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# Unirationality and rationality of algebraic varieties

Thomas Bourneuf

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# Unirationality and rationality of algebraic varieties

Bourneuf Thomas

July 23, 2021

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# 1 Introduction

A fundamental problem in algebraic geometry is to determine which varieties are rational, that is birational to the projective space. The study of this problem is very interesting because it uses vastly different methods. Moreover this problem has relations with the theory of fields as we will see. In this master thesis, we're going to be interested in a weaker notion than the rationality which is the unirationality of a variety  $X$ , that is, the existence of a dominant rational map from  $\mathbb{P}^n$  to  $X$ , where  $n$  is the dimension of  $X$ . We clearly have that if  $X$  is a rational variety, then  $X$  is unirational. At the end of the 19<sup>th</sup> century, Lüroth proves that if  $L$  is an intermediate field extension between  $k$  and  $k(X)$ , then there exists a rational function  $f$  in  $K(X)$  such that  $L$  is equal to  $K(f(X))$ . It means that if we have a dominant rational map from  $\mathbb{P}^1$  to an algebraic curve whose admits  $L$  as function field, then  $C$  is a rational curve. This theorem will be the object of the first part. Moreover, in the first part, we're going to introduce some tools which will turn out to be very useful in the study of the rationality, like the geometric genus and the arithmetic genus. In the second part, we're going to prove the Castelnuovo's theorem, which has for consequence the similar result than the Lüroth's theorem but for separable surfaces. In dimension 3, even over an algebraically closed field, it becomes false that a variety  $X$  is rational if and only if it is unirational, however we will see in the last part that any hypersurface of degree 3 in  $\mathbb{P}^n$ , with  $n$  at least 3, is unirational and we will prove it in the case of an algebraically closed field of characteristic zero. On the other hand, most of these cubic hypersurfaces are not rational. In fact, the mathematicians Clemens and Griffiths proved in 1972 that the general cubic hypersurface in  $\mathbb{P}^4$  is not rational.

## 2 Preliminaries

In all this master thesis, unless otherwise stated,  $k$  will be an algebraically closed field. Unless otherwise stated we will denote by variety an integral separated scheme over  $k$ . Let us first give some definitions in order to introduce the notion of rationality for a variety.

**Definition 2.0.1.** *If  $Y$  is a variety, we define the function field  $K(Y)$  of  $Y$  as follows: an element of  $K(Y)$  is an equivalence class of pairs  $\langle U, f \rangle$  where  $U$  is a nonempty open subset of  $Y$ ,  $f$  is a regular function on  $U$ , and where we identify two such pairs  $\langle U, f \rangle$  and  $\langle V, g \rangle$  if  $f = g$  on  $U \cap V$ .*

**Definition 2.0.2.** *If  $Y \subseteq \mathbb{A}^n$  is an affine algebraic set, we define the affine coordinate ring  $A(Y)$  of  $Y$ , to be  $A/I(Y)$ .*

**Definition 2.0.3.** *Let  $X, Y$  be varieties. A rational map  $f : X \dashrightarrow Y$  is an equivalence class of pairs  $\langle U, f_U \rangle$  where  $U$  is a nonempty open subset of  $X$ ,  $f_U$  is a morphism from  $U$  to  $Y$ , and where  $\langle U, f_U \rangle$  and  $\langle V, f_V \rangle$  are equivalent if  $f_U$  and  $f_V$  coincide on  $U \cap V$ . We say that the rational map  $f$  is dominant if for some pair  $\langle U, f_U \rangle$ , the image of  $f_U$  is dense in  $Y$ .*

The fact that the relation defined on the pairs  $\langle U, f_U \rangle$  actually is an equivalence relation result from the following lemma.

**Lemma 2.0.4.** *Let  $X$  and  $Y$  be varieties, let  $f$  and  $g$  be two morphisms from  $X$  to  $Y$ , and suppose there is a nonempty open subset  $U \subseteq X$  such that  $f_U = g_U$ . Then  $f = g$ .*

*Proof.* Let us consider the fiber product of  $Y$  with itself over  $S = \text{Spec}(k)$ . By the universal property of the fiber product, there exists a unique morphism  $h = (f, g) : X \rightarrow Y \times_S Y$  such that the following diagram

$$\begin{array}{ccc} & & Y \\ & \nearrow f & \uparrow p \\ X & \xrightarrow{h} & Y \times Y \\ & \searrow g & \downarrow q \\ & & Y \end{array}$$

is commutative. Let  $\Delta = \Delta_{Y/S}$  be the diagonal morphism of  $Y \rightarrow S$ . Then we have the following commutative diagram

$$\begin{array}{ccccc} & & & & Y \\ & & & & \uparrow p \\ X & \xrightarrow{f} & Y & \xrightarrow{\Delta} & Y \times Y \\ & \searrow f & & & \downarrow q \\ & & & & Y \end{array}$$

Hence, we have  $\Delta \circ f = (f, f)$ . Since  $f$  and  $g$  coincide on  $U$ , we have that  $\Delta \circ f$  and  $h$  coincide on  $U$ , by unicity of  $h$ . Hence we have that  $U \subseteq h^{-1}(\Delta(Y))$ . Since  $Y$  is separated,  $\Delta(Y)$  is closed. Therefore we have that  $X = h^{-1}(\Delta(Y))$ . So for every  $x \in X$  we have  $f(x) = g(x)$ . Then we have shown that  $f = g$  as maps. It remains to show that  $f = g$  as morphisms. Let us work locally on  $X$  and suppose that  $X = \text{Spec}(B)$  and  $Y = \text{Spec}(A)$  are affine, and also that  $U$  contains a non-empty principal open subset  $D(b) \subseteq U \subseteq X$ . We then have two ring homomorphisms  $\phi, \psi : A \rightarrow B$  such that  $\phi(a) = \psi(a)$  in  $B_b$  for all  $a \in A$ , since  $f|_{D(b)} = g|_{D(b)}$  as morphisms of affine schemes. Since  $X$  is integral,  $B$  is an integral domain so the localization  $B \rightarrow B_b$  is injective. So  $\phi = \psi$ . Hence  $f = g$  as scheme morphisms.  $\square$

**Remark:** The composition of two rational maps isn't always defined. For example, if  $f$  is a constant map on  $X$  and  $g$  is not defined at the point  $f(X)$  then the composition  $g \circ f$  has no sense. On the other hand, if  $h : X \dashrightarrow Y$  is a dominant rational map, then for any open  $V \in Y$ , we have  $h^{-1}(V) \neq \emptyset$  since  $h$  is dominant and so, denoting  $W \subseteq Y$  an open subset on which  $g : Y \dashrightarrow X$  admits a representant, the composition  $g \circ h$  is well defined since the composition as morphisms is well-defined on  $W \cap V$ , where  $V$  is an open subset of  $Y$ .

