have $p(x,y,t) - G(u(x,y,t)) < 0$ which is of course not permitted, then it would have been sufficient to allocate u' at that point in space and time such that $p(x,y,t) - G(u'(x,y,t)) = 0$. Nevertheless, to motivate the trade-offs between production, consumption and abatement studied in the next sections, we present in Figure 6 the evolution of total health, total consumption (which is the total revenue from production minus the amount dedicated to abatement) and total pollution for several abatement policies that could be envisioned. Each of the latter corresponds to the reallocation across all points in space of 5% of the total production revenue, at each time period. We consider 4 example policies:

$$u_1(x,y,t) = 0 \quad \forall (x,y,t) \qquad u_2(x,y,t) = 0.05 * F(H(t)) \frac{1}{N_x N_y}$$

$$u_3(x,y,t) = 0.05 * F(H(t)) \frac{f(x,y,t)}{\sum_{(x,y)\in\mathcal{R}} f(x,y,t)} \qquad u_4(x,y,t) = 0.05 * F(H(t)) \frac{h(x,y,t)}{\sum_{(x,y)\in\mathcal{R}} h(x,y,t)}$$

where N_x and N_y correspond to the number of discretized locations along the x and y axis, respectively. The associated abatement policy is thus uniform across space. The third and fourth equations corresponds to concentrating abatement efforts in the industrial area or in the residential area, respectively.
Each of the policies with positive abatement expenses eventually allows to increase consumption, even though it takes up a share of revenue. Unsurprisingly, the policy cleaning up pollution in the residential area, u_4, is the most successful at improving the long-term (and short term) health level, although it reduces the total pollution stock across the region by the least amount with regards to the benchmark case (see the third graph of Figure 6). Finally, a policy tackling pollution at the source(s), as u_3, is poorly effective in improving economic and environmental outcomes: it underlines the fact that, in this model, pollution is not necessarily undesirable at all points in space and does not need to be fully eliminated, as it incurs no harm on health if there is no one to be exposed at its location.

3.2 Spatio-temporal dynamics in relevant geographical contexts

The aim of this section is to demonstrate the capacity of the model to accommodate more stylized settings and represent real-world geographies. The end goal would be to pursue a thorough calibration of the model, using hydro-geological and production maps, as well as demographic and pollution data to provide a realistic picture of the initial state of the problem and of the pollution transport process. This exercise is limited here to exploiting the flexibility offered by the production and population distributions and the spatially variable coefficients of the pollution transport PDE to fit the gridded spatial data used for numerical simulations to relevant types of realistic geographical settings. The production function, the emission intensity and the damage function in particular are only illustrative and could be tailored to the specific type of pollution issue faced in the region at study. Broadly, the underlying question is to explore, in each case study, the extent of the pollution problem faced by the economy we are modelling and how it relates to the description of the geographical configuration introduced in the model.

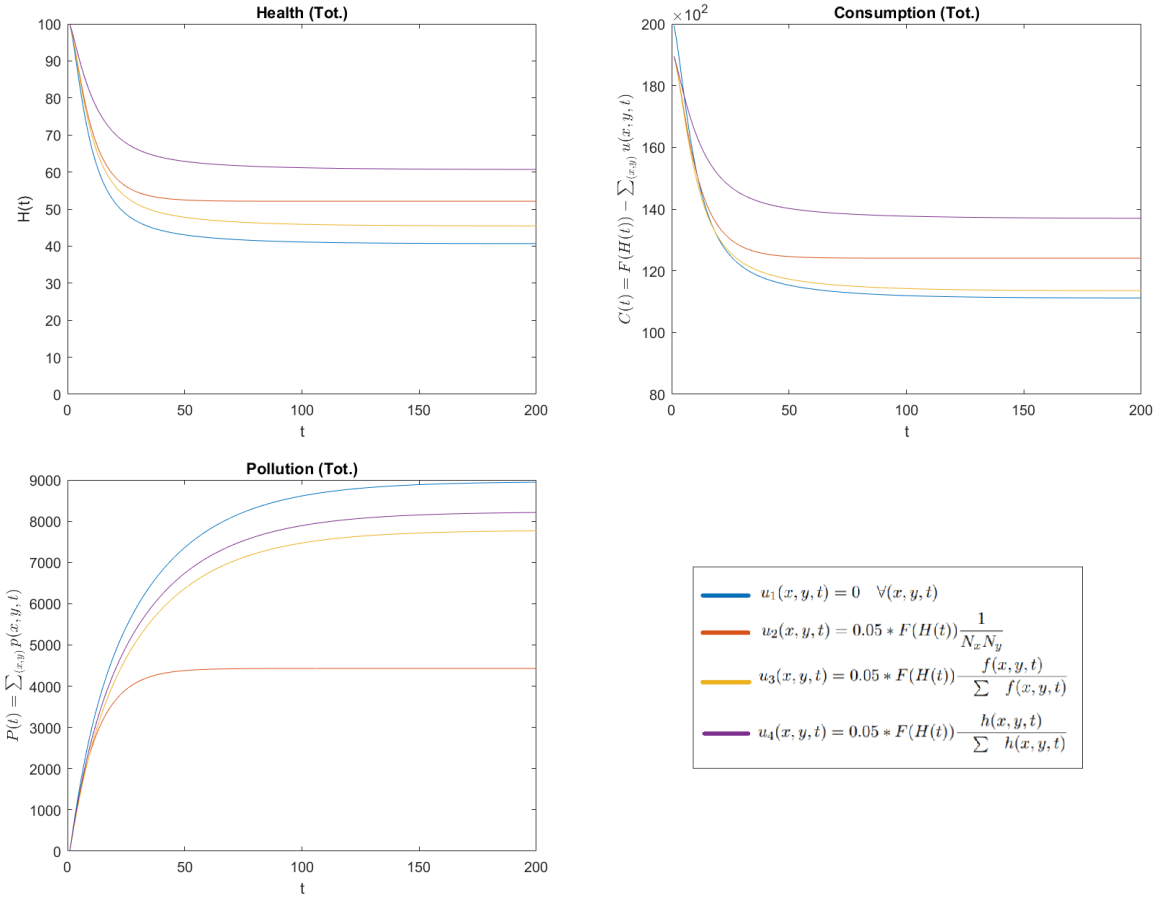


Figure 6: Evolution over time of aggregate healthy human capital, total production and total pollution stock in the benchmark case with different abatement policies and abatement technology $G(u) = u^{\nu}$ with $\nu = 0.5$.

3.2.1 *Case study 1*: A core-periphery setting

The core-periphery model, dating back to Friedmann (1966),Krugman (1991) and numerous other contributions, is a popular model in Economic Geography which seeks to explain the tendency of economic activities to be concentrated in some regional growth centers, and more generally the observation that socioeconomic development tends to be uneven across space. The most important factor that is put forward in this framework is the spatial distance to the core, which can be related to our previous discussion about the role of distance in the present model. It has been adapted to various spatial scales, but we will be particularly interested here in the so-called "four-stage model of regional development", describing patterns of urban growth across time. Broadly peaking, our setting will be a metropolitan monocentric area, as can be described the regions of Paris or London for example, that have gone past an industrial phase characterized by the dispersion of economic activity and the migration of people from rural areas to urban areas, yielding the following schematic organisation: a highly productive, prosperous and densely populated core, surrounded by residential suburbs at the bounds of which large industries and manufacturing units have been relegated. We will focus on the fact that polluting industries are located in a belt in the periphery while the core is dense and productive but with less polluting

economic activities, that is, with a higher $B(x, y)$ but a lower parameter $\theta(x, y)$ [8].

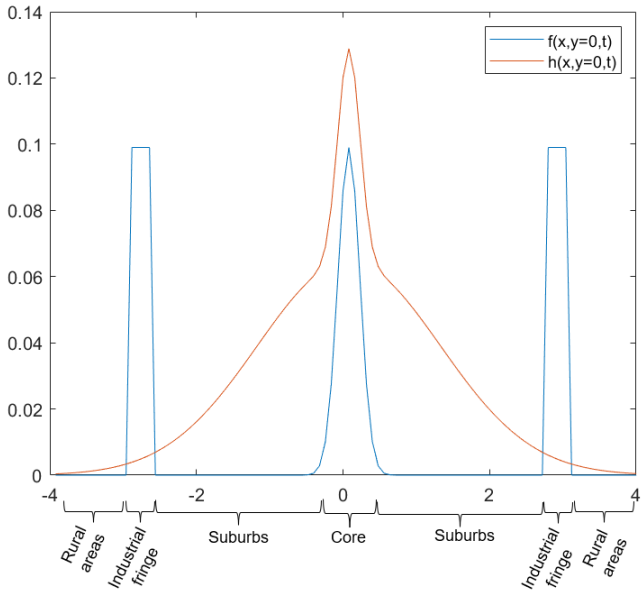


Figure 7: *Case study 1*: Core-periphery geographical setting. The profiles of the density functions along the x-dimension at $y = 0$ are plotted.

Figure 7 presents the corresponding geographical setting. The core is centered around point $(0, 0)$ in the region $\mathcal{R} = [-4, 4] \times [-4, 4]$, as before. The parameters of the diffusion process and of the production function are unchanged. However, similarly to the section related to the role of distance, the damage function is adapted:

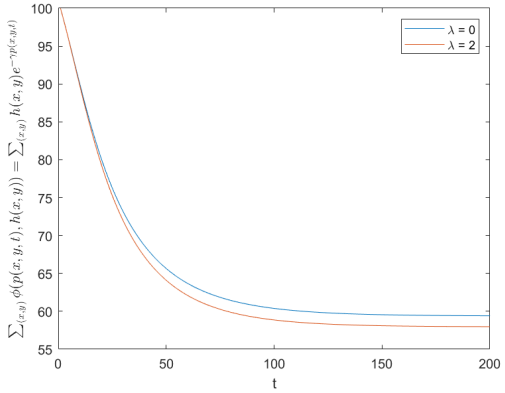
$$\phi(p(x, y, t), h(x, y, t), x, y) = h(x, y, t)e^{-\gamma p(x,y,t)}\mu e^{-\lambda l_h(x,y)}$$

where $l(x, y)$ stands for the distance of point (x, y) to the center: $l_h(x, y) = \sqrt{(x - x_h)^2 + (y - y_h)^2}$. μ is a scaling factor to allow comparison between cases with $\lambda = 0$ and $\lambda > 0$.
Indeed, the most interesting in this setting would be to incorporate the distribution of the productivity of workers, as the idea that residential areas are increasingly "specialized" in terms of worker types and namely that the productivity of workers declines as a function of the distance from the core are fundamental assumptions of the seminal core-periphery model. This is what we attempt to represent with the additional term $e^{-\lambda l_h(x,y)}$: if $\lambda > 0$, damages to the health of low productivity workers has a lower impact on total production than similar damages on highly productive workers, which is a crude way to put it but an interesting insight on spatial inequalities. We will thus study the evolution of production, health and pollution in this setting, assuming zero initial production.

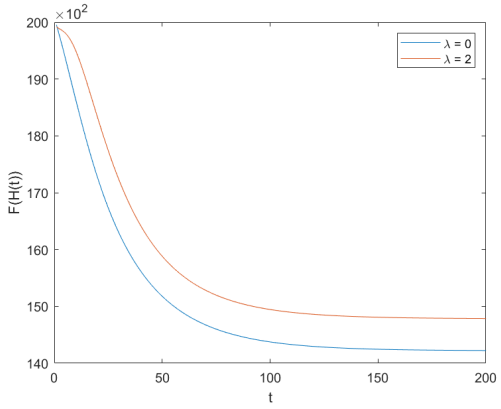
Results from numerical simulations are presented in Figure 8. The outcomes are obviously marked by strong spatial heterogeneity, stemming from the geographical configuration in which polluting industries are located at the city bounds. In contrast to the benchmark case with homogeneous diffusion and a central industrial area where the pollution profiles exhibited a

[8] This is of course only a poor approximation as in reality the source of pollution in crowded city centers is mostly related to traffic and everything that comes with population density rather than the economic activity. Also, while we consider these two spatially disconnected types of economic activity (mostly services in the core, industries further away in the periphery) we are not able to account for the correspondence between the location of workers and the location to which they supposedly go to work. This would actually yield a whole new set of questions regarding the environmental and economic cost of commuting in and out of each circle.

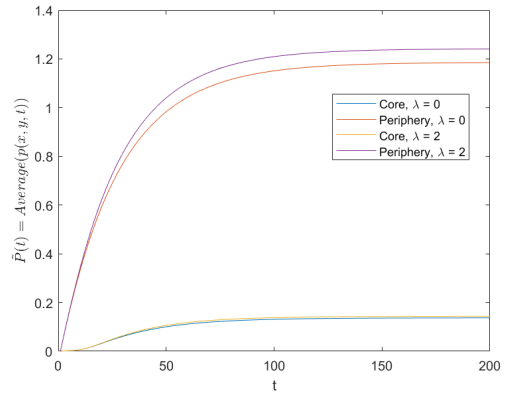
Gaussian profile across space, the core here enjoys much lower levels of pollution than the periphery. This geography is associated with better health outcomes in aggregate terms, but this is because the core is the most densely populated area. This may be seen as an additional source of spatial inequalities or, for the least, spatial disparities which is the main focus of the core-periphery model. Notice now, in Figure 8b, that accounting for the lower contribution to the productive activity of workers living in the suburban areas yields a higher production level and lower aggregate health (Figure 8a). This distribution of productivity, declining with the distance to the center, is beneficial for the economic activity as the larger amount of emissions associated with higher production strikes almost exclusively the low productivity workers living in the periphery (Figure 8c). This is an example of how economic incentives to perpetuate socio-environmental inequalities can arise.



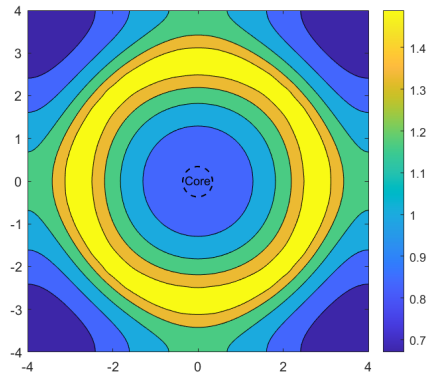
(a) Evolution over time of total human capital with pollution damages, without taking into account spatial discounting.



(b) Evolution over time of total production, with and without spatial discounting.



(c) Evolution over time of average pollution stock prevailing respectively in the core region and in the periphery, with and without spatial discounting.



(d) Long-term spatial distribution of pollution across space (for $\lambda = 2$).

Figure 8: Results from numerical simulations in *Case study 1*.
$\lambda = 2$ denotes a case in which the contribution to production of human capital located at one point in space declines with its distance to the center. $\lambda = 0$ is the case without spatial discounting.

3.2.2 *Case study 2*: A valley basin city

The flexibility of the model invites us to study contexts in which the geographical or geological nature of the region has a direct impact on the pollution transport process, hence on the accumulation of pollution. One type of such situations is cities that are confined in a mountain valley or a river basin, where the natural elevation characterizing this environment tends to prevent the transport of pollution and trap it above and around the city. This only underlines the relevance of carefully taking into account geographical specificities in pollution control problems. Infamous examples are for instance Mexico City, contained within the Valley of Mexico, which is one of the world's most polluted cities, or, as it is well-known in France, the metropolitan area of Grenoble, in the French Alps. The city is surrounded by high mountain ranges, carved by two river basins passing through Grenoble. The inhabitants of the valley suffer from high levels of air pollution as a consequence of this topography. The main part of this pollution is attributed to traffic, heating and other emission sources associated to a dense urban zone, but Grenoble remains a dynamic technology and research centre and industry still occupies a large part of the metropolis economy. Thus, we will recreate the geographical environment of the "Grenoble-Alpes Métropole" and study the outcomes of the model within this region.

The (gridded) spatially distributed data introduced in the model is plotted in Figure 9. We use elevation maps to construct heterogeneous diffusivity coefficients. The values are chosen as a illustration to reflect the variation in height across space and correspondingly how it can hinder the diffusion process, but they do not correspond to any physical reality. To construct the f distribution for industries, we use the geolocated entries of the BASIAS public database (*Inventaire historique de Sites Industriels et Activités de Service*) to produce an index of industrial density for each mesh of the spatial grid. The municipality of Grenoble corresponds to the central area, with the highest industrial and residential density. We focus this analysis on heterogeneous diffusion, and more specifically on natural barriers to diffusion, and we do not include advection into the problem, because winds in mountain valleys tend to be highly variable and the river streams are not relevant for the air pollution problem. We will compare the numerical solutions obtained with heterogeneous diffusion to those obtained with homogeneous diffusion to highlight the consequences of the topography of the Grenoble metropolitan area on its economic and environmental outcomes, which we are able to do because the model includes spatially variable coefficients.

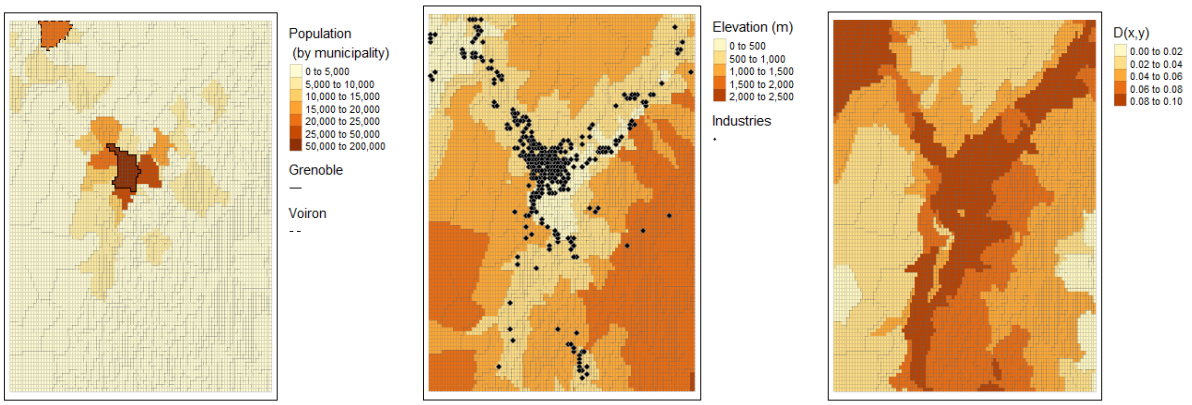


Figure 9: *Case study 2*: Geographical setting of the Grenoble Metropolitan area.[9]

[9] Although the region does not appear to be a square in these images, this is only a product of the cartographic projection. The mesh is equidistant and uniform, with the same number of points in each dimension, covering the same length in degrees in each direction.

Figure 10 shows the spread of pollution in the region across time, starting from a zero pollution initial state, with homogeneous and heterogeneous diffusion. At first glance, the spatial profiles may seem similar but this is because we consider the same distribution of industrial activity in each case which is obviously not random but constrained by the local geography, with a high concentration in the basins areas as illustrated in 9. However, we can indeed observe that the natural barriers to diffusion that the mountain ranges represent impact the distribution of pollution across space in short and long-term: it leads to a noticeably higher concentration of pollution in the central valley, over the city of Grenoble, with regards to what would arise with a uniform topography (see the homogeneous diffusion case in Figure 10). Figure 11 shows that this mechanism drives down the aggregate health level, hence the aggregate production, because the bowl in which pollution is confined is also the most densely populated area of the region. The second panel of Figure 11 further illustrates the disproportionate accumulation of pollution over time in the city of Grenoble, in comparison to a city that is not trapped within the valley, taking as an example the case of Voiron (in the northwest of the region, the municipality bounds are marked in Figure 9). While the total stock of pollution is almost identical with either type of diffusion (lower by barely 1.5% in steady-state for heterogeneous diffusion as a result of lower health, hence lower production), the heterogeneous diffusion process still considerably changes the distribution of pollution across space. It makes very little change for medium-size cities outside of the valley basin with industrial production, as Voiron, as they are mainly affected by their own activity. It makes things better for mountain villages or small cities further away from the dense Grenoble area, with no industries, as they can be spared (to a certain extent) from the excess pollution generated in Grenoble and its surroundings, where the health and pollution situation is clearly exacerbated by the topography of the environment.

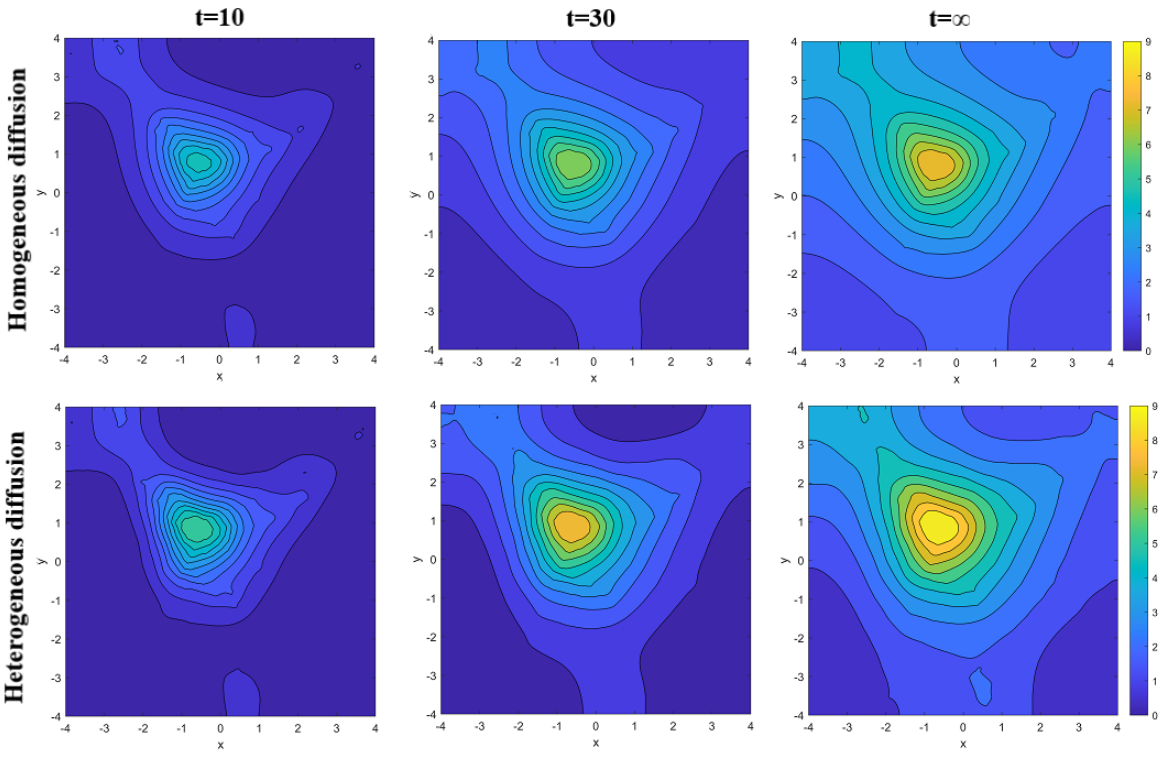


Figure 10: Numerical solutions for the spatio-temporal dynamics of pollution in *Case study 2* with homogeneous diffusion (top) and heterogeneous diffusion according to the geographical configuration (bottom).

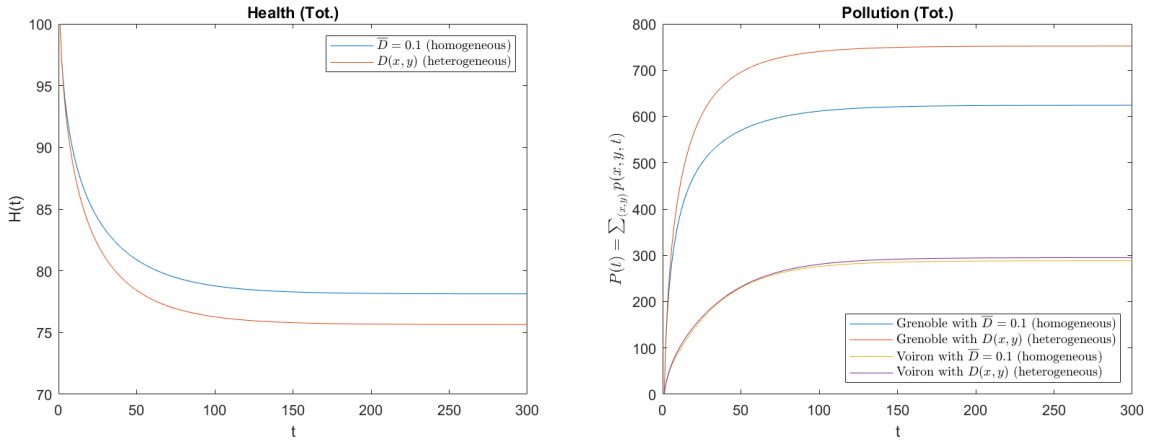


Figure 11: *Case study 2*: Evolution over time of aggregate healthy human capital and of the pollution stock in two municipalities, with homogeneous diffusion or heterogeneous diffusion.

3.2.3 *Case study 3*: An upstream-downstream situation

A classic pollution issue where the spatial dimension is of utmost relevance concerns conflicts of water-related interests between upstream and downstream locations. The link between upstream pollution and downstream water quality degradation is rather straightforward and has been evidenced in various areas of the world. In particular, it is a growing concern in dense regions of developing countries as China or India, that have known a booming, erratic, urban growth and a rapid industrialization, located for the main part in river basins, associated with a stringent lack of enforcement of environmental regulations, among which waste disposal regulations, leading to critical water and soil contamination.

We will consider as an example of an upstream-downstream setting the region of Hyderabad, South India, which represents the country's largest hub of bulk drug manufacturing industries and is known to suffer from a severe problem of pollution of water bodies (surface and groundwater) from pharmaceuticals[10]. Hyderabad city has spread on the banks of the Musi River which was a primary source of water for drinking, bathing and irrigation for many communities but is now loaded with pharmaceuticals and toxic wastes. This pollution has been directly associated with various health damages observed in Hyderabad or one of the hundred villages located in the river's drainage basin. The inappropriate discharge of industrial effluents has also lead to the occurrence of antibiotic resistance, sometimes in extremely dense municipalities and in locations that are far away from the (alleged) source of discharge. The region at study is presented in Figure 12. Patancheru and Banjara Hills are two major industrial areas. However, a public public project to deal with the critical pollution levels in Patancheru has led to the construction of a 18-km pipeline to divert all effluents from Patancheru to Amberpet, further downstream on the river, where the wastewater treatment plant is actually far too poorly equipped to treat this amount of waste before discharging it into the river. The model allows to study the outcomes associated with such policy choices: we will consider a scenario "before the pipeline", with two emission sources located respectively in Patancheru and Banjara Hills, and a scenario with the pipeline, where the two emission sources are now located in Banjara Hills and Amberpet. In either case, the distribution function f will correspond to the sum of two narrow normal distribution, each centered at one of the two points under consideration, with $\sigma_f = 0.3$ and the total density being normalized to 100. The population distribution h is based on population data, as plotted in Figure 12. We approximate the shape of the Musi river with two streams (a long horizontal segment and a shorter tilted one). Additionally, to illustrate the effect of con-

[10]See for instance on this subject Larsson et al. (2007),Fick et al. (2009),Lübbert et al. (2017).

sidering the advective transport in the model, we will compare the cases of two municipalities, Gandipet and Bhongiri, located respectively upstream and downstream of Hyderabad city. We also choose a low diffusivity homogeneous coefficient $D(x, y) = 0.01 \ \forall(x, y)$ to emphasize the advection dominated nature of the problem.[11]

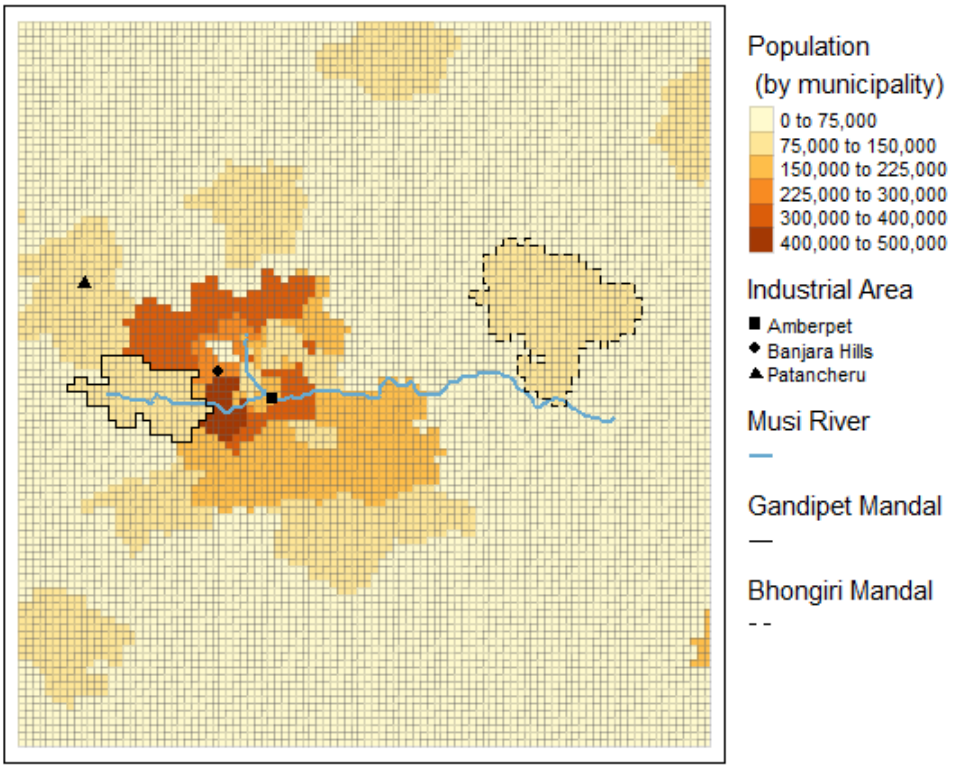
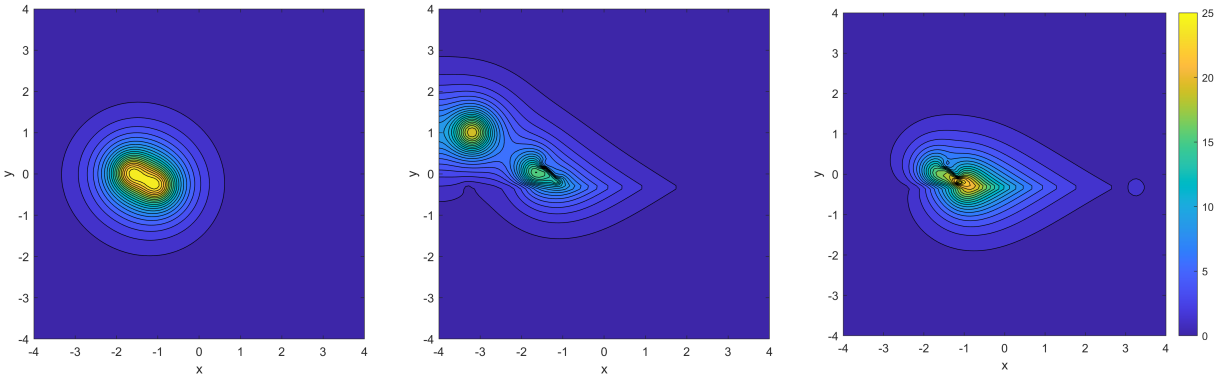


Figure 12: *Case study 3*: Hyderabad's geographical setting.

The results from the numerical solutions are shown in Figures 13 and 14. Figure 13 plots the distribution of pollution across space at the steady-state depending on the scenario. The displacement of waste emissions from Pantecheru to Amberpet after the construction of the pipeline obviously changes the geography of pollution, as it shifts the location of pollution sources. It would be interesting to study in more details the characteristics of the communities whose exposure directly depends on public policy projects like this one, potentially raising social justice issues. However, the role of advection in modifying the spatial profile of pollution is perhaps more striking, with important consequences as well as it brings pollution to regions that would not have been polluted if the river did not exist. It also drives pollution away from Banjara Hills (comparing 13c to 13a), making Amberpet the main and unique pollution hotspot.
The total health and total production in the region are however only affected to a very limited extent: we do not find an unambiguously positive effect of the advective field transporting pollution away, in comparison to our discussion about the role of advection in the benchmark case, because in the present case study the population density is not concentrated around a unique center but spread out along the river so that the total damage is unchanged with advection, unlike its spatial allocation. To look deeper into how advection can exacerbate spatial disparities,

[11] An advection dominated problem is a special type of PDE in which advection plays a significantly larger role than the other terms. A general condition to characterize advection dominated problems is $Pe >> 1$, where Pe is the Péclet number defined as : $Pe = \frac{v/L}{D/L^2} = \frac{\text{diffusion time}}{\text{advection time}}$ with v denoting the flow velocity, D the diffusion coefficient and L the characteristic length of the space.

we plot in Figure 14 the evolution of pollution over time in two municipalities, located upstream and downstream of Hyderabad city. We can observe that the advective transport is beneficial for upstream cities as Gandipet, which exhibit lower pollution levels than in the absence of this additional force. Note that the introduction of the pipeline also lowers pollution in this municipality which is not actively producing but still relatively close to Patancheru and a part of the dense central area affected by industrial development. In contrast, see in Figure 14b how the pipeline leads to a dramatic increase of the pollution flows reaching more remote rural areas like Bhongiri, because a large amount of waste is now dumped directly into the body of the river where it moves much faster than with diffusion only. This discussion is a simple example of how the model can be used to analyze public policies and perform counterfactual exercises.