**Definition 2.0.5.** Let  $X, Y$  be varieties and let  $f : X \dashrightarrow Y$  be a dominant rational map. We say that  $f$  is a birational map if there exists a rational map  $g : Y \dashrightarrow X$  such that  $f \circ g = id_Y$  and  $g \circ f = id_X$  as rational morphisms.

If there exists a birational map from  $X$  to  $Y$  then we say that  $X$  and  $Y$  are birationally equivalent, or simply birational.

**Definition 2.0.6.** Let  $X$  be a variety and let  $n$  be its dimension.

We say that  $X$  is rational if it is birational to  $\mathbb{P}^n$ .

We say that  $X$  is unirational if there exists a dominant rational map from  $\mathbb{P}^n$  to  $X$ .

From the definition, we see that if  $X$  is a rational variety, then  $X$  is also an unirational variety. The goal of this master thesis is to study when the converse is true, that is, if  $X$  is a unirational variety, then is  $X$  a rational variety. An important result to answer this question is the following Theorem.

**Theorem 2.0.7.** Let  $X, Y$  be two varieties. There is a bijection between

- i) the set of dominant rational maps from  $X$  to  $Y$ ,
- ii) the set of  $k$ -algebra homomorphisms from  $K(Y)$  to  $K(X)$ .

Moreover, this correspondance gives an equivalence of categories between the category of varieties, which morphisms are the dominant rational maps, and the category of finitely generated field extensions of  $k$ .

*Proof.* We will construct an inverse to the mapping given by the construction below. Let  $\theta : K(Y) \rightarrow K(X)$  be a homomorphism of  $k$ -algebras. We wish to define a dominant rational map from  $X$  to  $Y$ . The open affine subsets form a topological basis of  $Y$  so we can cover  $Y$  by affine open subsets. Hence we may assume that  $Y$  is affine.

Let  $A(Y)$  be its affine coordinate ring, and let  $y_1, \dots, y_n$  be generators for  $A(Y)$  as a  $k$ -algebra. Then  $\theta(y_1), \dots, \theta(y_n)$  are rational functions on  $X$ . We can find an open set  $U \subseteq X$  such that the functions  $\theta(y_i)$  are all regular on  $U$ . Then  $\theta$  defines an injective homomorphism of  $k$ -algebras  $A(Y) \rightarrow \mathcal{O}_Y(U)$ . Since  $Y$  is affine, the canonical map  $\rho : \text{Mor}(U, Y) \rightarrow \text{Hom}(A(Y), \mathcal{O}_U(U))$  is a bijection so this morphism corresponds to a morphism  $\phi : U \rightarrow Y$ , which gives us a dominant rational map from  $X$  to  $Y$ . That gives us a map of sets from the set of  $k$ -algebra homomorphisms from  $K(Y)$  to  $K(X)$ , to the set of dominant rational maps from  $X$  to  $Y$ , which is inverse to the one defined above.

It remains to see that we have an equivalence of categories as stated. For that, we need only to check that for any variety  $Y$ ,  $K(Y)$  is finitely generated over  $k$ , and conversely, if  $K/k$  is a finitely generated field extension, then  $K = K(Y)$  for some  $Y$ .

If  $Y$  is a variety, then  $K(Y) = K(U)$  for any open affine subset, so we may assume that  $Y$  is affine. For each  $P \in Y$ , there is a natural map  $A(Y)_{\mathfrak{m}_P} \rightarrow \mathcal{O}_P$ , where  $\mathfrak{m}_P = \{f \in A(Y), f(P) = 0\}$  is the maximal ideal corresponding to  $P$  and  $\mathcal{O}_P$  is the ring of germs of regular functions

on  $Y$  near  $P$ . This map is injective because we have an injective homomorphism  $A(Y) \rightarrow \mathcal{O}(Y)$  and it is surjective by definition of a regular function. So we have, for each  $P \in Y$ ,  $A(Y)_{\mathfrak{m}_P} \cong \mathcal{O}_P$ . Then the quotient field of  $A(Y)$  is isomorphic to the quotient field of  $\mathcal{O}_P$  for every  $P$ , and is equal to  $K(Y)$  because every rational function is in some  $\mathcal{O}_P$ . Since  $A(Y)$  is a finitely generated  $k$ -algebra,  $K(Y)$  is a finitely generated field extension of  $k$ .

On the other hand, let  $K$  be a finitely generated field extension of  $k$ . Let  $y_1, \dots, y_n \in K$  be a set of generators, and let  $B$  the sub- $k$ -algebra of  $K$  generated by  $y_1, \dots, y_n$ . Then  $B$  is a quotient of the polynomial ring  $A = k[x_1, \dots, x_n]$ , so  $B \cong A(Y)$  for some variety  $Y$  in  $\mathbb{A}^n$ . Then  $K \cong K(Y)$ .  $\square$

**Corollary 2.0.8.** *For any two varieties  $X, Y$  the following assertions are equivalent :*

- i)  $X$  and  $Y$  are birationally equivalent,*
- ii) there are open subsets  $U \subseteq X$  and  $V \subseteq Y$  with  $U$  isomorphic to  $V$ ,*
- iii)  $K(X) \cong K(Y)$  as  $k$ -algebras.*

*Proof.* • *(i)  $\Rightarrow$  (ii).* Let  $f : X \dashrightarrow Y$  and  $g : Y \dashrightarrow X$  be rational maps which are inverse to each other. Let  $f$  be represented by  $\langle U, f \rangle$  and  $g$  be represented by  $\langle V, g \rangle$ . Then  $g \circ f$  is represented by  $\langle f^{-1}(V), g \circ f \rangle$ . Since  $g \circ f = id_X$  as a rational map,  $g \circ f$  is the identity on  $f^{-1}(V)$  and similarly,  $f \circ g$  is the identity on  $g^{-1}(U)$ . So we can take  $f^{-1}(g^{-1}(U))$  as our open set in  $X$  and  $g^{-1}(f^{-1}(V))$  as our open set in  $Y$ . By construction, these two open sets are isomorphic.

- *(ii)  $\Rightarrow$  (iii).* Follows from the definition of function field.
- *(iii)  $\Rightarrow$  (i).* Follows from the previous theorem.  $\square$

## 3 Genus of an algebraic variety

In this section, we are going to define the geometric genus and the arithmetic genus of a variety. Those are two birational invariants which will help us to characterize the varieties which answer positively to our problem.

### 3.1 Geometric genus of an algebraic variety

#### 3.1.1 Sheaves of modules

**Definition 3.1.1.** *Let  $(X, \mathcal{O}_X)$  be a ringed topological space. A sheaf of  $\mathcal{O}_X$ -modules, or an  $\mathcal{O}_X$ -modules, is a sheaf  $\mathcal{F}$  on  $X$  such that for each open subset  $U \subseteq X$ ,  $\mathcal{F}(U)$  is an  $\mathcal{O}_X(U)$ -module, and for each inclusion of open subsets  $V \subseteq U$ , the restriction map  $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$*

is compatible with the module structures via the ring homomorphism  $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ .

**Definition 3.1.2.** Let  $(X, \mathcal{O}_X)$  be a ringed topological space. An  $\mathcal{O}_X$ -module  $\mathcal{F}$  is free if it is isomorphic to a direct sum of copies of  $\mathcal{O}_X$ . In that case, the rank of  $\mathcal{F}$  is the number of copies of  $\mathcal{O}_X$ .

It is locally free if  $X$  can be covered by open sets  $U_i$  for which  $\mathcal{F}|_{U_i}$  is free of rank  $n$  for every  $i$ . In that case, the rank of  $\mathcal{F}$  on such an open subset is the number of copies of the structure sheaf needed (finite or infinite).

**Remark:** If  $X$  is connected, the rank of a locally free sheaf is the same everywhere.

**Definition 3.1.3.** Let  $X$  be a scheme. We say that an  $\mathcal{O}_X$ -module  $\mathcal{L}$  is invertible if for every point  $x \in X$ , there exists an open neighbourhood  $U$  of  $x$  and an isomorphism of  $\mathcal{O}_U$ -modules  $\mathcal{O}_U|_U \simeq \mathcal{L}|_U$ . This comes down to saying that  $\mathcal{L}$  is a locally free sheaf of rank 1.

**Definition 3.1.4.** Let  $(X, \mathcal{O}_X)$  be a ringed topological space. A sheaf of ideals on  $X$  is a sheaf of modules  $\mathcal{I}$  which is a subsheaf of  $\mathcal{O}_X$  : for every open subset  $U \subseteq X$ ,  $\mathcal{I}(U)$  is an ideal in  $\mathcal{O}_X(U)$ .

**Definition 3.1.5.** Let  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  be a morphism of ringed topological spaces.

If  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module, then  $f_*\mathcal{F}$  is an  $f_*\mathcal{O}_X$ -module. Since we have the morphism  $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  of sheaves of rings on  $Y$ , this gives  $f_*\mathcal{F}$  a natural structure of  $\mathcal{O}_Y$ -module. We call it the direct image of  $\mathcal{F}$  by the morphism  $f$ .

Now let  $\mathcal{G}$  be a sheaf of  $\mathcal{O}_Y$ -modules. Then  $f^{-1}\mathcal{G}$  is an  $f^{-1}\mathcal{O}_Y$ -module. Because of the adjoint property of  $f^{-1}$ , we have a morphism  $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$  of sheaves of rings on  $X$ . We define  $f^*\mathcal{G}$  to be the tensor product of  $f^{-1}\mathcal{O}_Y$ -modules

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus  $f^*\mathcal{G}$  is an  $\mathcal{O}_X$ -module. We call it the inverse image of  $\mathcal{G}$  by the morphism  $f$ .

**Definition 3.1.6.** Let  $(X, \mathcal{O}_X)$  be a ringed topological space, and let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. We say that  $\mathcal{F}$  is finitely generated if for every  $x$  in  $X$ , there exists an open neighbourhood  $U$  of  $x$ , an integer  $n \geq 1$ , and a surjective homomorphism  $\mathcal{O}_X^n|_U \rightarrow \mathcal{F}|_U$ .

We say that  $\mathcal{F}$  is coherent if it is finitely generated, and if for every open subset  $U$  of  $X$ , and for every homomorphism  $\alpha : \mathcal{O}_X^n|_U \rightarrow \mathcal{F}|_U$ , the kernel  $\text{Ker } \alpha$  is finitely generated.

**Definition 3.1.7.** Let  $A$  be a ring and let  $M$  be an  $A$ -module. We define the sheaf associated to  $M$  on  $\text{Spec}(A)$ , denoted by  $\tilde{M}$ , as follows. For each  $\mathfrak{p} \subseteq A$ , let  $M_{\mathfrak{p}}$  be the localization of  $M$  at  $\mathfrak{p}$ . For any open subset  $U \subseteq \text{Spec}(A)$ , we define the group  $\tilde{M}(U)$  to be the set of functions  $s$  from  $U$  to  $\prod_{\mathfrak{p} \in U} M_{\mathfrak{p}}$  which are locally fractions. This means that for every  $\mathfrak{p} \in U$ , there is a neighbourhood  $V$  of  $\mathfrak{p} \in U$ , and there are elements  $m \in M$  and  $f \in S$ , such that for every  $\mathfrak{q} \in V$ , we have  $f \notin \mathfrak{q}$ , and  $s(\mathfrak{q}) = m/f$  in  $M_{\mathfrak{q}}$ . We make  $\tilde{M}$  into a sheaf with the obvious restriction maps.



**Proposition 3.1.8.** *Let  $A$  be a ring, let  $M$  be an  $A$ -module, and let  $\tilde{M}$  be the sheaf on  $X = \text{Spec}(A)$  associated to  $M$ .*

- (a)  $\tilde{M}$  is an  $\mathcal{O}_X$ -module.
- (b) For each  $\mathfrak{p} \in X$ , the stalk  $(\tilde{M})_{\mathfrak{p}}$  of the sheaf  $\tilde{M}$  at  $\mathfrak{p}$  is isomorphic to the localized module  $M_{\mathfrak{p}}$ .
- (c) For any  $f \in A$ , denoting  $D(f) = X \setminus V(f)$ , the  $A_f$ -module  $\tilde{M}(D(f))$  is isomorphic to the localized module  $M_f$ .
- (d) In particular,  $\Gamma(X, \tilde{M}) = M$ .

**Proposition 3.1.9.** *Let  $A$  be a ring and let  $X = \text{Spec}(A)$ .*

- (a) A sequence of  $A$ -modules  $L \rightarrow M \rightarrow N$  is exact if and only if the associated sequence of  $\mathcal{O}_X$ -modules  $\tilde{L} \rightarrow \tilde{M} \rightarrow \tilde{N}$  is exact.
- (b) Let  $M, N$  be two  $A$ -modules. Then we have a canonical isomorphism  $(M \otimes_A N) \simeq \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ .
- (c) Let  $\{M_i\}_i$  be a family of  $A$ -modules. Then  $(\bigoplus_i M_i) \simeq \bigoplus_i \tilde{M}_i$ .