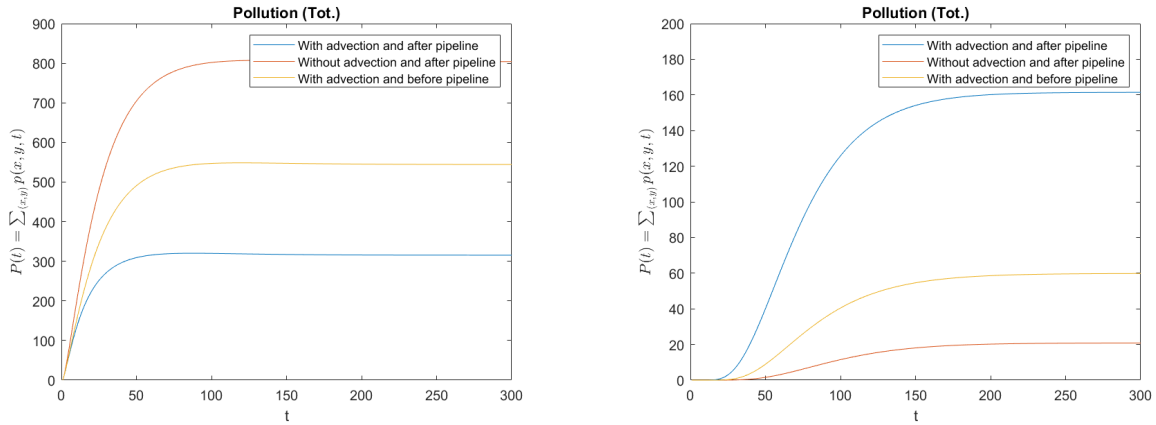


(a) Without advection and after pipeline.

(b) With advection and before pipeline.

(c) With advection and after pipeline.

Figure 13: Steady-state pollution levels across space in *Case study 3* for different scenarios.



(a) Gandipet Mandal (upstream).

(b) Bhongiri Mandal (downstream).

Figure 14: Evolution over time of the stock of aggregate pollution in two municipalities (i.e. mandals) for different scenarios.

4 An optimal control problem

In the Solow-type framework, we find that pollution always settles at a positive level of harm but we cannot say anything about the (sub)optimality of the outcomes nor design efficient consumption and abatement policies as the gains derived from production do not enter in determining the steady state level. We thus turn to a Ramsey-type version of the model in which economic and environmental policies (i.e. consumption, abatement and consequently production) are endogenously determined. Accordingly, the economy will now be described by an optimal control problem where the state equations is the partial differential equation we have studied in the previous sections.

In the same framework, we assume that the policy maker collects all production and reallocates it across locations at no cost, yielding the following budget constraint:

$$\iint_{\mathcal{R}} [c(x,y,t)h(x,y,t) + u(x,y,t)]\,dx\,dy = F(H(t)) = BH(t)^{\alpha} \tag{5}$$

which may seem like a strong assumption but actually enlarges the set of feasible consumption and abatement decisions by allowing the spatial reallocation of production, as noted in Camacho and Pérez-Barahona (2015). This is all the more true in our setting where the ability of each location to produce is determined by the industrial density distribution f, which in principle exhibits strong spatial heterogeneity, in such a way that assuming that each location consumes at most what it produces would drastically reduce the set of possibilities.

The policy maker chooses the level of consumption per capita $c(x,y,t)$, that is the consumption level at time t of a representative household located at (x,y), and the amount allocated to cleaning activities $u(x,y,t)$, again at each location and each point in time, to maximize the discounted welfare of the entire population:

$$\max_{c,u} \int_0^{\infty} \iint_{\mathcal{R}} \mathcal{U}(c(x,y,t))h(x,y,t)e^{-\rho t}\,dx\,dy\,dt \tag{6}$$

subject to:

$$\mathcal{P}_c \begin{cases} p_t - \frac{\partial}{\partial x}[Dp_x] - \frac{\partial}{\partial y}[Dp_y] = \theta BH(t)^{\alpha} f(x,y,t) - \frac{\partial}{\partial x}[v_1 p] - \frac{\partial}{\partial y}[v_2 p] - \delta p - G(u) \\ \iint_{\mathcal{R}} c(x,y,t)h(x,y,t) + u(x,y,t)\,dx\,dy = BH(t)^{\alpha} \\ H(t) = \iint_{\mathcal{R}} \phi(p(x,y,t), h(x,y,t), x, y)\,dx\,dy \\ p(x,y,0) = p_0(x,y) \qquad , (x,y) \in \mathbb{R}^2, \\ \lim\limits_{x \to \partial R_X} p_x = \lim\limits_{y \to \partial R_Y} p_y = 0 \qquad , \forall t \in [0, +\infty) \end{cases} \tag{7}$$

considering a standard instantaneous utility function $\mathcal{U} : \mathbb{R}^+ \to \mathbb{R}$ that is increasing and concave and satisfies the Inada conditions.

As in the classical Ramsey model and as in Boucekkine et al. (2009, 2013a); Camacho and Pérez-Barahona (2015), we work in the classical Benthamian case, in the sense that the utility function is defined as the sum of individual intertemporal utilities, for the representative households located in every point in space. However, this sum is weighted by the population density function h, which can be seen as a type of space discounting in which the relative importance attributed to each location by the policy maker depends on the share of the population it represents. Equivalently, we consider that the residents are homogeneous across space and that the policy maker weighs each of them identically. The standard exponential time discounting $e^{-\rho t}$, with $\rho > 0$, is included as well.

As the "spatial" policy maker re-allocates production across locations at no cost, and does not have any subjective spatial preferences outside of population density patterns, the consumption per capita $c(x,y,t)$ should be identical across all locations, i.e., $c(x,y,t) = c(t)$. Then, recalling

that the total population is normalized to 1, the feasibility condition can be rewritten as follows:

$$c(t) + \iint_{\mathcal{R}} u(x,y,t)\,dx\,dy = BH(t)^{\alpha}$$

It might also be noted in (6) that we restrict ourselves to a case where the utility function is a function of consumption only. In particular, pollution could enter the utility function directly, in addition to its effect on health and, ultimately, on the total amount of revenue available. For instance, La Torre et al. (2015) introduces a pollution disutility term which trades off the relative importance of environmental quality with respect to consumption. Nevertheless, because our underlying goal is to set the basis of a currently non-existent framework to account for pollution impacts on health and labor productivity in a spatial and dynamic way, we choose to focus on this channel only in the present dissertation.

4.1 Optimal trajectories

An optimal solution to problem (6)-(7) is defined as follows:

Definition 1. *A trajectory $(c(x,y,t),u(x,y,t),p(x,y,t))$, with $c(x,y,t),u(x,y,t)$ and $p(x,y,t) \in C^{2,1}(\mathbb{R}^2 \times [0,+\infty))$, is admissible if $p(x,y,t)$ is a solution to problem (7) with controls $u(x,y,t)$ and $c(x,y,t)$ on $t \geq 0$, $(x,y) \in \mathbb{R}^2$, and if the integral objective function (6) converges. A trajectory $(c^*(x,y,t),u^*(x,y,t),p^*(x,y,t))$ is an optimal solution of problem (6) and (7) if it is admissible and it it is optimal in the set of admissible trajectories, i.e. the value of the objective (6) corresponding to $(c^*(x,y,t),u^*(x,y,t),p^*(x,y,t))$ is greater than its value for any admissible trajectory.*

If an optimal solution exists, we may derive the first-order necessary optimality conditions fitting our problem by adopting a variational approach, following Boucekkine et al. (2009) or Camacho and Pérez-Barahona (2015). More specifically, we use the Ekeland variational principle (Ekeland (1974)) to obtain a Pontryagin maximum principle for the problem, involving a system of PDEs. Following this procedure, we write the associated value function $\mathcal{V}$ as a function of c, u, p and B as follows:

$$\begin{aligned}\mathcal{V}(c,u,p,H) = &\int_0^{\infty} \iint_{\mathcal{R}} \mathcal{U}(c(x,y,t))h(x,y,t)e^{-\rho t}\,dx\,dy\,dt \qquad (8)\\
&- \int_0^{\infty} \iint_{\mathcal{R}} q(x,y,t)\Big[p_t - \frac{\partial}{\partial x}[Dp_x] - \frac{\partial}{\partial y}[Dp_y] + \frac{\partial}{\partial x}[v_1 p] + \frac{\partial}{\partial y}[v_2 p] - \theta BH(t)^{\alpha} f(x,y,t) + \delta p + G(u)\Big]\,dx\,dy\,dt\\
&- \int_0^{\infty} m(t)[H(t) - \iint_{\mathcal{R}} \phi(h,p)\,dx\,dy]\,dt - \int_0^{\infty} n(t)[\iint_{\mathcal{R}} c(x,y,t)h(x,y,t) + u(x,y,t)\,dx\,dy - BH(t)^{\alpha}]\,dt\end{aligned}$$

where q, m and n are auxiliary functions.

Proposition 1 presents the corresponding set of necessary optimal conditions. The derivation is detailed in Appendix B.

Proposition 1. *The set of Pontryagin conditions of the problem are given by:*

$$\begin{cases} p_t - Dp_{xx} - Dp_{yy} - D_x p_x - D_y p_y = \theta B H(t)^\alpha f(x,y,t) - (v_1 p_x + v_2 p_y) - p(v_{1x} + v_{2y}) - \delta p - G(u) \\ Q_t + DQ_{xx} + DQ_{yy} + Q_x[D_x + v_1] + Q_y[D_y + v_2] \\ \qquad + \Big[\mathcal{U}'(c(x,y,t)) + \theta \iint_{\mathcal{R}} Q(x',y',t) f(x',y')\, dx'\, dy'\Big] \alpha B H^{\alpha-1} \phi_p(p,h) = (\delta+\rho) Q(x,y,t), \\ Q(x,y,t) = -\frac{\mathcal{U}'(c(x,y,t))}{G'(u(x,y,t))}, \\ \iint_{\mathcal{R}} c(x,y,t) h(x,y,t) + u(x,y,t)\, dx\, dy = B H(t)^\alpha \\ H(t) = \iint_{\mathcal{R}} \phi(p,h)\, dx\, dy \\ p(x,y,0) = p_0(x,y), \lim_{x \to \partial R_X} p_x(x,y,t) = \lim_{y \to \partial R_Y} p_y(x,y,t) = 0, \\ \lim_{x \to \partial R_X} Q_x(x,y,t) p(x,y,t) = \lim_{y \to \partial R_Y} Q_y(x,y,t) p(x,y,t) = 0, \\ \lim_{t \to +\infty} Q(x,y,t) p(x,y,t) = 0, \end{cases} \tag{9}$$

for $(x,y,t) \in R \times \mathbb{R}^+$.

The first condition describes the law of motion of the state variable, describing to the spatio-temporal dynamics of pollution in our problem. The second expression is the adjoint equation corresponding to the shadow price (co-state variable) of pollution, which is ruled as well by a parabolic PDE, as it also varies in time and space because pollution moves in time and space, but this one has no initial value. The proposition includes boundary conditions for both PDEs and the transversality condition $\lim_{t \to +\infty} Q(x,y,t) p(x,y,t) = 0$. As in Boucekkine et al. (2009) and Camacho and Pérez-Barahona (2015), if we focus on interior solutions, we can consider analogous standard boundary conditions for the shadow price of pollution: $\lim_{x \to \partial R_X} Q_x(x,y,t) = \lim_{y \to \partial R_Y} Q_y(x,y,t) = 0$, stemming from the Neumann boundary conditions constraining the pollution flow at borders in (7).

The third condition in (9) represents the trade-off between the gains from consumption, in terms of discounted marginal utility, and the gains from abatement in terms of reduced pollution. An increase of the revenue dedicated to consumption will also imply higher marginal social cost due to the lower investment in abatement activities, hence higher pollution. Finally, as previously explained, the fourth equation describes the budget constraint of the policy maker allowing for the re-allocation of production across locations.

Provided that we have assumed $\mathcal{U}'(\cdot) > 0$ and $G'(\cdot) > 0$, one can note that the condition $\mathcal{U}'(c(t)) = -Q(x,y,t) G'(u(x,y,t))$ implies that $Q(x,y,t)$ is negative and allows us to identify a lower bound: $-Q(x,y,t) \geq \mathcal{U}'(c(x,y,t))$. Since, c and u are negatively related by the condition $\iint_{\mathcal{R}} [c(t) h(x,y,t) + u(x,y,t)]\, dx\, dy = B H(t)^\alpha$, we get that, when $c \to 0$, $\mathcal{U}'(c(t)) \to +\infty$ while $-Q(x,y,t) G'(u(x,y,t))$ is equal to a non-negative value for $c = 0$, and when $c \to +\infty$, $\mathcal{U}'(c(t)) \to 0^+$ while $-Q(x,y,t) G'(u(x,y,t)) \to +\infty$. Following the same reasoning, note also that $\frac{\partial \mathcal{U}'(c(x,y,t))}{\partial c} \leq 0$ and $\frac{\partial(-Q(x,y,t) G'(u(x,y,t)))}{\partial c} \geq 0$ as $G''(\cdot) \leq 0$, so that both sides of the equation are monotone functions and they cross only once which proves that, $\forall t > 0$, $c(x,y,t) = c(t)$ is uniquely determined by $q(x,y,t)$ and $u(x,y,t)$.

The dynamic system (9) induced by the Pontryagin conditions is a forward-backward system of partial differential equations yielding an ill-posed problem well described in Boucekkine et al. (2009, 2013b), meaning that neither the existence nor the uniqueness of the solutions can be ensured. The issue stands more specifically in the backward PDE corresponding to the adjoint equation when considering a strictly concave utility function and an infinite time horizon, as

it cannot be reversed in time in this case. Boucekkine et al. (2013a) finds that the source of ill-posedness is due to the fact that the first-order optimality conditions derived by applying the Pontryagin maximum principle are only necessary in the infinite time horizon case, contrary to the finite horizon case where they are necessary and sufficient under usual regularity conditions. Several options have been proposed in the literature to circumvent the ill-posedness problem: Boucekkine et al. (2009) considers a spatial Ramsey model with linear utility, as a way to make the adjoint equation degenerate, while Boucekkine et al. (2013a) or Cornet (2020) adopt dynamic programming methods allowing to study infinite time horizon problems while avoiding the use of adjoint equations. We leave the latter approach for future research and resort to reducing the time horizon to $[0, T]$ with $T < \infty$ to be able to explore our optimal control problem and its dynamics into further details with numerical methods.

The set of first-order conditions for the finite time horizon case can be derived similarly to system (9) with $T = \infty$, but particular attention should be dedicated to the specification of the transversality condition. Indeed, when the time horizon is finite, the remaining cost of the pollution at the end of the planning period needs to be taken into account, while in principle when T goes to infinity, this scrap value should vanish.

We proceed to rewrite the control problem in finite time as follows:

$$\max_{c,u} \int_0^T \iint_{\mathcal{R}} \mathcal{U}(c(x,y,t))h(x,y,t)e^{-\rho t}\,dx\,dy\,dt + \iint_{\mathcal{R}} \psi(p(x,y,T),x,y)e^{-\rho T}\,dx\,dy \tag{10}$$

subject to:

$$\mathcal{P}_T \begin{cases} p_t - \frac{\partial}{\partial x}[Dp_x] - \frac{\partial}{\partial y}[Dp_y] = \theta BH(t)^\alpha f(x,y,t) - \frac{\partial}{\partial x}[v_1 p] - \frac{\partial}{\partial y}[v_2 p] - \delta p - G(u) \\ \iint_{\mathcal{R}} c(x,y,t)h(x,y,t) + u(x,y,t)\,dx\,dy = BH(t)^\alpha \\ H(t) = \iint_{\mathcal{R}} \phi(p(x,y,t),h(x,y,t),x,y)\,dx\,dy \\ p(x,y,0) = p_0(x,y) \qquad , (x,y) \in \mathbb{R}^2, \\ \lim\limits_{x \to \partial R_X} p_x = \lim\limits_{y \to \partial R_Y} p_y = 0 \qquad , \forall t \in [0,T] \end{cases} \tag{11}$$

where $\psi(\cdot)$ is a bounded and continuously differentiable function with $\frac{\partial \psi}{\partial p} = \psi_p \leq 0$ and $\frac{\partial^2 \psi}{\partial p^2} = \psi_{pp} \geq 0$. This second integral term in (10) can be interpreted as the total cost of cleaning the remaining amount of pollution at T, as for instance in Augeraud-Véron et al. (2021), or as the concern of the policy maker for the pollution left behind at the end of the planning period (Camacho and Pérez-Barahona (2015)). Then, the set of necessary conditions is modified by imposing a terminal condition on the shadow price of pollution:

$$Q(x,y,T) = \psi_p(p(x,y,T),x,y) \quad \forall (x,y) \in \mathcal{R}$$

Proposition 2 thus presents the amended necessary (and sufficient) optimal conditions for the finite time optimal control problem $\mathcal{P}_T$.