*Proof.* (a) If the sequence  $L \rightarrow M \rightarrow N$  is exact then for every  $\mathfrak{p} \in \text{Spec}(A)$ , the sequence  $L_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$  is exact since  $A_{\mathfrak{p}}$  is flat over  $A$ . Since the exactness of the sequence  $\tilde{L} \rightarrow \tilde{M} \rightarrow \tilde{N}$  can be verified at the stalks, we deduce that this sequence is exact. Conversely, let us suppose that we have an exact sequence

$$\tilde{L} \xrightarrow{\alpha} \tilde{M} \xrightarrow{\beta} \tilde{N}.$$

For any prime ideal  $\mathfrak{p} \in \text{Spec}(A)$ , we have a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\alpha(X)} & M & \xrightarrow{\beta(X)} & N \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathfrak{p}} & \longrightarrow & M_{\mathfrak{p}} & \longrightarrow & N_{\mathfrak{p}} \end{array}$$

where the vertical arrows are localizations, and where the second horizontal sequence is exact. This allows to deduce that  $(\text{Ker}(\beta(X))/\text{Im}(\alpha(X)))_{\mathfrak{p}} = 0$ . It follows from this  $\text{Ker}(\beta(X)) = \text{Im}(\alpha(X))$  and hence that the sequence  $L \rightarrow M \rightarrow N$  is exact.

- (b) Let  $L = M \otimes_A N$ . For any principal open subset  $D(f)$  of  $X$ , we have a canonical isomorphism of  $\mathcal{O}_X(D(f))$ -modules :

$$\begin{aligned} \tilde{L}(D(f)) &= (M \otimes_A N) \otimes_A A_f \simeq (M \otimes_A A_f) \otimes_{A_f} (N \otimes_A A_f) \\ &= \tilde{M}(D(f)) \otimes_{\mathcal{O}_X(D(f))} \tilde{N}(D(f)), \end{aligned}$$

compatible with the restriction homomorphisms. Since the principal open subsets form a base for the topology of  $X$ , this induces an isomorphism of  $\mathcal{O}_X$ -modules  $\tilde{L} \simeq \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ .

- (c) It results from the fact that the localization commutes with the direct sum. □

**Definition 3.1.10.** Let  $(X, \mathcal{O}_X)$  be a scheme. A sheaf of  $\mathcal{O}_X$ -modules  $\mathcal{F}$  is quasi-coherent if  $X$  can be covered by open affine subsets  $U_i = \text{Spec}(A_i)$ , such that for each  $i$ , there is an  $A_i$ -module  $M_i$  with  $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$ .

**Example:** For any  $A$  module  $M$ , the sheaf  $\tilde{M}$  is quasi-coherent.

**Definition 3.1.11.** Let  $S$  be a graded ring and let  $M$  be a graded  $S$ -module. We define the sheaf associated to  $M$  on  $\text{Proj}(S)$ , denoted by  $\tilde{M}$ , as follows. For each  $\mathfrak{p} \in \text{Proj}(S)$ , let  $M_{(\mathfrak{p})}$  be the group of the elements of degree 0 in the localization  $T^{-1}M$ , where  $T$  is the multiplicative system of homogeneous elements of  $S$  not in  $\mathfrak{p}$ . For any open subset  $U \subseteq \text{Proj}(S)$ , we define  $\tilde{M}(U)$  to be the set of functions  $s$  from  $U$  to  $\prod_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$  which are locally fractions. This means that for every  $\mathfrak{p} \in U$ , there is a neighbourhood  $V$  of  $\mathfrak{p} \in U$ , and homogeneous elements  $m \in M$  and  $f \in S$  of the same degree, such that for every  $\mathfrak{q} \in V$ , we have  $f \notin \mathfrak{q}$ , and  $s(\mathfrak{q}) = m/f$  in  $M_{(\mathfrak{q})}$ . We make  $\tilde{M}$  into a sheaf with the obvious restriction maps.

**Proposition 3.1.12.** Let  $S$  be a graded ring, and  $M$  a graded  $S$ -module. Let  $X = \text{Proj}(S)$ .

- (a) For any  $\mathfrak{p} \in X$ , the stalk  $(\tilde{M})_{\mathfrak{p}} = M_{(\mathfrak{p})}$ .
- (b) For any homogeneous  $f \in S_+$ , denoting  $D_+(f) = X \setminus V((f))$ , we have  $\tilde{M}|_{D_+(f)} \simeq \widetilde{(M_{(f)})}$  via the isomorphism  $D_+(f) \simeq \text{Spec } S_{(f)}$ , where  $M_{(f)}$  denotes the group of the elements of degree 0 in the localized module  $M_f$ .

**Remark:** If  $S$  is noetherian and  $M$  is finitely generated, then  $\tilde{M}$  is coherent.

**Definition 3.1.13.** Let  $S$  be a graded ring, and let  $X = \text{Proj}(S)$ . For any  $n \in \mathbb{Z}$ , we define the sheaf  $\mathcal{O}_X(n)$  to be  $\widetilde{S(n)}$ . We call  $\mathcal{O}_X(1)$  the twisting sheaf of Serre. For any sheaf of  $\mathcal{O}_X$ -modules,  $\mathcal{F}$ , we denote by  $\mathcal{F}(n)$  the twisted sheaf  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ .

For any scheme  $Y$ , we define the twisted sheaf  $\mathcal{O}(1)$  on  $\mathbb{P}_Y^r$  to be  $g^*(\mathcal{O}(1))$ , where  $g: \mathbb{P}_Y^r \rightarrow \mathbb{P}_{\mathbb{Z}}^r$  is the natural map.

**Proposition 3.1.14.** Let  $S$  be a graded ring and let  $X = \text{Proj}(S)$ . Assume that  $S$  is generated by  $S_1$  as an  $S_0$ -algebra.

- (a) The sheaf  $\mathcal{O}_X(n)$  is an invertible sheaf on  $X$ .
- (b) For any graded  $S$ -module  $M$ ,  $\tilde{M}(n) \simeq \widetilde{(M(n))}$ . In particular,  $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \simeq \mathcal{O}_X(n+m)$ .

*Proof.* (a) We want to show that  $\mathcal{O}_X(n)$  is a locally free sheaf of rank 1. Let  $f \in S_1$ . We consider the restriction  $\mathcal{O}_X(n)|_{D_+(f)}$ . By the previous Proposition, we have  $\mathcal{O}_X(n)|_{D_+(f)} \simeq (S(n)_{(f)})$  on  $\text{Spec}(S_{(f)})$ . Let us show that this restriction is free of rank 1. First notice that  $S(n)_f$  is a free  $S_{(f)}$ -module of rank 1.  $S(n)_{(f)}$  is the group of the elements of degree  $n$  in  $S_f$  and  $S_{(f)}$  is the group of elements of degree 0 in  $S_f$  so we obtain a morphism from  $S_{(f)}$  to  $S(n)_{(f)}$  by sending  $s \mapsto f^n s$ . Since  $f$  is invertible in  $S_f$ , this morphism is well-defined for any  $n \in \mathbb{Z}$  and is in fact an isomorphism. Since  $S$  is generated by  $S_1$  as an  $S_0$ -algebra,  $X$  is covered by the the open sets  $D_+(f)$  for  $f \in S_1$ . Hence  $\mathcal{O}_X(n)$  is invertible.

(b) Let  $M, N$  be two graded  $S$ -modules. For any  $f \in S_1$ , we have  $(M \otimes_S N)_{(f)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ . The results follows from this and the fact that  $S$  is generated by  $S_1$  as an  $S_0$ -algebra.  $\square$

### 3.1.2 Kähler Differentials and sheaves of differentials

Let  $A$  be a commutative ring with identity, let  $B$  be an  $A$ -algebra and let  $M$  be a  $B$ -module.

**Definition 3.1.15.** *An  $A$ -derivation of  $B$  into  $M$  is a map  $d : B \rightarrow M$  such that*

*i)  $d$  is additive,*

*ii) for any  $b, b' \in B$  we have*

$$d(bb') = bd(b') + d(b)b',$$

*iii) for any  $a \in A$  we have  $d(a) = 0$ .*

**Definition 3.1.16.** *We define the module of relative differential forms of  $B$  over  $A$  to be a  $B$ -module  $\Omega_{B/A}$ , together with an  $A$ -derivation  $d : B \rightarrow \Omega_{B/A}$ , which satisfies the following universal property : for any  $B$ -module  $M$ , and for any  $A$ -derivation  $d' : B \rightarrow M$ , there exists a unique  $B$ -module homomorphism  $f : \Omega_{B/A} \rightarrow M$  such that the following diagram*

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B/A} \\ & \searrow d' & \downarrow f \\ & & M \end{array}$$

*commutes.*

*We denote the set of these derivations by  $\text{Der}_A(B, M)$ .*

*Proof.* Existence. Let  $F$  be the free  $B$ -module generated by the symbols  $\{d(b), b \in B\}$ . Let  $E$  be the submodule of  $F$  generated by the elements of the form  $d(a)$ ,  $a \in A$ ,  $d(b+b') - d(b) - d(b')$  for  $b, b' \in B$ , and  $d(bb') - d(b)b' - bd(b')$  for  $b, b' \in B$ . Let us set  $\Omega_{B/A} = F/E$  and  $d : B \rightarrow \Omega_{B/A}$  which sends  $b$  to the image of  $d(b)$  in  $\Omega_{B/A}$ . Then  $(\Omega_{B/A}, d)$  satisfies the required properties. The uniqueness follows from the definition.  $\square$

**Remark:** By construction,  $\Omega_{B/A}$  is generated as a  $B$ -module by  $\{d(b), b \in B\}$ .

**Remark:** For any  $B$ -module  $M$ , we have a map

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \rightarrow \mathrm{Der}_A(B, M)$$

defined by  $\phi \mapsto \phi \circ d$ . The universal property of  $\Omega_{B/A}$  consists of saying that this map is an isomorphism of  $A$ -modules.

**Proposition 3.1.17.** *Let  $B$  be an algebra over a ring  $A$ . Let  $C$  be an  $A$ -algebra and let us set  $B' = B \otimes_A C$ . There exists a canonical isomorphism of  $B'$ -modules  $\Omega_{B'/C} \simeq \Omega_{B/A} \otimes_B B'$ .*

*Proof.* The canonical derivation  $d : B \rightarrow \Omega_{B/A}$  induces a  $C$ -derivation defined by  $d' = d \otimes \mathrm{Id}_C : B' \rightarrow \Omega_{B/A} \otimes_A C = \Omega_{B/A} \otimes_B B'$ . We easily see that  $(\Omega_{B/A} \otimes_B B', d')$  verifies the universal property of  $\Omega_{B'/C}$ .  $\square$

**Proposition 3.1.18.** *Let  $A \xrightarrow{\Phi} B \xrightarrow{\Psi} C$  be rings and homomorphisms. Then there is a natural exact sequence of  $C$ -modules*

$$\Omega_{B/A} \otimes_B C \xrightarrow{u} \Omega_{C/A} \xrightarrow{v} \Omega_{C/B} \longrightarrow 0.$$

*Proof.* We define the maps  $v$  and  $u$  by

$$u : \begin{array}{ccc} \Omega_{B/A} \otimes_B C & \longrightarrow & \Omega_{C/A} \\ d_{B/A}(b) \otimes c & \longmapsto & c \cdot d_{C/A}(\psi(b)), \end{array}$$

$$v : \begin{array}{ccc} \Omega_{C/A} & \longrightarrow & \Omega_{C/B} \\ c \cdot d_{C/A}(c') & \longmapsto & c \cdot d_{C/B}(c') \end{array}$$

where  $b \in B$  and  $c, c' \in C$ . The map  $v$  is surjective. Since  $d_{C/B}(\psi(b)) = 0$  we have  $v \circ u = 0$ . So we have to prove that  $\mathrm{Ker}(v) = \mathrm{Im}(u)$ . To do this, it is enough to prove that

$$\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, T) \longleftarrow \mathrm{Hom}_C(\Omega_{C/A}, T) \longleftarrow \mathrm{Hom}_C(\Omega_{C/B}, T) \longleftarrow 0$$

is exact for any  $C$ -module  $T$ . But, with the previous Proposition and the previous Remark, we have canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, T) &\simeq \mathrm{Der}_A(B, T), \\ \mathrm{Hom}_C(\Omega_{C/A}, T) &\simeq \mathrm{Der}_A(C, T), \\ \mathrm{Hom}_C(\Omega_{C/B}, T) &\simeq \mathrm{Der}_B(C, T), \end{aligned}$$

so we can identify the last sequence with

$$\mathrm{Der}_A(B, T) \longleftarrow \mathrm{Der}_A(C, T) \longleftarrow \mathrm{Der}_B(C, T) \longleftarrow 0.$$

It follows from the definition of a derivation that this sequence is exact, which proves the Proposition.  $\square$

Let  $f : X \rightarrow Y$  be a morphism of schemes. We consider the diagonal morphism  $\Delta : X \rightarrow X \times_Y X$ . Then  $\delta$  gives an isomorphism of  $X$  onto its image  $\Delta(X)$ , which is a locally closed subscheme of  $X \times_Y X$ , that is a closed subscheme of an open subset  $W$  of  $X \times_Y X$ .

**Definition 3.1.19.** Let  $\mathcal{I}$  be the sheaf of ideals of  $\Delta(X)$  on  $W$ . We define the sheaf of relative differentials of  $X$  over  $Y$  to be the sheaf  $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$  on  $X$ .

**Theorem 3.1.20.** Let  $B$  be a local ring containing a field  $k$  isomorphic to its residue field. Assume furthermore that  $k$  is perfect, and that  $B$  is a localization of a finitely generated  $k$ -algebra. Then  $\Omega_{B/k}$  is a free  $B$ -module of rank  $\dim(B)$  if and only if  $B$  is a regular local ring.