Proposition 2. *The Pontryagin conditions of the problem with finite time horizon $T < \infty$ are given by:*

$$\begin{cases} p_t - Dp_{xx} - Dp_{yy} - D_x p_x - D_y p_y = \theta B H(t)^\alpha f(x,y,t) - (v_1 p_x + v_2 p_y) - p(v_{1x} + v_{2y}) - \delta p - G(u) \\ Q_t + DQ_{xx} + DQ_{yy} + Q_x[D_x + v_1] + Q_y[D_y + v_2] \\ \qquad + \Big[\mathcal{U}'(c(x,y,t)) + \theta \iint_{\mathcal{R}} Q(x',y',t) f(x',y')\, dx'\, dy'\Big] \alpha B H^{\alpha-1} \phi_p(p,h) = (\delta + \rho) Q(x,y,t), \\ Q(x,y,t) = -\frac{\mathcal{U}'(c(x,y,t))}{G'(u(x,y,t))}, \\ \iint_{\mathcal{R}} c(x,y,t) h(x,y,t) + u(x,y,t)\, dx\, dy = B H(t)^\alpha \\ H(t) = \iint_{\mathcal{R}} \phi(p,h)\, dx\, dy \\ p(x,y,0) = p_0(x,y), \lim\limits_{x \to \partial R_X} p_x(x,y,t) = \lim\limits_{y \to \partial R_Y} p_y(x,y,t) = 0, \\ Q(x,y,T) = \psi_p(p(x,y,T),x,y), \lim\limits_{x \to \partial R_X} Q_x(x,y,t) = \lim\limits_{y \to \partial R_Y} Q_y(x,y,t) = 0, \end{cases} \tag{12}$$

for $(x,y,t) \in R \times [0,T]$.

The proof of existence and uniqueness of an optimal solution for the problem in finite time are not replicated in this work, but we can build on the results from Augeraud-Véron et al. (2019b, 2021), Camacho et al. (2008) or Camacho and Pérez-Barahona (2015), for similar problems and with the technical assumptions presented in Section 2, to ensure that there exists at least a solution to the social optimum.

4.2 Existence of a stationary solution

If a time-invariant solution $(\overline{p}, \overline{q})$ exists, that is if an equilibrium where all variables do not change over time exists, then it verifies the following system:

$$\mathcal{S} \begin{cases} -D(p_{xx} + p_{yy}) - D_x p_x - D_y p_y + v_1 p_x + v_2 p_y + p(v_{1x} + v_{2y}) = \theta B H^\alpha f(x,y) - \delta p(x,y) - G(u(x,y)) \\ D(Q_{xx} + Q_{yy}) + D_x Q_x + D_y Q_y + v_1 Q_x + v_2 Q_y = -\Big[\mathcal{U}'(c) + \theta \iint_{\mathcal{R}} Q(x',y',t) f(x',y')\, dx'\, dy'\Big] \alpha B H^{\alpha-1} \phi_p(p,h) \\ \qquad\qquad + (\delta + \rho) Q(x,y) \end{cases}$$

where $H = \iint_{\mathcal{R}} \phi(p,h,x,y)\, dx\, dy$, and $u(x,y)$ is the unique solution to the condition

$$\mathcal{U}'(c) = -Q(x,y) G'(u(x,y))$$

with $c = BH^\alpha - \iint_{\mathcal{R}} u(x,y)\, dx\, dy$.

Following Camacho and Pérez-Barahona (2015), we can derive sufficient conditions for existence and uniqueness of the solution to system $\mathcal{S}_D$, denoting the system $\mathcal{S}$ with $V = 0$, that is, considering diffusion only. The case with advection requires more involved mathematical analysis and is out of the scope of this project for the time being.

Proposition 3. *Given a spatial population distribution $h(x,y,t)$, we define a set Z of time-invariant functions*

$$Z = \{(\overline{p}, \overline{q}) : \phi_{pp}(\overline{p}, h) \geq 0 \text{ and } G(u) \leq \theta B \overline{H}^\alpha f(x,y)\}.$$

Then, under previous assumptions, the solution $(\overline{p}, \overline{q})$ to system $\mathcal{S}_D$ is unique in set Z.

The first condition indicates that the damage function ϕ exhibits diminishing marginal damages: holding everything else constant, as pollution increases, it decreases health by less and

less. The condition $G(u) \leq \theta BH^{\alpha} f(x,y)$ states that, in equilibrium, in each period of time and in in each location, the amount of abated pollution is not higher than the quantity of pollution newly emitted at this point in space and time, given the abatement and production technologies. This is actually quite restrictive for the spatial profile of abatement expenses, as, in our model, zones with $\theta BH^{\alpha} f(x,y) \geq 0$ do not necessarily span the entire region, and in particular they do not necessarily overlap with residential areas where it could be more interesting to undertake pollution cleaning than in sparsely populated industrial areas.
In any case, note that we cannot guarantee the non-emptiness of set Z, nor have we established whether the trajectories converge to this unique equilibrium in Z if it exists. The complexity of the Pontryagin conditions for this problem prevents us to characterize analytically an equilibrium or even optimal paths. Therefore, in the next section we will use numerical methods to solve and simulate the optimal control problem and explore spatio-temporal transition dynamics and, notably, the convergence to a potential time-invariant equilibrium. In this case, the conditions presented in Proposition 3 may be helpful to conclude about the uniqueness of the time-invariant equilibrium unveiled in the numerical simulations for the particular scenario.

5 Numerical solutions to the optimal control problem

5.1 Resolution algorithm

The numerical algorithm aims at solving (12) and it is adapted to our model specification from Camacho et al. (2008); Camacho and Pérez-Barahona (2015). It is first helpful to rewrite the Pontryagin conditions of the finite time problem with the change of variable $\bar{t} = T - t$, that is, reversing time in the adjoint equation, turning it into a forward problem with an initial condition:

$$\begin{cases} p_t - Dp_{xx} - Dp_{yy} - D_x p_x - D_y p_y = \theta BH(t)^{\alpha} f(x,y,t) - (v_1 p_x + v_2 p_y) - p(v_{1x} + v_{2y}) - \delta p - G(u) \\ Q_t - DQ_{xx} - DQ_{yy} - Q_x[D_x + v_1] - Q_y[D_y + v_2] = \\ \qquad \Big[\mathcal{U}'(c(\bar{t})) + \theta \iint_{\mathcal{R}} Q(x',y',t) f(x',y')\, dx'\, dy'\Big] \alpha BH^{\alpha-1} \phi_p(p(x,y,\bar{t}),h) - (\delta+\rho) Q(x,y,t), \\ Q(x,y,\bar{t}) = -\frac{\mathcal{U}'(c(x,y,t))}{G'(u(x,y,t))}, \\ c(t) = BH(t)^{\alpha} - \iint_{\mathcal{R}} u(x,y,t)\, dx\, dy \\ H(t) = \iint_{\mathcal{R}} \phi(p,h)\, dx\, dy \\ p(x,y,0) = p_0(x,y), \lim\limits_{x \to \partial R_X} p_x(x,y,t) = \lim\limits_{y \to \partial R_Y} p_y(x,y,t) = 0, \\ Q(x,y,0) = Q_0(x,y) = \psi_p(p(x,y,T),x,y), \lim\limits_{x \to \partial R_X} Q_x(x,y,t) = \lim\limits_{y \to \partial R_Y} Q_y(x,y,t) = 0, \end{cases} \tag{13}$$

for $(x,y,t) \in R \times [0,T]$.

These equations are then discretized and approximated using the finite difference scheme detailed in Appendix A for the pollution PDE.
The algorithm consists in an iterative process starting from initial guesses for the reverse-time shadow price of pollution and for the abatement expenses. The optimal solution to (13) will then be found as the fixed point of this iterative process. The algorithm proceeds as follows:

- **Step 1:** We choose two tolerance parameters ε_1,ε_2 and an initial distribution for pollution $p_0 = \{p_{0_{i,j}}\}_{i,j=1\ldots J}$ where J is the number of grid points in each space dimension. We make an initial guess of a sequence $\{\hat{q}^n_{i,j}\}^{n=1\ldots N}_{i,j=1\ldots J}$ for the adjoint variable, as well as for abatement: $\{\hat{u}^n_{i,j}\}^{n=1\ldots N}_{i,j=1\ldots J}$, where $N = T/\Delta t$. Note that these two guesses uniquely determine the initial consumption matrix as well.

- **Step 2:** As advised in Camacho and Pérez-Barahona (2015), we run an intermediate loop to speed up the convergence of the algorithm by improving the initial guesses for consumption and abatement. Namely, for each time step n, we can compute:

$$c^n = BH^\alpha - \sum_{i,j} \hat{u}^n_{i,j} \quad \text{or, to be more precise,} \quad c^n = \max(0, BH^\alpha - \sum_{i,j} \hat{u}^n_{i,j})$$

using our initial guess for the abatement decisions and the pollution levels of the previous period, that is $\{p^{n-1}_{i,j}\}_{i,j=1...J}$, to compute H^{n-1} which is only an approximation but arguably one without any significant consequence as p, hence H, are continuous so the variation between H^{n-1} and H^n should be infinitesimally small.
Then, the guess for $\{\hat{u}'^n_{i,j}\}_{i,j=1...J}$ is updated using the value for c^n and the following relation:

$$u^n_{i,j} = (G')^{-1}[-\frac{\mathcal{U}'(c^n)}{\hat{q}'^n_{i,j}}]$$

and c^n can be recalculated using the new guess for u and the budget constraint as before. This two computations process is repeated until the euclidean distance between two consecutive outcomes for c^n becomes smaller than ε_1.
Finally, using $\{p^{n-1}_{i,j}\}_{i,j=1...J}$, c^n and $\{u^n_{i,j}\}_{i,j=1...J}$, the values of the pollution prevailing at time n for each point in space $\{p^n_{i,j}\}_{i,j=1...J}$ are computed by applying the Crank-Nicolson simulation scheme presented in the appendix to the pollution PDE, which is the first equation of (13).

- **Step 3:** Once the previous step is completed for all n from 1 to N, we input the values $\{p^n_{i,j}\}^{n=1...N}_{i,j=1...J}$, $\{c^n\}^{n=1...N}$ and $\{u^n_{i,j}\}^{n=1...N}_{i,j=1...J}$ into the parabolic PDE corresponding to the shadow price of pollution Q (i.e. the second equation in (13)) to obtain a new guess for $\{q^n_{i,j}\}^{n=1...N}_{i,j=1...J}$. If the euclidean distance between the new values and the previous guess $\{\hat{q}^n_{i,j}\}^{n=1...N}_{i,j=1...J}$ is larger than ε_2, then Step 2 is iterated with our updated values for q and u as initial guesses.

5.2 The benchmark case

In this section, the above numerical algorithm is applied to the benchmark setting presented in Section 3.1. The spatial setting is identical and the time horizon T is taken large enough not to affect significantly the spatial extent of the problem under consideration, as we will try to do for each case study. The parameter values and the functional form assumptions are listed in Table 2. We run the simulations with $T = 300$, zero initial pollution and the following initial guesses[12]: $\hat{q}^n_{i,j} = -1 \ \forall n, i, j$ and $\hat{u}^n_{i,j} = 0 \ \forall n, i, j$.

Figure 15 compares the optimally planned outcomes to the results of the Solow-type model, in the same benchmark case. Notice first that both the state variables (pollution, hence health) and the controls, consumption and abatement, eventually reach a constant level through time. Given the parameter values, we find that, in the benchmark setting, the policymaker is able to address the economy toward a sustainable outcome by implementing intensive environmental efforts in the first 50 periods of time before reaching a stable situation in which there is still 25% of the total revenue from production being continually allocated to abatement. However, the uniqueness of this seemingly time-invariant solution cannot be ensured as the condition $G(u) \leq \theta B \overline{H}^\alpha f(x, y)$ stated in Proposition 3 is verified almost but not everywhere in $\mathcal{R}$.
In addition to the outcomes at the end of the planning period, the optimal controls also affects the dynamics of the state variables. Accordingly, the health level is improved at any time and

[12]Having repeated the numerical exercises with a range of different initial guesses for q and u, we can be confident that our algorithm is robust with respect to the initial guesses in the sense that it ends up generating the same optimal trajectories in any case.

Table 2: Parameter values, functional assumptions and initial conditions for the optimal control problem in the benchmark case

$\mathcal{R} = [X_1, X_2] \times [Y_1, Y_2]$	$[-4, 4] \times [-4, 4]$	B	10
(x_f, y_f)	$(0, 0)$	α	0.65
(x_h, y_h)	$(1, -1)$	θ	0.02
$\overline{D}$	0.06	$u(x, y, t)$	0
p_l	0	δ	0.025
p_h	5		
$\phi(p(x, y, t), h(x, y, t)) = h(x, y, t)e^{-\gamma p(x,y,t)}$		γ	0.5
$f(x, y, t) = e^{-\frac{(x-x_f)^2+(y-y_f)^2}{2\sigma_f^2}}$		σ_f	0.3
$h(x, y, t) = e^{-\frac{(x-x_h)^2+(y-y_h)^2}{2\sigma_h^2}}$		σ_h	0.5
$\mathcal{U}(c) = \log(1 + c)$		ρ	0.03
$G(u) = Au^\nu$		ν	0.5
		A	0.012
$\psi(p(x, y, T)) = \chi p(x, y, T)$		χ	-5

it stabilizes at a higher steady-state level in the optimal control problem than in the problem without (optimal) environmental policy, and, consequently, so does production. Consumption exhibits a decreasing profile over time, similarly as production, because of the growing stock of pollution, but the change in consumption over time actually remains quite low because when the production is very high initially (as we start from zero initial pollution), abatement expenses are high as well as it is shown in the third panel of Figure 15, meaning that consumption starts off at a moderate rate, not so high above its steady-state level.

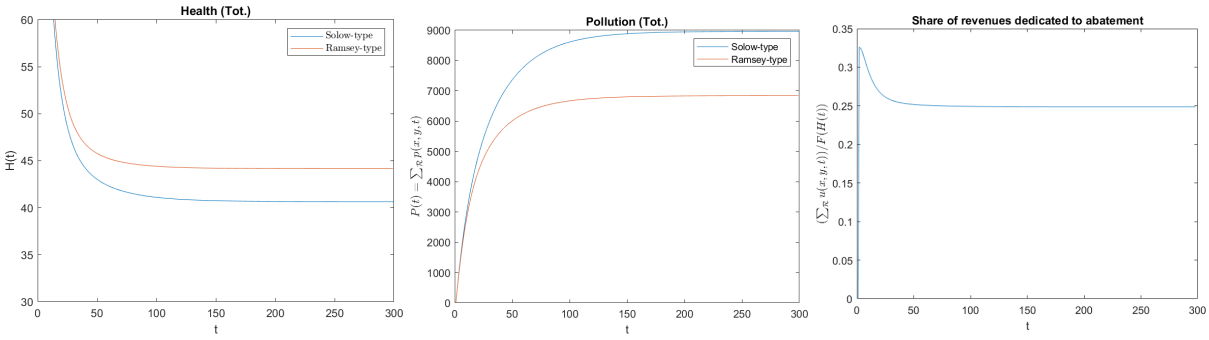


Figure 15: Evolution over time of aggregate healthy human capital, total pollution stock and share of the total revenue dedicated to abatement in the benchmark case with optimal consumption and abatement policies (Ramsey-type model) or without optimization and abatement (Solow-type).