**Proposition 3.1.21.** Let  $f : X \rightarrow Y$  be a morphism of schemes. Let  $Y'$  be a  $Y$ -scheme. Let  $X' = X \times_Y Y'$  and let  $p : X' \rightarrow X$  be the first projection. Then  $\Omega_{X'/Y'} \simeq p^*\Omega_{X/Y}$ .

*Proof.* It follows from the Proposition 3.1.17.  $\square$

**Proposition 3.1.22.** Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be morphisms of schemes. Then there is an exact sequence of sheaves on  $X$

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

*Proof.* It follows from the Proposition 3.1.18.  $\square$

**Theorem 3.1.23.** Let  $A$  be a ring, let  $Y = \mathrm{Spec}(A)$  and  $X = \mathbb{P}_A^n$ . Then there is an exact sequence of sheaves on  $X$

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

where  $\mathcal{O}_X(-1)^{n+1}$  means a direct sum of  $n+1$  copies of  $\mathcal{O}_X(-1)$ .

*Proof.* Let  $S = A[x_0, \dots, x_n]$  be the homogeneous coordinate ring of  $X$  and let  $E$  be the graded  $S$ -module  $S(-1)^{n+1}$ , with basis  $e_0, \dots, e_n$  in degree 1. We define a homomorphism of graded  $S$ -modules  $E \rightarrow S$  by sending  $e_i \mapsto x_i$  and let us denote  $M$  the kernel of this homomorphism. We have the following exact sequence

$$0 \rightarrow M \rightarrow E \rightarrow S.$$

This exact sequence of graded  $S$ -modules gives us an exact sequence of sheaves on  $X$

$$0 \rightarrow \tilde{M} \rightarrow \mathcal{O}_X(-1)^{n+1} \rightarrow \mathcal{O}_X \rightarrow 0.$$

The last arrow is exact because  $E \rightarrow S$  is surjective in all degrees  $n \geq 1$ . Let us show that  $\tilde{M} \simeq \Omega_{X/Y}$ . Taking the localizations of  $E$  and  $S$  at  $x_i$  we obtain a surjective homomorphism  $E_{x_i} \rightarrow S_{x_i}$  of free  $S_{x_i}$ -modules so  $M_{x_i}$  is free of rank  $n$  and is generated by  $\{e_j - (x_j/x_i)e_i, j \neq i\}$ . It follows that if  $U_i$  is the open set in  $X$  defined by  $x_i$ , then  $\tilde{M}|_{U_i}$  is a free  $\mathcal{O}_{U_i}$ -module generated by the sections  $(1/x_i)e_j - (x_j/x_i^2)e_i$ , for  $j \neq i$ . Let  $\phi_i : \Omega_{X/Y}|_{U_i} \rightarrow \tilde{M}|_{U_i}$  be defined as follows. Since  $\Omega_{X/Y}|_{U_i}$  is a free  $\mathcal{O}_{U_i}$ -module generated by  $d(x_0/x_i), \dots, d(x_n/x_i)$ , we define  $\phi_i$  by  $\phi_i(d(x_j/x_i)) = (1/x_i)e_j - (x_j/x_i^2)e_i$ . Then  $\phi_i$  is an isomorphism. It remains to show that we can glue the  $\phi_i$  to have an isomorphism  $\phi : \Omega_{X/Y} \rightarrow \tilde{M}$ . In order to do that, we have to verify that  $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ . For any  $k$  we have, on  $U_i \cap U_j$ ,

$$d\left(\frac{x_k}{x_i}\right) = d\left(\frac{x_k}{x_j} \cdot \frac{x_j}{x_i}\right) = \frac{x_k}{x_j} \cdot d\left(\frac{x_j}{x_i}\right) + \frac{x_j}{x_i} \cdot d\left(\frac{x_k}{x_j}\right).$$

Hence, on  $U_i \cap U_j$ , we obtain that

$$\begin{aligned} \phi_i\left(d\left(\frac{x_k}{x_i}\right)\right) &= \frac{x_j}{x_i} \cdot \left(\phi_i\left(d\left(\frac{x_k}{x_i}\right)\right) - \frac{x_k}{x_j}\phi_i\left(d\left(\frac{x_j}{x_i}\right)\right)\right) \\ &= \frac{1}{x_j}e_k - \frac{x_k}{x_i x_j}e_i - \frac{x_k}{x_j^2}e_j + \frac{x_k}{x_j x_i}e_i \\ &= \frac{1}{x_j}e_k - \frac{x_k}{x_j^2}e_j = \phi_j\left(d\left(\frac{x_k}{x_j}\right)\right). \end{aligned}$$

Thus the isomorphisms  $\phi_i$  glue, which completes the proof.  $\square$

**Definition 3.1.24.** Let  $X$  be a nonsingular variety over a field  $k$ . We define the canonical sheaf of  $X$  to be  $\omega_X = \Lambda^n \Omega_{X/k}$  the  $n$ th exterior power of the sheaf of differentials, where  $n = \dim X$ .

If  $X$  is projective and nonsingular, we define the geometric genus of  $X$  to be  $p_g = p_g(X) = \dim_k \Gamma(X, \omega_X)$ .

**Remark:**  $\omega_X$  is an invertible sheaf on  $X$ .

**Remark:** When it is defined,  $p_g$  is a nonnegative integer.

**Definition 3.1.25.** Let  $A \subseteq B$  be local rings. We say that  $B$  dominates  $A$  if the inclusion  $A \rightarrow B$  is a local ring homomorphism.

**Lemma 3.1.26.** Let  $\mathcal{O}_K$  be a valuation ring,  $K = \text{Frac}(\mathcal{O}_K)$ , and  $A$  a local subring of  $K$  which dominates  $\mathcal{O}_K$ . Then  $A = \mathcal{O}_K$ .

*Proof.* Let  $\nu$  be the valuation of  $K$  and  $\mathfrak{m}_A$  be the maximal ideal of  $A$ . By contradiction, suppose that there exists  $a \in A \setminus \mathcal{O}_K$ . Then  $\nu(a^{-1}) > 0$ . Consequently  $a^{-1}$  is contained in the maximal ideal of  $\mathcal{O}_K$  and we have  $a^{-1} \in \mathfrak{m}_A$ . So  $1 = a \cdot (a^{-1}) \in \mathfrak{m}_A$ , which is impossible.  $\square$

**Theorem 3.1.27.** Let  $X$  be a proper scheme over a valuation ring  $\mathcal{O}_K$ , and let  $K = \text{Frac}(\mathcal{O}_K)$ . Then the canonical map  $X(\mathcal{O}_K) \rightarrow X_K(K)$  is a bijection.

*Proof.* Since  $X \rightarrow \text{Spec}(\mathcal{O}_K)$  is separated and  $\text{Spec}(K)$  is dense in  $\text{Spec}(\mathcal{O}_K)$ , the morphism  $X(\mathcal{O}_K) \rightarrow X_K(K)$  is injective. Let  $\pi : \text{Spec}(K) \rightarrow X_K$  be a section of  $X_K$  and let  $\eta$  be the generic point of  $\text{Spec}(\mathcal{O}_K)$ . Let  $x = \pi(\eta)$ . Let  $Z = \overline{\{x\}} \subseteq X$  be the closed subset endowed with the structure of a reduced subscheme. Since the closed immersions are proper and the composition of two proper morphisms is a proper morphism,  $Z$  is proper over  $\mathcal{O}_K$ . The point  $x$  is closed in  $X_K$ , and dense in  $Z_K$ . Therefore  $Z_K = \{x\}$ . The image of  $Z \rightarrow \text{Spec}(\mathcal{O}_K)$  is closed and contains  $\eta$  so it is equal to  $\text{Spec}(\mathcal{O}_K)$ . Let  $s$  be the closed point of  $\text{Spec}(\mathcal{O}_K)$  and let  $t \in Z_s$ . Then  $\mathcal{O}_{Z,t}$  is a local ring dominated  $\mathcal{O}_K$ , with field of fractions  $\mathcal{O}_{Z,x} = K$ . By the previous Lemma, we have that  $\mathcal{O}_{Z,t} = \mathcal{O}_K$ . So we have an extension of  $\text{Spec}(K) \rightarrow X$ , to  $\text{Spec}(\mathcal{O}_K) = \text{Spec}(\mathcal{O}_{Z,t}) \rightarrow Z \rightarrow X$ .  $\square$

**Corollary 3.1.28** (Valuative Criterion of Properness). *Let  $f : X \rightarrow Y$  be a proper morphism. Then for any  $Y$ -scheme  $\text{Spec}(\mathcal{O}_K)$ , where  $\mathcal{O}_K$  is a valuation ring with field of fractions  $K$ , the canonical map  $X(\mathcal{O}_K) \rightarrow X(K)$  is bijective.*

*Proof.* Let us set  $Z = X \times_Y \text{Spec}(\mathcal{O}_K)$ . Then we have a canonical bijection  $X(\mathcal{O}_K) = Z(\mathcal{O}_K)$  and  $X(K) = Z_K(K)$ . Applying the previous theorem to  $Z \rightarrow \text{Spec}(\mathcal{O}_K)$  we obtain that the map  $Z(\mathcal{O}_K) = X(\mathcal{O}_K) \rightarrow Z_K(K) = X(K)$  is bijective.  $\square$

**Definition 3.1.29.** *Let  $A$  denote a noetherian ring and  $M$  be an  $A$ -module. We say that a prime ideal  $\mathfrak{p}$  of  $A$  is an associated prime of  $M$  if one the following equivalent conditions holds :*

- (i) *there exists an element  $x \in M$  with  $\text{Ann}(x) = \mathfrak{p}$ ,*
- (ii)  *$M$  contains a submodule isomorphic to  $A/\mathfrak{p}$ .*

*The set of the associated primes of  $M$  is denoted by  $\text{Ass}(M)$ .*

**Definition 3.1.30.** *Let  $A$  be a noetherian domain with quotient field  $k$ . For any non zero ideal  $I$  of  $A$ , we put  $I^{-1} = \{x \in k, xI \subseteq A\}$ .*

**Lemma 3.1.31.** *Let  $A$  be a noetherian integral domain and  $I$  a non-zero ideal of  $A$ . Then  $I^{-1}$  is a subring of  $\text{Frac}(A)$ , finite over  $A$ .*

**Definition 3.1.32.** *Let  $A$  be a ring and  $\mathfrak{p}$  a prime ideal of  $A$ . The height of  $\mathfrak{p}$ ,  $ht(\mathfrak{p})$ , is the supremum of the lengths of the strictly ascending chains of prime ideals contained in  $\mathfrak{p}$ . For an arbitrary ideal  $I$  of  $A$ ,  $ht(I)$  is the infimum of the heights of the prime ideals containing  $I$ .*

**Theorem 3.1.33** (Krull's theorem). *Let  $A$  be a noetherian ring,  $I$  an ideal of  $A$ , and  $M$  a finitely generated  $A$ -module. Then*

$$\bigcap_{n \geq 0} (I^n M)$$

*is the set of elements  $x \in M$  for which there exists an  $\alpha \in I$  such that  $(1 + \alpha)x = 0$ .*

**Lemma 3.1.34.** *Let  $(A, \mathfrak{m})$  be a noetherian local ring. The following conditions are equivalent :*

(i)  $\dim A = 0$ ,

(ii) every descending sequence of ideals of  $A$  is stationary.

**Theorem 3.1.35** (Krull's principal ideal theorem). *Let  $A$  be a noetherian ring and  $f \in A$  be a non invertible element. Then for any prime ideal  $\mathfrak{p}$  that is minimal among those containing  $f$ , we have  $ht(\mathfrak{p}) \leq 1$ .*

*Proof.* Localizing in  $\mathfrak{p}$ , if necessary, we may assume that  $A$  is local with maximal ideal  $\mathfrak{p}$ . Moreover, if necessary, replacing  $A$  by its quotient by a minimal prime ideal, we may assume that  $A$  is an integral domain. Let  $\mathfrak{q} \subsetneq \mathfrak{p}$  be a prime ideal. We have to show that  $\mathfrak{q} = 0$ . By assumption,  $\mathfrak{p}/fA$  is the unique prime ideal of  $A/fA$ ; hence  $\dim A/fA = 0$ . By the Lemma above, every descending sequence of ideals of  $A/fA$  is stationary. Let us consider the sequence of ideals  $\mathfrak{q}_n := \mathfrak{q}^n A_f \cap A$ . The image of this sequence in  $A/fA$  is stationary. Therefore there exists an  $n_0$  such that for every  $n \geq n_0$ , we have  $\mathfrak{q}_n \subseteq \mathfrak{q}_{n+1} + fA$ . Let  $n \geq n_0$  and let  $x \in \mathfrak{q}_n$ . There exists  $y \in A$  such that  $x - fy \in \mathfrak{q}_{n+1}$ . It follows that  $fy \in \mathfrak{q}_n$  and therefore  $y \in \mathfrak{q}_n$ . That is we have  $\mathfrak{q}_n \subseteq \mathfrak{q}_{n+1} + f\mathfrak{q}_n \subseteq \mathfrak{q}_{n+1} + \mathfrak{p}\mathfrak{q}_n$ . It follows from Nakayama's Lemma that  $\mathfrak{q}_n = \mathfrak{q}_{n+1}$ . As  $\mathfrak{q}_n A_f = \mathfrak{q}^n A_f$ , we have, by Krull's theorem,

$$\mathfrak{q}^{n_0} A_f = \bigcap_{n \geq n_0} \mathfrak{q}^n A_f.$$

It follows from that  $\mathfrak{q} = 0$ . □

**Theorem 3.1.36.** *Let  $A$  be a normal noetherian ring of dimension greater or equal to 1. Then we have the equality*

$$A = \bigcap_{\substack{\mathfrak{p} \in \text{Spec}(A), \\ ht(\mathfrak{p})=1}} A_{\mathfrak{p}}.$$