Moreover, we can explore the spatial distribution of the environment abatement decisions, shown for different scenarios and time steps in Figure 16. We find that $u(x, y, t)$ is characterized by quite a strong spatial homogeneity in our benchmark case with homogeneous technology (see

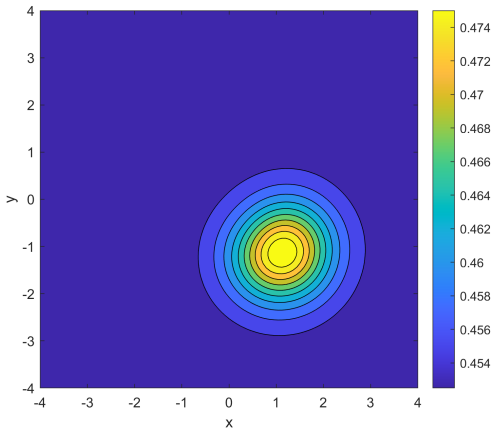
18a and 18b[13]). The spatial variance of $u(x, y, t)$ in fact exhibits a similar profile over time as the sum of abatement expenses : in the first 50 periods the "extra" revenue dedicated to abatement, above the time-invariant level, is primarily allocated across space according to the location of potential damages described by h, the population density; then, over time, this mild concentration pattern dissipates and the spatial variance of $u(x, y, t)$ goes to zero.
To investigate further the drivers of the distribution of abatement across space, Figures 16c and 18b replicate the previous one in a scenario in which the efficiency $A(x, y)$ of the cleaning technology described by $G(u(x, y, t)) = A(x, y)u(x, y, t)^{\nu}$ is heterogeneous across space, and more specifically it decreases as we get away from the center of the region $(0, 0)$. In this case, abatement expenses are heterogeneous across space, concentrating where the efficiency of cleaning pollution is higher, without regards for the population distribution. It thus seem that the distribution of $u(x, y, t)$ is driven by the total abatement potentially achievable more than by where health improvements could benefit a larger population. However, the total health gains in this second scenario, where abatement efforts are concentrated, are lower than in the first one, where abatement is spread out across the whole region, due to the decreasing marginal returns embedded in function G. It is still an encouraging finding that if, on the contrary, the most populated area happens to be the one with the best technology as well, the impact of abatement activities on global health will be larger as they are concentrated in this highly developed and dense pole.

5.3 Geographical case studies

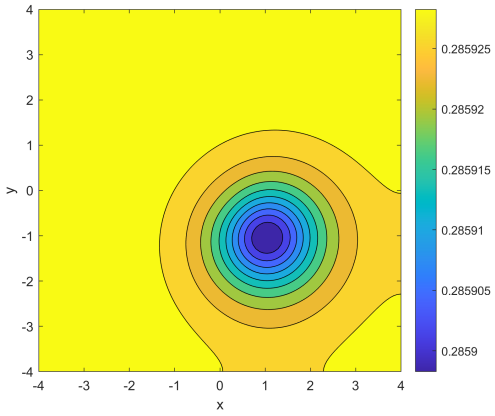
We apply the same numerical algorithm to solve the optimal control problem in the spatial settings presented in Section 3.2. The results in either of these case studies are in reality similar to the benchmark case to a surprising extent. Figure 17 plots, for the three case studies, the evolution of total health in the optimal control problem as well as the evolution of the revenue being dedicated to abatement activities (total amount and share of total revenue). Interestingly, the observation that around 25 to 30% of production revenues are allocated to abatement in steady-state seems to be robust to the spatial setting, while the fact that a time-invariant equilibrium is reached means that optimal policy allowed the economy to follow a sustainable development path. Furthermore, notice that the difference in total abatement expenses are to be related to differences in financial capacity: the society described in *Case study 2* has the lowest health level, hence the lowest production level, which can explain that it also spends less in total for abatement. However, we can observe that this economy not only produces less but devotes a relatively lower budget share to abatement than what is observed in the other examples. As we will get back to at the end of this section, this may be a consequence of the convexity of the damage function (with $\phi_{pp} > 0$) which entails that the marginal damage of pollution goes down with the prevailing level of pollution. Therefore, in a situation characterized by a spatial concentration of pollution, thus summing up to high levels (as a consequence of the surrounding mountain ranges in the case of Grenoble), it becomes less profitable, in terms of the marginal improvement in health one can hope for, to divert resources away from consumption in order to engage in abatement in comparison to less polluted economies, which is a questionable outcome stemming from the model specification.

The magnitude of the global health gains with optimally planned policies in comparison to the outcomes of the Solow-type model is of the same order in every case, although, in each example, the abatement activities seem to modify the outcomes of some areas more than others, as illustrated in Figure 18. Note that, as in the benchmark case, the abatement expenses exhibit a strong homogeneity across space, except in the very beginning of the time period where it is slightly higher in more populated areas. Nonetheless, we still observe that the optimal policy

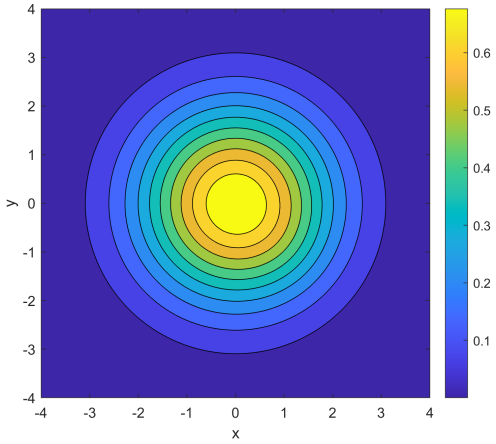
[13]The color variations in these figures may be misleading but note that they actually span a very small range of values



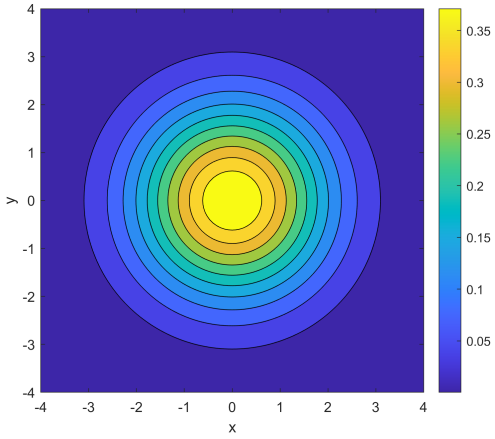
(a) $u(x, y, t = 10)$ with homogeneous cleaning efficiency.



(b) $u(x, y, t = T)$ with homogeneous cleaning efficiency.



(c) $u(x, y, t = 10)$ with heterogeneous cleaning efficiency.



(d) $u(x, y, t = T)$ with heterogeneous cleaning efficiency.

Figure 16: Spatial distribution of abatement costs $u(x, y, t)$ in the benchmark case for the abatement technology $G(u(x, y, t)) = A(x, y)u(x, y, t)^{\nu}$ with homogeneous $A(x, y) = 0.012$ or heterogeneous $A(x, y) = 0.012 \times e^{-\frac{x^2+y^2}{2*2^2}}$.

benefits some areas more than other, such as Voiron, the city outside of the Grenoble valley, or Bhongiri, the municipality which is far away downstream in our third example, while the effects are sizeable for the core and the periphery in the first example. Again this does not arise from spatially differentiated policies, but from the shape of the damage function and from the impacts on the transport process. Regarding the latter point, notice in Figure 18c how the abatement policy drastically reduces pollution downstream in Bhongiri, changing not only the level but the dynamics of health over time: this is partly due to the local cleaning, but also to the reduction of pollution accruing from the river stream, due to the abatement efforts undertaken upstream. On the other hand, the functional form of the damage function (which is such that $\phi_p(p, h) < 0$ and $\phi_{pp}(p, h) > 0$) implies that a unit of pollution is more harmful if it adds up to a low stock of pollution than in a highly polluted area. Consequently, under homogeneous abatement efficiency, if the same amount u is allocated to two points in space with the exact same population density, it will have a larger positive effect on health in the location that was initially the least polluted. This last comment may be a potential explanation to understand the puzzling finding that abatement optimal policies end up being almost perfectly homogeneous across space, despite the significant spatial heterogeneity that characterizes the production and population distributions, as well as the dispersal of pollution across time, in every example that we have studied. The alternative justification suspected by the author is an inadequate

specification of the abatement function G, implying a very narrow range for u, hence making the differences across space seemingly negligible. This will thus be the subject of continuing research.

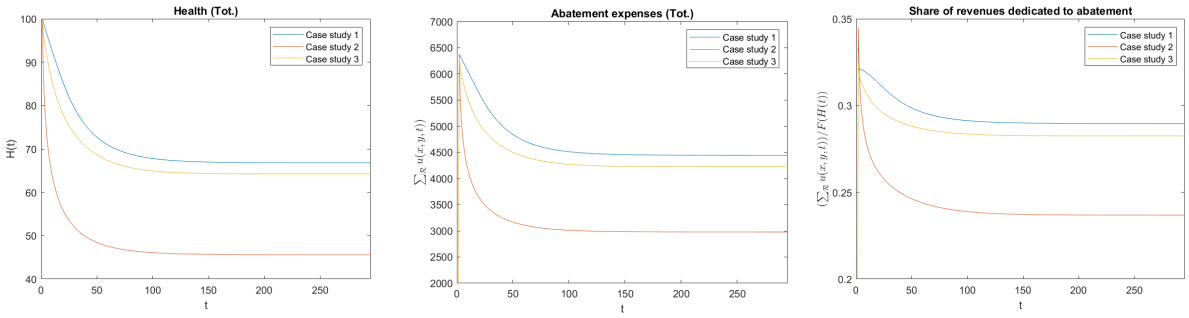
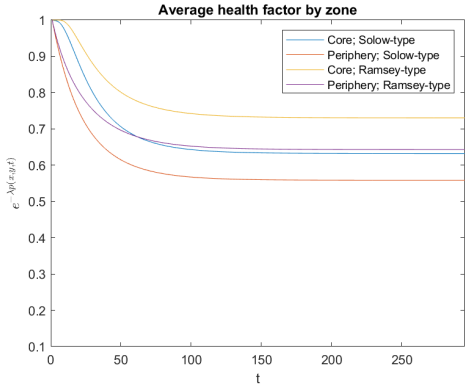
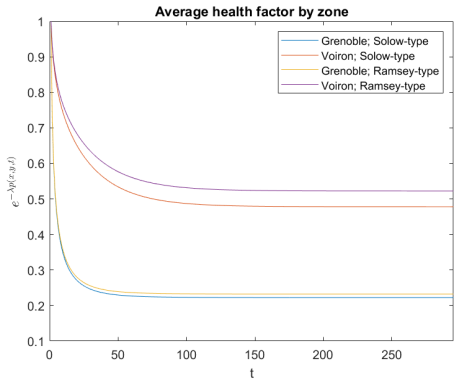


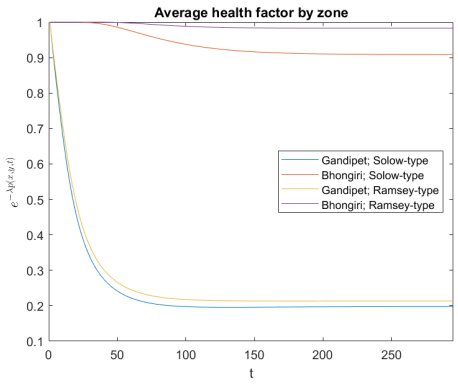
Figure 17: Evolution over time of aggregate healthy human capital and of the revenue dedicated to abatement (in terms of amount and in terms of share of total production) in our three case studies, with optimal consumption and abatement policies (Ramsey-type model)



(a) *Case study 1*



(b) *Case study 2*



(c) *Case study 3*

Figure 18: Evolution of the average health discount factor $e^{-\lambda p(x,y,t)}$ prevailing in various regions with and without optimal policy, denoted respectively by Ramsey-type and Solow-type problems (a result equal to 1 indicating zero pollution).

6 Conclusion

This master's thesis project has encompassed plural objectives with theoretical and technical contributions. A model was developed to study the relationship of industrial production and pollution with health externalities in continuous time and two-dimensional space and the role of geographical factors in this process. Appropriate numerical tools were then elaborated to provide insightful illustrations of the behavior of the model, including the role of relevant spatial features such as distance, agglomeration, natural pollution decay and advective transport, which constitutes a novelty of this work with regards to the spatial economic growth literature, in addition to the consideration of two space dimensions and spatially variable coefficients, in a constant search for realism. The analysis of the spatio-temporal dynamics of production, pollution and health was undertaken in a benchmark schematic framework before being applied to more stylized geographical settings representing popular economic geography models (*Case study 1*) or real world contexts in their specific economic and ecological environment (*Case studies 2 & 3*). These examples come together to show how geography matters for the sustainable development issue as well as for social inequalities with spatial characteristics, and it all the more emphasizes the applicability of the model to a variety of problems and spatial configurations.
An extension of the model into an optimal control problem with consumption and abatement optimal policies was then studied. First order necessary optimality conditions were analytically derived and the optimal control problem was numerically solved in the previously defined example settings, allowing for the comparison of optimally planned outcomes to the baseline outcomes without optimization. This exercise suggests that spatially dependent policies are essential to enhance welfare and achieve global environmental objectives. As previously discussed, the results of the optimal control problems may however seem confounding and the eventual improvement of its specification will be the subject of further research. Nevertheless, the mere fact that we are able to obtain results, despite the technical issues induced by the two-dimensional heterogeneous space structure and the resulting complexity of the optimality conditions, even more in scenarios with very different spatial settings and pollution transport characteristics, remains a very encouraging outcome. It proves that the economic–ecological modeling framework proposed throughout this dissertation indeed allows to represent realistic situations and shows flexibility to be potentially tailored in more details to pollution problems like the Grenoble or Hyderabad contexts and to provide a thoroughly calibrated empirical study. Applications will be the subject of continuing research, and constitutes perhaps the most striking originality of this work with regards to the related spatial economic growth literature. Of course, the interest in representing relevant, meaningful, but highly-stylized scenarios is of course to be traded-off with technical complexity and generalizability.

In terms of other future research prospects, the most appealing or at least more obvious extension of the model is to consider population spatio-temporal dynamics in addition to that of pollution, including potentially population growth, mortality and mobility. It would be particularly interesting for instance to consider irreversibility in the impact of pollution on health, including long-term effects on life expectancy of the population in the vein of Marsiglio et al. (2019) for example. A second set of considerations revolve around the role of the policy maker in this setting and the relevance of various policy options. This framework could be exploited to explore a variety of spatially differentiated optimal policies to deal with the spatial externalities generated by production, from simple taxation schemes to optimal land settlement regulations or territorial development policies (see for instance an interesting take on the optimal design of buffer zones in Augeraud-Véron et al. (2021)). The possibility of decentralized policy implementation and the appropriate level of governance (local decision-makers versus a global policy-maker) are also, of course, popular questions of interest in the literature.

References

S. Aniţa and V. Capasso. A stabilization strategy for a reaction–diffusion system modelling a class of spatially structured epidemic systems (think globally, act locally). *Nonlinear Analysis: Real World Applications*, 10(4):2026–2035, Aug. 2009.

S. Aniţa, V. Capasso, H. Kunze, and D. La Torre. Optimal control and long-run dynamics for a spatial economic growth model with physical capital accumulation and pollution diffusion. *Applied Mathematics Letters*, 26(8):908–912, Aug. 2013.

S. Aniţa, V. Capasso, H. Kunze, and D. La Torre. Dynamics and optimal control in a spatially structured economic growth model with pollution diffusion and environmental taxation. *Applied Mathematics Letters*, 42:36–40, Apr. 2015.

E. Augeraud-Véron, C. Choquet, and Comte. Optimal Control for a Groundwater Pollution Ruled by a Convection–Diffusion–Reaction Problem. *Journal of Optimization Theory and Applications*, 173(3):941–966, June 2017.

E. Augeraud-Véron, R. Boucekkine, and V. M. Veliov. Distributed optimal control models in environmental economics: a review. *Mathematical Modelling of Natural Phenomena*, 14(1): 106, 2019a.

E. Augeraud-Véron, C. Choquet, and Comte. Existence, uniqueness and asymptotic analysis of optimal control problems for a model of groundwater pollution. *ESAIM: Control, Optimisation and Calculus of Variations*, 25:53, 2019b.

E. Augeraud-Véron, C. Choquet, and Comte. Optimal buffer zone for the control of groundwater pollution from agricultural activities. *Applied Mathematics Optimization*, 84, 08 2021.

R. Boucekkine, C. Camacho, and B. Zou. Bridging the Gap between Growth Theory and the New Economic Geography: the Spatial Ramsey Model. *Macroeconomic Dynamics*, 13(1): 20–45, Feb. 2009.

R. Boucekkine, C. Camacho, and G. Fabbri. Spatial dynamics and convergence: The spatial AK model. *Journal of Economic Theory*, 148(6):2719–2736, Nov. 2013a.

R. Boucekkine, C. Camacho, and F. Giorgio. On the optimal control of some parabolic partial differential equations arising in economics. *Serdica Mathematical Journal*, 39(3):331–354, 2013b.

R. Boucekkine, G. Fabbri, S. Federico, and F. Gozzi. Growth and agglomeration in the heterogeneous space: a generalized AK approach. *Journal of Economic Geography*, 19(6):1287–1318, 08 2018.

R. Boucekkine, G. Fabbri, S. Federico, and F. Gozzi. Geographic environmental Kuznets curves: the optimal growth linear-quadratic case. *Mathematical Modelling of Natural Phenomena*, 14 (1):105, 2019.

P. Brito. The Dynamics of Growth and Distribution in a Spatially Heterogeneous World. Feb. 2004.

W. Brock and A. Xepapadeas. Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control. *Journal of Economic Dynamics and Control*, 32(9):2745–2787, 2008.

W. Brock and A. Xepapadeas. Pattern formation, spatial externalities and regulation in coupled economic–ecological systems. *Journal of Environmental Economics and Management*, 59(2): 149 – 164, 2010.

W. Brock, A. Xepapadeas, and A. Yannacopoulos. Optimal Control in Space and Time and the Management of Environmental Resources. *Annual Review of Resource Economics*, 6(1): 33–68, Nov. 2014.

W. A. Brock and M. S. Taylor. The Green Solow model. *Journal of Economic Growth*, 15(2): 127–153, 2010.

C. Camacho and A. Pérez-Barahona. Land use dynamics and the environment. *Journal of Economic Dynamics and Control*, 52:96–118, Mar. 2015.

C. Camacho, B. Zou, and M. Briani. On the dynamics of capital accumulation across space. *European Journal of Operational Research*, 186(2):451–465, Apr. 2008.

V. Capasso, R. Engbers, and D. La Torre. On a spatial Solow model with technological diffusion and nonconcave production function. *Nonlinear Analysis: Real World Applications*, 11(5): 3858–3876, Oct. 2010.

A. Cornet. *Dynamics and interactions in natural resources economics: land use, soil pollution and renewable energy*. PhD thesis, 2020. Thèse de doctorat dirigée par Mandel, Antoine Économie Paris 1 2020.

J. de Frutos and G. Martín-Herrán. Spatial effects and strategic behavior in a multiregional transboundary pollution dynamic game. *Journal of Environmental Economics and Management*, 97:182–207, Sept. 2019.

K. Desmet and E. Rossi-Hansberg. On the spatial economic impact of global warming. *Journal of Urban Economics*, 88:16–37, July 2015.

I. Ekeland. On the variational principle. *Journal of Mathematical Analysis and Applications*, 47 (2):324–353, 1974.

G. Fabbri. Geographical structure and convergence: A note on geometry in spatial growth models. *Journal of Economic Theory*, 162:114–136, Mar. 2016.

J. Fick, H. Söderström, R. H. Lindberg, C. Phan, M. Tysklind, and D. G. J. Larsson. Contamination of surface, ground, and drinking water from pharmaceutical production. *Environmental Toxicology and Chemistry*, 28(12):2522–2527, 2009.

J. Friedmann. *Regional development policy: a case study of Venezuela*. M.I.T. Press, Cambridge, Mass., 1966.

P. Krugman. *Geography and Trade*. Gaston Eyskens Lectures. MIT Press, Cambridge, MA, USA, July 1991.

D. La Torre, D. Liuzzi, and S. Marsiglio. Pollution diffusion and abatement activities across space and over time. *Mathematical Social Sciences*, 78:48–63, Nov. 2015.

D. G. J. Larsson, C. de Pedro, and N. Paxeus. Effluent from drug manufactures contains extremely high levels of pharmaceuticals. *Journal of Hazardous Materials*, 148(3):751–755, Sept. 2007.

C. Lübbert, C. Baars, A. Dayakar, N. Lippmann, A. C. Rodloff, M. Kinzig, and F. Sörgel. Environmental pollution with antimicrobial agents from bulk drug manufacturing industries in Hyderabad, South India, is associated with dissemination of extended-spectrum beta-lactamase and carbapenemase-producing pathogens. *Infection*, 45(4):479–491, Aug. 2017.

A. Marshall. *Principles of Economics*. Palgrave Macmillan UK, Basingstoke, 1920.

S. Marsiglio, D. La Torre, and D. Liuzzi. Population and Geography Do Matter for Sustainable Development. *Environment and Development Economics*, 24:201–223, Apr. 2019.

C. V. Pao. *Nonlinear parabolic and elliptic equations*. The Language of science. Plenum Press, New York, 1992.

V. M. Veliov. Optimal control of heterogeneous systems: Basic theory. *Journal of Mathematical Analysis and Applications*, 346(1):227–242, Oct. 2008.

B. Zou and C. Camacho. The spatial Solow model. *Economics Bulletin*, 18:1–11, Mar. 2004.

Appendices

A Numerical scheme for PDE solving

We numerically solve our pollution PDE using a Crank–Nicolson finite difference approximation. The idea underlying this method is to replace the second derivatives with respect to space with an average of the FTCS (Forward Time Centered Space) explicit and implicit schemes, and replace the derivative with respect to time with a forward difference in time. This is a second-order method in space and time, implicit in time, and it is known to be unconditionally stable and highly accurate.
For this purpose, we set up a rectangular, equidistant grid for the space domain, with $(J+1) \times (J+1)$ grid points and mesh size $h = \Delta x = \Delta y = \frac{1}{J+1}$, such that $\mathcal{R}_h := \{(ih, jh) : i, j = 1, ..., J-1\}$. Then, if v is a function defined on the grid, we denote $v(ih, jh, n\Delta t) = v^n_{i,j}$.

$$\begin{aligned} p_t &= \theta F(H) f(x,y,t) + \nabla \cdot (D \nabla p) - \nabla \cdot (Vp) - \delta p - G(u) \\ &= \theta F(H) f(x,y,t) + D\nabla^2 p + (\nabla \cdot D)(\nabla \cdot p) - V^T(\nabla p) - p(\nabla \cdot V) - \delta p - G(u) \\ &= \theta F(H) f(x,y,t) + D(p_{xx} + p_{yy}) + D_x p_x + D_y p_y - (v_1 p_x + v_2 p_y) - p(v_{1x} + v_{2y}) - \delta p - G(u) \end{aligned}$$

Then, we will use second-order centred-space differencing to obtain a discrete approximation of the Laplace operators inside the grid, namely:

$$\begin{aligned} \nabla^2 p(x,y,t) &= p_{xx} + p_{yy} \\ &= \frac{1}{2h^2}[(p^{n+1}_{i+1,j} + p^{n+1}_{i-1,j} + p^{n+1}_{i,j-1} + p^{n+1}_{i,j+1} - 4p^{n+1}_{i,j}) + (p^n_{i+1,j} + p^n_{i-1,j} + p^n_{i,j-1} + p^n_{i,j+1} - 4p^n_{i,j})] \end{aligned}$$

and first-order centered differencing[14] to approximate the spatial derivatives:

$$\frac{\partial p(x,y,t)}{\partial x} = p_x = \frac{1}{2h}[\frac{p^n_{i+1,j} - p^n_{i-1,j}}{2} + \frac{p^{n+1}_{i+1,j} - p^{n+1}_{i-1,j}}{2}]$$

Similarly, the space derivatives of D, v_1 and v_2 are approximated by central differences. For instance, we compute: $D^{i,j}_x = \frac{D^{i+1,j} - D^{i-1,j}}{2h}$.

With the Crank–Nicolson discretization, the finite difference analog to our PDE problem can then be written:

$$\begin{cases} \frac{p^{n+1}_{i,j} - p^n_{i,j}}{\Delta t} = \frac{D^{i,j}}{2h^2}[(p^{n+1}_{i+1,j} + p^{n+1}_{i-1,j} + p^{n+1}_{i,j-1} + p^{n+1}_{i,j+1} - 4p^{n+1}_{i,j}) + (p^n_{i+1,j} + p^n_{i-1,j} + p^n_{i,j-1} + p^n_{i,j+1} - 4p^n_{i,j})] \\ \qquad + \frac{D^{i,j}_x - v_1^{i,j}}{4h}[(p^n_{i+1,j} - p^n_{i-1,j}) + (p^{n+1}_{i+1,j} - p^{n+1}_{i-1,j})] + \frac{D^{i,j}_y - v_2^{i,j}}{4h}[(p^n_{i,j+1} - p^n_{i,j-1}) + (p^{n+1}_{i,j+1} - p^{n+1}_{i,j-1})] \\ \qquad + \theta B H(n\Delta t)^\alpha f(ih, jh, t) - p^n_{i,j}(\delta + v_1{}^{i,j}_x + v_2{}^{i,j}_y) - G(u^n_{i,j})) \\ \text{with } H(t) = \sum_{(x,y)} \phi(p(x,y,t), h(x,y,t), x, y), \\ p^0_{i,j} \text{ given}, \\ p(X_1 - h, jh, n\Delta t) = p(X_1 + h, jh, n\Delta t), p(X_2 - h, jh, n\Delta t) = p(X_2 + h, jh, n\Delta t) \quad \forall n \text{ and } \forall j = 0, ..., J, \\ p(ih, Y_1 - h, n\Delta t) = p(ih, Y_1 + h, n\Delta t), p(ih, Y_2 - h, n\Delta t) = p(ih, Y_2 + h, n\Delta t) \quad \forall n \text{ and } \forall i = 0, ..., J \end{cases}$$

Note that the last expressions correspond the Neumann boundary conditions, while the first equation only holds inside the grid, that is in $R_h := \{(ih, jh) : i, j = 1, ..., J-1\}$ and it defines a linear system of $(J-1) \times (J-1)$ equations with $(J+1) \times (J+1)$. The boundary conditions

[14] We will however prefer to use an upwind differencing scheme ($p_x = \frac{1}{h}[\frac{p^n_{i,j} - p^n_{i-1,j}}{2} + \frac{p^{n+1}_{i,j} - p^{n+1}_{i-1,j}}{2}]$) for advection-dominated equations, in which case it is less accurate but more stable than centered differencing, provided that the Courant–Friedrichs–Lewy holds: $\frac{max(V)\Delta t}{h} \leq 1$.

supply the four missing equations.
The first equation can be rearranged as such:

$$\begin{aligned}
p_{i,j}^{n+1} &= p_{i,j}^n + \frac{D^{i,j}\Delta t}{2h^2}[(p_{i+1,j}^{n+1} + p_{i-1,j}^{n+1} + p_{i,j-1}^{n+1} + p_{i,j+1}^{n+1} - 4p_{i,j}^{n+1}) + (p_{i+1,j}^n + p_{i-1,j}^n + p_{i,j-1}^n + p_{i,j+1}^n - 4p_{i,j}^n)] \\
&\quad + \frac{(D_x^{i,j} - v_1^{i,j})\Delta t}{4h}[(p_{i+1,j}^n - p_{i-1,j}^n) + (p_{i+1,j}^{n+1} - p_{i-1,j}^{n+1})] + \frac{(D_y^{i,j} - v_2^{i,j})\Delta t}{4h}[(p_{i,j+1}^n - p_{i,j-1}^n) + (p_{i,j+1}^{n+1} - p_{i,j-1}^{n+1})] \\
&\quad + \Delta t[\theta BH(n\Delta t)^\alpha f(ih, jh, t) - p_{i,j}^n(\delta + {v_1}_x^{i,j} + {v_2}_y^{i,j}) - G(u_{i,j}^n))] \\
\iff (1 + 2\mu^{i,j})p_{i,j}^{n+1} &- \frac{\mu^{i,j} + \alpha^{i,j}}{2}p_{i+1,j}^{n+1} - \frac{\mu^{i,j} - \alpha^{i,j}}{2}p_{i-1,j}^{n+1} - \frac{\mu^{i,j} + \beta^{i,j}}{2}p_{i,j+1}^{n+1} - \frac{\mu^{i,j} - \beta^{i,j}}{2}p_{i,j-1}^{n+1} = \\
&\quad (1 - 2\mu - \delta\Delta t)p_{i,j}^n + \frac{\mu^{i,j} + \alpha^{i,j}}{2}p_{i+1,j}^n + \frac{\mu^{i,j} - \alpha^{i,j}}{2}p_{i-1,j}^n + \frac{\mu^{i,j} + \beta^{i,j}}{2}p_{i,j+1}^n + \frac{\mu^{i,j} - \beta^{i,j}}{2}p_{i,j-1}^n \\
&\quad + \Delta t[\theta BH(n\Delta t)^\alpha f(ih, jh, t) - p_{i,j}^n(\delta + {v_1}_x^{i,j} + {v_2}_y^{i,j}) - G(u_{i,j}^n))]
\end{aligned}$$

with $\mu^{i,j} = \frac{D^{i,j}\Delta t}{h^2}$, $\alpha^{i,j} = \frac{(D_x^{i,j} - v_1^{i,j})\Delta t}{2h}$ and $\beta^{i,j} = \frac{(D_y^{i,j} - v_2^{i,j})\Delta t}{2h}$.

The equation can be expressed in matrix-vector form

$$A_h p_h^{n+1} = B_h p_h^n + f_h^n$$

where p_h^{n+1} is the vector

$$p_h^{n+1} := (p_{1,1}^{n+1}, ..., p_{1,J-1}^{n+1}, p_{2,1}^{n+1}, ..., p_{2,J-1}^{n+1}, p_{J-1,1}^{n+1}, ..., p_{J-1,J-1}^{n+1})^T$$

such that the numbering scheme verifies $p_{i,j}^n = p_{(i-1)J+j}^n$ and f_h is defined in a similar way as the vector of the remaining of the right-hand sides: $f_{i,j}^n = \Delta t[\theta BH(n\Delta t)^\alpha f(ih, jh, t) - G(u_{i,j}^n))]$.
Hence, the left-hand side can be rewritten as:

$$(1+2\mu^{i,j})p_{(i-1)J+j}^{n+1} - \frac{\mu^{i,j} + \alpha^{i,j}}{2}p_{iJ+j}^{n+1} - \frac{\mu^{i,j} - \alpha^{i,j}}{2}p_{(i-2)J+j}^{n+1} - \frac{\mu^{i,j} + \beta^{i,j}}{2}p_{(i-1)J+j+1}^{n+1} - \frac{\mu^{i,j} - \beta^{i,j}}{2}p_{(i-1)J+j-1}^{n+1}$$

Then, A_h is the following $(J-1)^2 \times (J-1)^2$ sparse matrix :

$$\begin{pmatrix}
(1+2\mu) & -\frac{\mu+\beta}{2} & \dots & -\frac{\mu+\alpha}{2} & 0 & \dots & \dots & \dots & \dots & 0 \\
-\frac{\mu-\beta}{2} & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \vdots \\
-\frac{\mu-\alpha}{2} & \dots & -\frac{\mu-\beta}{2} & (1+2\mu) & -\frac{\mu+\beta}{2} & \dots & -\frac{\mu+\alpha}{2} & 0 & \dots & 0 \\
0 & -\frac{\mu-\alpha}{2} & \dots & -\frac{\mu-\beta}{2} & (1+2\mu) & -\frac{\mu+\beta}{2} & \dots & -\frac{\mu+\alpha}{2} & \dots & 0 \\
\vdots & & & & & & & & \ddots & \vdots \\
0 & \dots & \dots & \dots & \dots & 0 & -\frac{\mu-\alpha}{2} & \dots & -\frac{\mu-\beta}{2} & (1+2\mu)
\end{pmatrix}$$

where $\mu = \mu^{i,j}$ in row $(i-1)J + j$. Similarly, for the right-hand side matrix, we get:

$$B_h = \begin{pmatrix}
(1-2\mu-\delta\Delta t) & \frac{\mu+\beta}{2} & \dots & \frac{\mu+\alpha}{2} & 0 & \dots & \dots & \dots & \dots & 0 \\
\frac{\mu-\beta}{2} & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \vdots \\
\frac{\mu-\alpha}{2} & \dots & \frac{\mu-\beta}{2} & (1-2\mu-\delta\Delta t) & \frac{\mu+\beta}{2} & \dots & \frac{\mu+\alpha}{2} & 0 & \dots & 0 \\
0 & \frac{\mu-\alpha}{2} & \dots & \frac{\mu-\beta}{2} & (1-2\mu-\delta\Delta t) & \frac{\mu+\beta}{2} & \dots & \frac{\mu+\alpha}{2} & \dots & 0 \\
\vdots & & & & & & & & \ddots & \vdots \\
0 & \dots & \dots & \dots & \dots & 0 & \frac{\mu-\alpha}{2} & \dots & \frac{\mu-\beta}{2} & (1-2\mu-\delta\Delta t)
\end{pmatrix}$$

Then, using these two matrices which can be computed using the initial data of the problem, we can iterate the following recurrence relation to obtain the value of p at any time and each space point:

$$p_h^{n+1} = A_h^{-1}(B_h p_h^n + f_h^n)$$

The previous matrices need, however, to be corrected on the borders to incorporate our Neumann boundary conditions, and ultimately convert our system into a system of equations involving a square matrix, namely, as a system with $(J+1)\times(J+1)$ equations and $(J+1)\times(J+1)$ unknowns.

Applying the finite difference scheme at the boundary points, for instance at $x = X_2$, yields:

$$\begin{aligned}
&\nabla\cdot(D\nabla p) - \nabla\cdot(Vp) = \\
&\quad \frac{D^{J,j}}{2h^2}[(p^{n+1}_{J+1,j} + p^{n+1}_{J-1,j} + p^{n+1}_{J,j-1} + p^{n+1}_{J,j+1} - 4p^{n+1}_{J,j}) + (p^{n}_{J+1,j} + p^{n}_{J-1,j} + p^{n}_{J,j-1} + p^{n}_{J,j+1} - 4p^{n}_{J,j})] \\
&\quad + \frac{D_x^{J,j} - v_1^{J,j}}{4h}[(p^n_{J+1,j} - p^n_{J-1,j}) + (p^{n+1}_{J+1,j} - p^{n+1}_{J-1,j})] + \frac{D_y^{J,j} - v_2^{J,j}}{4h}[(p^n_{J,j+1} - p^n_{J,j-1}) + (p^{n+1}_{J,j+1} - p^{n+1}_{J,j-1})] \\
&\quad = \frac{D^{J,j}}{2h^2}[2p^{n+1}_{J-1,j} + p^{n+1}_{J,j-1} + p^{n+1}_{J,j+1} - 4p^{n+1}_{J,j}) + (2p^{n}_{J-1,j} + p^{n}_{J,j-1} + p^{n}_{J,j+1} - 4p^{n}_{J,j})] \\
&\qquad + \frac{D_y^{J,j} - v_2^{J,j}}{4h}[(p^n_{J,j+1} - p^n_{J,j-1}) + (p^{n+1}_{J,j+1} - p^{n+1}_{J,j-1})]
\end{aligned}$$

using the boundary conditions.
Accordingly, we can define the additional four equations of the system:

$$\begin{cases}
p^{n+1}_{0,j} = p^n_{0,j} + \frac{D^{0,j}\Delta t}{2h^2}[(2p^{n+1}_{1,j} + p^{n+1}_{0,j-1} + p^{n+1}_{0,j+1} - 4p^{n+1}_{0,j}) + (2p^n_{1,j} + p^n_{0,j-1} + p^n_{0,j+1} - 4p^n_{0,j})] \\
\qquad\qquad +\Delta t\frac{D_y^{0,j} - v_2^{0,j}}{4h}[(p^n_{0,j+1} - p^n_{0,j-1}) + (p^{n+1}_{0,j+1} - p^{n+1}_{0,j-1})] + f^n_{0,j} \\
p^{n+1}_{J,j} = p^n_{J,j} + \frac{D^{J,j}\Delta t}{2h^2}[2p^{n+1}_{J-1,j} + p^{n+1}_{J,j-1} + p^{n+1}_{J,j+1} - 4p^{n+1}_{J,j}) + (2p^n_{J-1,j} + p^n_{J,j-1} + p^n_{J,j+1} - 4p^n_{J,j})] \\
\qquad\qquad +\Delta t\frac{D_y^{J,j} - v_2^{J,j}}{4h}[(p^n_{J,j+1} - p^n_{J,j-1}) + (p^{n+1}_{J,j+1} - p^{n+1}_{J,j-1})] + f^n_{J,j} \\
p^{n+1}_{i,0} = p^n_{i,0} + \frac{D^{i,0}\Delta t}{2h^2}[(2p^{n+1}_{i,1} + p^{n+1}_{i-1,0} + p^{n+1}_{i+1,0} - 4p^{n+1}_{i,0}) + (2p^n_{i,1} + p^n_{i-1,0} + p^n_{i+1,0} - 4p^n_{i,0})] \\
\qquad\qquad +\Delta t\frac{D_x^{i,0} - v_1^{i,0}}{4h}[(p^n_{i+1,0} - p^n_{i-1,0}) + (p^{n+1}_{i+1,0} - p^{n+1}_{i-1,0})] + f^n_{i,0} \\
p^{n+1}_{i,J} = p^n_{i,J} + \frac{D^{i,J}\Delta t}{2h^2}[(2p^{n+1}_{i,J-1} + p^{n+1}_{i-1,J} + p^{n+1}_{i+1,J} - 4p^{n+1}_{i,J}) + (2p^n_{i,J-1} + p^n_{i-1,J} + p^n_{i+1,J} - 4p^n_{i,J})] \\
\qquad\qquad +\Delta t\frac{D_x^{i,J} - v_1^{i,J}}{4h}[(p^n_{i+1,J} - p^n_{i-1,J}) + (p^{n+1}_{i+1,J} - p^{n+1}_{i-1,J})] + f^n_{i,J}
\end{cases}$$

$$\iff \begin{cases}
(1+2\mu^{0,j})p^{n+1}_{0,j} - \mu^{0,j}p^{n+1}_{1,j} - \frac{\mu^{0,j}+\beta^{0,j}}{2}p^{n+1}_{0,j+1} - \frac{\mu^{0,j}-\beta^{0,j}}{2}p^{n+1}_{0,j-1} = \\
\qquad (1-2\mu^{0,j} - \delta\Delta t)p^n_{0,j} + \mu^{0,j}p^n_{1,j} + \frac{\mu^{0,j}+\beta^{0,j}}{2}p^n_{0,j+1} + \frac{\mu^{0,j}-\beta^{0,j}}{2}p^n_{0,j-1} + f^n_{0,j} \\
(1+2\mu^{J,j})p^{n+1}_{J,j} - \mu^{J,j}p^{n+1}_{J-1,j} - \frac{\mu^{J,j}+\beta^{J,j}}{2}p^{n+1}_{J,j+1} - \frac{\mu^{J,j}-\beta^{J,j}}{2}p^{n+1}_{J,j-1} = \\
\qquad (1-2\mu^{J,j} - \delta\Delta t)p^n_{J,j} + \mu^{J,j}p^n_{J-1,j} + \frac{\mu^{J,j}+\beta^{J,j}}{2}p^n_{J,j+1} + \frac{\mu^{J,j}-\beta^{J,j}}{2}p^n_{J,j-1} + f^n_{J,j} \\
(1+2\mu^{i,0})p^{n+1}_{i,0} - \mu^{i,0}p^{n+1}_{i,1} - \frac{\mu^{i,0}+\alpha^{i,0}}{2}p^{n+1}_{i+1,0} - \frac{\mu^{i,0}-\alpha^{i,0}}{2}p^{n+1}_{i-1,0} = \\
\qquad (1-2\mu^{i,0} - \delta\Delta t)p^n_{i,0} - \mu^{i,0}p^n_{i,1} - \frac{\mu^{i,0}+\alpha^{i,0}}{2}p^n_{i+1,0} - \frac{\mu^{i,0}-\alpha^{i,0}}{2}p^n_{i-1,0} + f^n_{i,0} \\
(1+2\mu^{i,J})p^{n+1}_{i,J} - \mu^{i,J}p^{n+1}_{i,J-1} - \frac{\mu^{i,J}+\alpha^{i,J}}{2}p^{n+1}_{i+1,J} - \frac{\mu^{i,J}-\alpha^{i,J}}{2}p^{n+1}_{i-1,J} = \\
\qquad (1-2\mu^{i,J} - \delta\Delta t)p^n_{i,J} - \mu^{i,J}p^n_{i,J-1} - \frac{\mu^{i,J}+\alpha^{i,J}}{2}p^n_{i+1,J} - \frac{\mu^{i,J}-\alpha^{i,J}}{2}p^n_{i-1,J} + f^n_{i,J}
\end{cases}$$

So, we redefine our matrix expression $AA_h p_h^{n+1} = BB_h p_h^n + f_h^n$ where p_h^{n+1} is the vector

$$p_h^{n+1} := (p^{n+1}_{0,0}, ..., p^{n+1}_{0,J}, p^{n+1}_{1,0}, ..., p^{n+1}_{1,J}, \dots, p^{n+1}_{J,0}, ..., p^{n+1}_{J,J})^T$$

and f_h is still defined as the vector of the remaining of the right-hand sides, with the same

numbering format: $f^n_{i,j} = \Delta t[\theta BH(n\Delta t)^\alpha f(ih, jh, t) - p^n_{i,j}(\delta + v_1{}^{i,j}_x + v_2{}^{i,j}_y) - G(u^n_{i,j}))]$.

Then, AA_h is the following $(J+1)^2 \times (J+1)^2$ sparse matrix:

$$\begin{pmatrix}
(1+2\mu^{0,0}) & -\mu^{0,0} & \overset{\cdots}{\underset{J-1}{}} & -\mu^{0,0} & \cdots & \cdots & \cdots & \cdots & \cdots \\
-\frac{\mu^{0,1}-\beta^{0,1}}{2} & (1+2\mu) & -\frac{\mu^{0,1}+\beta^{0,1}}{2} & \overset{\cdots}{\underset{J-1}{}} & -\mu^{0,1} & \cdots & \cdots & \cdots & \cdots \\
\overset{\vdots}{}_{J-1} & & & & & & & & \\
\cdots & \cdots & -\mu^{0,J} & (1+2\mu^{0,J}) & 0 & \overset{\cdots}{\underset{J-1}{}} & -\mu^{0,J} & \cdots & \cdots \\
\cdots & -\frac{\mu^{1,0}-\alpha^{1,0}}{2} & \overset{\cdots}{\underset{J-1}{}} & 0 & (1+2\mu^{1,0}) & -\mu^{1,0} & \overset{\cdots}{\underset{J-1}{}} & -\frac{\mu^{1,0}+\alpha^{1,0}}{2} & \cdots \\
0.\overset{\cdots}{}_{J} & -\frac{\mu^{1,1}-\alpha^{1,1}}{2} & \cdots & -\frac{\mu^{1,1}-\beta^{1,1}}{2} & (1+2\mu^{1,1}) & -\frac{\mu^{1,1}+\beta^{1,1}}{2} & \cdots & -\frac{\mu^{1,1}+\alpha^{1,1}}{2} & \cdots \\
\overset{\vdots}{}_{J-1} & & & & & & & & \\
\cdots & -\frac{\mu^{1,J}-\alpha^{1,J}}{2} & \cdots & -\mu^{1,J} & (1+2\mu^{1,J}) & -\mu^{1,J} & \cdots & -\frac{\mu^{1,1}+\alpha^{1,J}}{2} & \cdots \\
\vdots & & & & \vdots & & & & \vdots \\
\cdots & \cdots & \cdots & \cdots & -\mu^{J,J-1} & \overset{\cdots}{\underset{J-1}{}} & -\frac{\mu^{J,J-1}-\beta^{J,J-1}}{2} & (1+2\mu^{J,J-1}) & -\frac{\mu^{J,J-1}+\beta^{J,J-1}}{2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & -\mu^{J,J} & \overset{\cdots}{\underset{J-1}{}} & -\mu^{J,J} & (1+2\mu^{J,J})
\end{pmatrix}$$

and BB_h is modified in the same way. Here again, one can then find the value of p at any time by iterating the following equation: $p^{n+1}_h = AA^{-1}_h(BB_h p^n_h + f^n_h)$.
This adapted Crank–Nicolson method is unconditionnally stable, provided that the source term verifies $(\delta + v_{1x} + v_{2y})\Delta t \leq 2$. Note that this essentially constrains the amount of spatial variation contained in V, and similarly, it was observed in our various simulation exercises that the scheme also requires low enough variation in the diffusion coefficients (that is, low enough D_x and D_y) to be numerically stable.

B Proposition 1 Proof

We use the Ekeland variational principle to obtain the Pontryagin conditions of the problem. Following this procedure, we write the value function $\mathcal{V}$ associated to the problem as a function of c, u, p and H as follows:

$$\begin{aligned}
\mathcal{V}(c,u,p,H) &= \int_0^\infty \iint_{\mathcal{R}} \mathcal{U}(c(x,y,t))h(x,y,t)e^{-\rho t}\,dx\,dy\,dt \\
&- \int_0^\infty \iint_{\mathcal{R}} q(x,y,t)\Big[p_t - \frac{\partial}{\partial x}[Dp_x] - \frac{\partial}{\partial y}[Dp_y] + \frac{\partial}{\partial x}[v_1 p] + \frac{\partial}{\partial y}[v_2 p] - \theta BH(t)^\alpha f(x,y,t) + \delta p + G(u)\Big]\,dx\,dy\,dt \\
&- \int_0^\infty m(t)\Big[H(t) - \iint_{\mathcal{R}} \phi(h,p)\,dx\,dy\Big]\,dt - \int_0^\infty n(t)\Big[\iint_{\mathcal{R}} c(x,y,t)h(x,y,t) + u(x,y,t)\,dx\,dy - BH(t)^\alpha\Big]\,dt
\end{aligned}$$

where q, m and n are auxiliary functions.

Using integration by parts, we re-arrange some integral terms in V :

$$\int_0^\infty \iint_{\mathcal{R}} q(x,y,t)p_t(x,y,t)\,dx\,dy\,dt = \iint_{\mathcal{R}} q(x,y,t)p(x,y,t)\Big|_0^\infty\,dx\,dy - \int_0^\infty \iint_{\mathcal{R}} q_t(x,y,t)p(x,y,t)\,dx\,dy\,dt$$

$$\int_0^\infty \iint_R q(x,y,t)\frac{\partial}{\partial x}[Dp_x]\,dx\,dy\,dt = \int_0^\infty \int_{Y_1}^{Y_2} q(x,y,t)Dp_x(x,y,t)\Big|_{X_1}^{X_2} dy\,dt - \int_0^\infty \iint_R q_x(x,y,t)Dp_x(x,y,t)\,dx\,dy\,dt$$
$$= \int_0^\infty \int_{Y_1}^{Y_2} q(x,y,t)Dp_x(x,y,t)\Big|_{X_1}^{X_2} dy\,dt - \int_0^\infty \int_{Y_1}^{Y_2} q_x(x,y,t)Dp(x,y,t)\Big|_{X_1}^{X_2} dy\,dt$$
$$+ \int_0^\infty \iint_R \frac{\partial}{\partial x}[q_x D]p(x,y,t)\,dx\,dy\,dt$$
$$= \int_0^\infty \int_{Y_1}^{Y_2} q(x,y,t)Dp_x(x,y,t)\Big|_{X_1}^{X_2} dy\,dt - \int_0^\infty \int_{Y_1}^{Y_2} q_x(x,y,t)Dp(x,y,t)\Big|_{X_1}^{X_2} dy\,dt$$
$$+ \int_0^\infty \iint_R [q_{xx}D + q_x D_x]p(x,y,t)\,dx\,dy\,dt$$

$$\int_0^\infty \iint_{\mathcal{R}} q(x,y,t)\frac{\partial}{\partial x}[v_1 p]\,dx\,dy\,dt = \int_0^\infty \int_{Y_1}^{Y_2} q(x,y,t)v_1 p(x,y,t)\Big|_{X_1}^{X_2} dy\,dt - \int_0^\infty \iint_{\mathcal{R}} q_x(x,y,t)v_1 p(x,y,t)\,dx\,dy\,dt$$

$$\int_0^\infty \iint_R q(x,y,t)\frac{\partial}{\partial y}[Dp_y]\,dx\,dy\,dt = \int_0^\infty \int_{X_1}^{X_2} q(x,y,t)Dp_y(x,y,t)\Big|_{Y_1}^{Y_2} dy\,dt - \int_0^\infty \iint_R q_y(x,y,t)Dp_y(x,y,t)\,dx\,dy\,dt$$
$$= \int_0^\infty \int_{X_1}^{X_2} q(x,y,t)Dp_y(x,y,t)\Big|_{Y_1}^{Y_2} dy\,dt - \int_0^\infty \int_{X_1}^{X_2} q_y(x,y,t)Dp(x,y,t)\Big|_{Y_1}^{Y_2} dy\,dt$$
$$+ \int_0^\infty \iint_R \frac{\partial}{\partial y}[q_y D]p(x,y,t)\,dx\,dy\,dt$$
$$= \int_0^\infty \int_{X_1}^{X_2} q(x,y,t)Dp_y(x,y,t)\Big|_{Y_1}^{Y_2} dy\,dt - \int_0^\infty \int_{X_1}^{X_2} q_y(x,y,t)Dp(x,y,t)\Big|_{Y_1}^{Y_2} dy\,dt$$
$$+ \int_0^\infty \iint_R [q_{yy}D + q_y D_y]p(x,y,t)\,dx\,dy\,dt$$

$$\int_0^\infty \iint_{\mathcal{R}} q(x,y,t)\frac{\partial}{\partial y}[v_2 p]\,dx\,dy\,dt = \int_0^\infty \int_{X_1}^{X_2} q(x,y,t)v_2 p(x,y,t)\Big|_{Y_1}^{Y_2} dy\,dt - \int_0^\infty \iint_{\mathcal{R}} q_y(x,y,t)v_2 p(x,y,t)\,dx\,dy\,dt$$

We then obtain:

$$\mathcal{V} = \int_0^\infty \iint_{\mathcal{R}} \mathcal{U}(c(x,y,t))h(x,y,t)e^{-\rho t}\,dx\,dy\,dt - \iint_{\mathcal{R}} q(x,y,t)p(x,y,t)\Big|_0^\infty dx\,dy + \int_0^\infty \iint_{\mathcal{R}} q_t(x,y,t)p(x,y,t)\,dx\,dy\,dt$$
$$+ \int_0^\infty \int_{Y_1}^{Y_2} q(x,y,t)Dp_x(x,y,t)\Big|_{X_1}^{X_2} - q_x(x,y,t)Dp(x,y,t)\Big|_{X_1}^{X_2} - q(x,y,t)v_1 p(x,y,t)\Big|_{X_1}^{X_2} dy\,dt$$
$$+ \int_0^\infty \iint_{\mathcal{R}} [q_{xx}D + q_x[D_x + v_1]]p(x,y,t)\,dx\,dy\,dt$$
$$+ \int_0^\infty \int_{X_1}^{X_2} q(x,y,t)Dp_y(x,y,t)\Big|_{Y_1}^{Y_2} - q_y(x,y,t)Dp(x,y,t)\Big|_{Y_1}^{Y_2} dx\,dt - q(x,y,t)v_2 p(x,y,t)\Big|_{Y_1}^{Y_2} dy\,dt$$
$$+ \int_0^\infty \iint_{\mathcal{R}} [q_{yy}D + q_y[D_y + v_2]]p(x,y,t)\,dx\,dy\,dt$$
$$- \int_0^\infty \iint_{\mathcal{R}} q(x,y,t)[-\theta B H(t)^\alpha f(x,y,t) + G(u(x,y,t)) + \delta p(x,y,t)]\,dx\,dy\,dt$$
$$- \int_0^\infty m(t)[H(t) - \iint_{\mathcal{R}} \phi(p,h)\,dx\,dy]\,dt - \int_0^\infty n(t)[\iint_{\mathcal{R}} c(x,y,t)h(x,y,t) + u(x,y,t)\,dx\,dy - BH^\alpha]\,dt$$

Then, assuming there exists at least a solution to the social optimum, the Ekeland variational principle ensures the existence of a maximum value for $\mathcal{V}$ (if $\mathcal{V}$ is upper semi-continuous). Hence, any feasible solution satisfying (7) can be written as a deviation from the optimal solution

(c^*, u^*, p^*, H^*):

$$
\begin{aligned}
c(x,y,t) &= c^*(x,y,t) + \epsilon\kappa(x,y,t) \\
u(x,y,t) &= u^*(x,y,t) + \epsilon\Upsilon(x,y,t) \\
p(x,y,t) &= p^*(x,y,t) + \epsilon\Pi(x,y,t) \\
H(t) &= H^*(t) + \epsilon\Omega(t)
\end{aligned}
$$

where κ, Υ, Π, Ω are real functions of $(x,y,t) \in \mathcal{R} \times \mathbb{R}^+$.The optimal solution then results from minimizing $\mathcal{V}$ with respect to ϵ, which is the deviation of the trajectory from the optimal solution. Hence, we take the first order derivative of the value function V with respect to ϵ:

$$
\begin{aligned}
&\frac{\partial \mathcal{V}(c,u,p,H)}{\partial \epsilon} = \int_0^\infty \iint_{\mathcal{R}} \mathcal{U}'(c(x,y,t))h(x,y,t)e^{-\rho t}\kappa(x,y,t)\,dx\,dy\,dt - \iint_{\mathcal{R}} q(x,y,t)\Pi(x,y,t)\Big|_0^\infty dx\,dy \\
&+ \int_0^\infty \iint_{\mathcal{R}} \Pi(x,y,t)[q_t(x,y,t) + q_{xx}D + q_x[D_x + v_1] + q_{yy}D + q_y[D_y + v_2]]\,dx\,dy\,dt \\
&- \int_0^\infty \int_{Y_1}^{Y_2} Dq_x(x,y,t)\Pi(x,y,t)\Big|_{X_1}^{X_2} dy\,dt - \int_0^\infty \int_{Y_1}^{Y_2} v_1 q(x,y,t)\Pi(x,y,t)\Big|_{X_1}^{X_2} dy\,dt \\
&- \int_0^\infty \int_{X_1}^{X_2} Dq_y(x,y,t)\Pi(x,y,t)\Big|_{Y_1}^{Y_2} dx\,dt - \int_0^\infty \int_{X_1}^{X_2} v_2 q(x,y,t)\Pi(x,y,t)\Big|_{Y_1}^{Y_2} dx\,dt \\
&- \int_0^\infty \iint_{\mathcal{R}} q(x,y,t)[-\theta B\alpha H^{\alpha-1}\Omega(t)f(x,y,t) + \Upsilon(x,y,t)G'(u(x,y,t)) + \delta\Pi(x,y,t)]\,dx\,dy\,dt \\
&- \int_0^\infty m(t)[\Omega(t) - \iint_{\mathcal{R}} \Pi(x,y,t)\phi_p(p,h)\,dx\,dy]\,dt \\
&- \int_0^\infty n(t)[\iint_{\mathcal{R}} (\kappa(x,y,t)h(x,y,t) + \Upsilon(x,y,t))\,dx\,dy - B\alpha H^{\alpha-1}\Omega(t)]\,dt
\end{aligned}
$$

To get the necessary conditions for $\frac{\partial \mathcal{V}(c,u,p,H)}{\partial \epsilon} = 0$, we group all the elements multiplying κ, Υ, Π and Ω, and equate each of them to zero. Rhus, we need:

$$
\begin{cases}
\kappa : \mathcal{U}'(c(x,y,t))e^{-\rho t} = n(t) \\
\Upsilon : q(x,y,t)G'(u(x,y,t)) + n(t) = 0 \\
\Pi : q_t(x,y,t) + q_{xx}D + q_x[D_x + v_1] + q_{yy}D + q_y[D_y + v_2] + m(t)\phi_p(p,h) = \delta q(x,y,t) \\
\Omega : m(t) - n(t)\alpha B H^{\alpha-1} = \theta\alpha B H^{\alpha-1} \iint_{\mathcal{R}} q(x,y,t)f(x,y,t)\,dx\,dy
\end{cases}
$$

Then, detrending co-state variables, with $Q(x,y,t) := q(x,y,t)e^{\rho t}$, $N(x,y,t) := n(x,y,t)e^{\rho t}$, $M(x,y,t) := m(x,y,t)e^{\rho t}$, we obtain the following necessary conditions:

$$
\begin{cases}
\kappa : \mathcal{U}'(c(x,y,t)) = N(t) \\
\Upsilon : Q(x,y,t)G'(u(x,y,t)) + N(t) = 0 \\
\Pi : Q_t(x,y,t) + DQ_{xx} + Q_x[D_x + v_1] + Q_{yy}D + Q_y[D_y + v_2] + M(t)\phi_p(p,h) = (\delta + \rho)Q(x,y,t) \\
\Omega : M(t) = \alpha B H^{\alpha-1}(N(t) + \theta \iint_{\mathcal{R}} Q(x,y,t)f(x,y,t)\,dx\,dy)
\end{cases}
$$

To these conditions, we add the spatial boundary and transversality necessary conditions which can be written as follows:

$$
\begin{cases}
\lim\limits_{x\to\partial R_X} e^{-\rho t}DQ_x(x,y,t)p(x,y,t) = 0,\ \lim\limits_{x\to\partial R_X} e^{-\rho t}v_1 Q(x,y,t)p(x,y,t) = 0 & , \forall y \in [Y_1, Y_2],\ \forall t \in \mathbb{R}^+ \\
\lim\limits_{y\to\partial R_Y} e^{-\rho t}DQ_y(x,y,t)p(x,y,t) = 0,\ \lim\limits_{y\to\partial R_Y} e^{-\rho t}v_2 Q(x,y,t)p(x,y,t) = 0 & , \forall x \in [X_1, X_2],\ \forall t \in \mathbb{R}^+, \\
\lim\limits_{t\to+\infty} e^{-\rho t}Q(x,y,t)p(x,y,t) = 0 & , \forall (x,y) \in R.
\end{cases}
$$

Finally, after substituting the co-state variables into the dynamic equation for q (using the expression for $M(t)$) and recalling the assumption that V and D are essentially bounded, we obtain the set of first-order necessary optimal conditions presented in Proposition 1:

$$\begin{cases} p_t - Dp_{xx} - Dp_{yy} - D_x p_x - D_y p_y = \theta BH(t)^\alpha f(x,y,t) - (v_1 p_x + v_2 p_y) - p(v_{1x} + v_{2y}) - \delta p - G(u) \\ Q_t + DQ_{xx} + DQ_{yy} + Q_x[D_x + v_1] + Q_y[D_y + v_2] \\ \qquad + \Big[\mathcal{U}'(c(x,y,t)) + \theta \iint_{\mathcal{R}} Q(x',y',t) f(x',y')\, dx'\, dy'\Big] \alpha BH^{\alpha-1} \phi_p(p,h) = (\delta+\rho) Q(x,y,t), \\ Q(x,y,t) = -\frac{\mathcal{U}'(c(x,y,t))}{G'(u(x,y,t))}, \\ \iint_{\mathcal{R}} c(x,y,t) h(x,y,t) + u(x,y,t)\, dx\, dy = BH(t)^\alpha \\ H(t) = \iint_{\mathcal{R}} \phi(p,h)\, dx\, dy \\ p(x,y,0) = p_0(x,y), \lim\limits_{x \to \partial R_X} p_x(x,y,t) = \lim\limits_{y \to \partial R_Y} p_y(x,y,t) = 0, \\ \lim\limits_{x \to \partial R_X} e^{-\rho t} Q_x(x,y,t) p(x,y,t) = \lim\limits_{y \to \partial R_Y} e^{-\rho t} Q_y(x,y,t) p(x,y,t) = 0, \\ \lim\limits_{t \to +\infty} e^{-\rho t} Q(x,y,t) p(x,y,t) = 0, \end{cases}$$

for $(x,y,t) \in R \times \mathbb{R}^+$.

C Proposition 3 Proof

To determine sufficient conditions for the existence and uniqueness of the solution to system $\mathcal{S}$, we restrict the problem to a case with diffusion only (system $\mathcal{S}_D$ with $V = 0$) and we make use of the Theorem 1 proved in Camacho and Pérez-Barahona (2015), based on Pao (1992).
Since all theorems apply to positive solutions and the negativity of Q is known from the optimality condition $Q(x,y,t) = -\frac{\mathcal{U}'(c(x,y,t))}{G'(u(x,y,t))}$, we will study the existence and uniqueness of time-invariant solutions to a modified system with $\tilde{q} = -Q$:

$$\tilde{\mathcal{S}_D} \begin{cases} -\frac{\partial}{\partial x}[Dp_x](x,y) - \frac{\partial}{\partial y}[Dp_y](x,y) = \theta BH^\alpha f(x,y) - \delta p(x,y) - G(u(x,y)) \\ -\frac{\partial}{\partial x}[D\tilde{q}_x] - \frac{\partial}{\partial y}[D\tilde{q}_y] = -\Big[\mathcal{U}'(c) - \theta \iint_{\mathcal{R}} \tilde{q}(x',y') f(x',y')\, dx'\, dy'\Big] \alpha BH^{\alpha-1} \phi_p(p,h) - (\delta+\rho)\tilde{q}(x,y) \end{cases}$$

We want to compute the derivatives of the reaction terms:

$$\begin{aligned} f_1 &= \theta BH^\alpha f(x,y) - \delta p(x,y) - G(u(x,y)) \qquad \text{and} \\ f_2 &= -\Big[\mathcal{U}'(c) - \theta \iint_{\mathcal{R}} \tilde{q}(x',y') f(x',y')\, dx'\, dy'\Big] \alpha BH^{\alpha-1} \phi_p(p,h) - (\delta+\rho)\tilde{q}(x,y) \end{aligned}$$

with respect to p and $\tilde{q}$. First,

$$\frac{\partial f_1}{\partial \tilde{q}} = -G' \frac{\partial u}{\partial \tilde{q}}$$

To get the derivatives of u, we rewrite the condition that uniquely defines it:

$$J(u,p,\tilde{q}) = \mathcal{U}'(c) + Q(x,y) G'(u(x,y)) = \mathcal{U}'(c) - \tilde{q} G'$$

By the Implicit Function Theorem, we have:

$$\begin{aligned} \frac{\partial u}{\partial \tilde{q}} &= -\frac{\partial J / \partial \tilde{q}}{\partial J / \partial u} = \frac{G'}{-\tilde{q} G''} \geq 0 \\ \frac{\partial u}{\partial p} &= -\frac{\partial J / \partial p}{\partial J / \partial u} = 0 \end{aligned}$$

Therefore we know $\frac{\partial f_1}{\partial \tilde{q}} \leq 0$ and we derive conditions under which $\frac{\partial (f_1/p)}{\partial p} \leq 0$, in order to satisfy the Theorem's hypothesis:

$$\frac{f_1}{p} = \frac{\theta B H^\alpha f(x,y) - G(u)}{p} - \delta \implies \frac{\partial (f_1/p)}{\partial p} = \frac{p[\theta \alpha B H^{\alpha-1} f(x,y) \frac{\partial H}{\partial p} - G'(u)\frac{\partial u}{\partial p}] - [\theta B H^\alpha \boldsymbol{\delta_0} - G(u)]}{p^2}$$
$$= \frac{-\theta B H^\alpha f(x,y) + G(u)}{p^2}$$

Then, $\frac{\partial (f_1/p)}{\partial p} \leq 0$ if $G(u) \leq \theta B H^\alpha f(x,y)$.

Furthermore, under the problem assumptions, we have :

$$\frac{f_2}{\tilde{q}} = -\Big[\mathcal{U}'(c) - \theta \iint_{\mathcal{R}} \tilde{q}(x',y') f(x',y')\, dx'\, dy'\Big] \frac{\alpha B H^{\alpha-1} \phi_p(p,h)}{\tilde{q}} - (\delta + \rho)$$

$$\frac{\partial (f_2/\tilde{q})}{\partial \tilde{q}} = \frac{\Big[\mathcal{U}'(c) - \theta \iint_{\mathcal{R}} \tilde{q}(x',y') f(x',y')\, dx'\, dy'\Big] \alpha B H^{\alpha-1} \phi_p(p,h)}{\tilde{q}^2}$$

As $\phi_p(p,h) < 0$ by assumption, the sign $\frac{\partial (f_2/\tilde{q})}{\partial \tilde{q}}$ has the opposite sign of the term in brackets. We now want to derive conditions under which $\frac{\partial f_2}{\partial p}$ has the same sign as $\frac{\partial (f_2/\tilde{q})}{\partial \tilde{q}}$:

$$\frac{\partial f_2}{\partial p} = -\Big[\mathcal{U}'(c) - \theta \iint_{\mathcal{R}} \tilde{q}(x',y') \boldsymbol{\delta_0}(x',y')\, dx'\, dy'\Big] \alpha B H^{\alpha-1} \phi_{pp}(p,h)$$

Hence, $\frac{\partial f_2}{\partial p}$ has the same sign as $\frac{\partial (f_2/\tilde{q})}{\partial \tilde{q}}$ if $\phi_{pp}(p,h) \geq 0$.

So we have shown that we have either

$$\begin{cases} \frac{\partial f_1}{\partial \tilde{q}} \leq 0 \\ \frac{\partial (f_1/p)}{\partial p} \leq 0 \end{cases} \text{ and } \begin{cases} \frac{\partial (f_2/\tilde{q})}{\partial \tilde{q}} \geq 0 \\ \frac{\partial f_2}{\partial p} \geq 0 \end{cases}, \text{ or } \begin{cases} \frac{\partial f_1}{\partial \tilde{q}} \leq 0 \\ \frac{\partial (f_1/p)}{\partial p} \leq 0 \end{cases} \text{ and } \begin{cases} \frac{\partial (f_2/\tilde{q})}{\partial \tilde{q}} \leq 0 \\ \frac{\partial f_2}{\partial p} \leq 0 \end{cases},$$

if the following conditions hold:

$$\begin{cases} G(u) \leq \theta B H^\alpha f(x,y), \\ \phi_{pp}(p,h) \geq 0. \end{cases}$$

Then, if we can prove the existence of at least an upper and a lower solution to the stationary problem $\tilde{\mathcal{S}_D}$, and if f_1 and f_2 verify the above conditions, then we can ensure the existence of a unique solution to the stationary problem $\tilde{\mathcal{S}_D}$, hence to S_D.