*Proof.* Let  $A'$  denote the right-hand side in the equality we want to prove, and let us proceed by contradiction: suppose that  $A' \neq A$ . For every  $s \in A' \setminus A$ , let  $I_s$  be the proper ideal  $\{a \in A, as \in A\}$ . Since  $A$  is noetherian, the set of  $I_s$  with  $s \in A' \setminus A$  admits a maximal element  $\mathfrak{q} := I_t$  for some  $t \in A' \setminus A$ . Let us first show that  $\mathfrak{q}$  is a prime ideal of  $A$ . Let  $a, b \in A$  such that  $ab \in \mathfrak{q}$  and  $b \notin \mathfrak{q}$ . Then  $bt \in A' \setminus A$  and we have  $\mathfrak{q} \in I_{bt}$  and  $a \in I_{bt}$ . Since  $\mathfrak{q}$  is a maximal element of the set  $\{I_s, s \in A' \setminus A\}$ , we have  $a \in I_{bt} = \mathfrak{q}$ , and hence  $\mathfrak{q}$  is prime.

Let us now consider the ideal  $t\mathfrak{q}A_{\mathfrak{q}}$  of  $A_{\mathfrak{q}}$ . If it is equal to  $A_{\mathfrak{q}}$ , then  $\mathfrak{q}A_{\mathfrak{q}} = t^{-1}A_{\mathfrak{q}}$ , and  $ht(\mathfrak{q}) = 1$  by the previous Theorem. Therefore  $t \in A_{\mathfrak{q}}$ , and there exists an  $s \in A \setminus \mathfrak{q}$  such that  $st \in A$ . So  $s \in I_t = \mathfrak{q}$ , which is a contradiction. Consequently we need  $t\mathfrak{q}A_{\mathfrak{q}} \subsetneq \mathfrak{q}A_{\mathfrak{q}}$ . Applying the Lemma 3.1.31 to the ideal  $\mathfrak{q}A_{\mathfrak{q}}$  of  $A_{\mathfrak{q}}$ , we see that  $t$  is integral over, and therefore belongs to,  $A_{\mathfrak{q}}$  which is impossible. Hence  $A' = A$ . □

**Theorem 3.1.37.** *Let  $X$  and  $X'$  be two birationally equivalent nonsingular projective varieties over  $k$ . Then  $p_g(X) = p_g(X')$ .*

*Proof.* Let us consider the rational map from  $X$  to  $X'$  and let  $V \subseteq X$  be the largest open set for which there is a morphism  $f : V \rightarrow X'$  representing this rational map. Since  $X$  and  $X'$  are birationally equivalent,



these are locally free sheaves of the same rank  $n = \dim X$ , so we get an induced map on the exterior powers  $f^* : \omega_{X'} \rightarrow \omega_V$ . This map induces a map on the spaces of global sections  $f^* : \Gamma(X', \omega_{X'}) \rightarrow \Gamma(V, \omega_V)$ . Since  $f$  is birational, Corollary 2.0.8. tells us that there exists an open set  $U \subseteq V$  such that  $f(U)$  is open in  $X'$ , and  $f$  induces an isomorphism from  $U$  to  $f(U)$ . Thus we have an isomorphism of sheaves between  $\omega_V|_U$  and  $\omega_{X'}|_{f(U)}$  via  $f$ . So  $f^* : \Gamma(X', \omega_{X'}) \rightarrow \Gamma(V, \omega_V)$  must be injective since a nonzero global section of an invertible sheaf cannot vanish on a dense open subset and an open subset in a variety is dense.

Now we are going to compare  $\Gamma(X, \omega_X)$  with  $\Gamma(V, \omega_V)$ . In order to show that, we are going to prove that  $X \setminus V$  has codimension at least equal to 2. This follows from the Valuative Criterion of Properness. If  $P \in X$  is a point of codimension 1, then since  $X$  is nonsingular,  $\mathcal{O}_{P,X}$  is a discrete valuation ring. We already have a map from the generic point of  $X$  to  $X'$  and  $X$  is projective, hence proper over  $k$ , so there exists a unique morphism  $\text{Spec}(\mathcal{O}_{P,X}) \rightarrow X'$  compatible with the given birational map. This extends to a morphism of some neighborhood of  $P$  to  $X'$  and since  $V$  is the largest open set for which there is a morphism  $f : V \rightarrow X'$  representing this rational map, we have  $P \in V$ , so  $X \setminus V$  has codimension at least 2.

Now we are going to show that the restriction map  $\Gamma(X, \omega_X) \rightarrow \Gamma(V, \omega_V)$  is bijective. It is enough to show that for any open affine subset  $U \subseteq X$  such that  $\omega_X|_U \simeq \mathcal{O}_U$ , the map  $\Gamma(U, \mathcal{O}_U) \rightarrow \Gamma(U \cap V, \mathcal{O}_{U \cap V})$  is bijective. For any prime ideal  $\mathfrak{p} \subset \mathcal{O}_U(U)$  of height 1, we have  $\mathfrak{p} \in U \setminus (U \cap V)$ . Since  $X$  is nonsingular, so is normal, and since  $U \setminus (U \cap V)$  has codimension at least 2 in  $U$ , the results follows immediately from the previous Theorem.

Combining our results, we have that  $p_g(X') \leq p_g(X)$ . By symmetry, we obtain the reverse inequality, that shows that  $p_g(X') = p_g(X)$ .  $\square$

## 3.2 Arithmetic genus of an algebraic variety

### 3.2.1 Arithmetic genus of a projective algebraic set

**Theorem 3.2.1** (Projective Dimension Theorem). *Let  $Y, Z$  be varieties of dimensions  $r, s$  in  $\mathbb{P}^n$ . Then every irreducible component of  $Y \cap Z$  has dimension equal or greater than  $r + s - n$ . Furthermore, if  $r + s - n$  is positive, then  $Y \cap Z$  is nonempty.*

**Definition 3.2.2.** *A numerical polynomial is a polynomial  $P \in \mathbb{Q}[z]$  such that  $P(n) \in \mathbb{Z}$  for all integer  $n \gg 0$ .*

**Proposition 3.2.3.** *1) If  $P \in \mathbb{Q}[z]$  is a numerical polynomial, then there are integers  $c_0, c_1, \dots, c_r$ , uniquely determined, such that*

$$P(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \dots + c_r,$$

where  $\binom{z}{r} = \frac{1}{r!} z(z-1) \cdots (z-r+1)$  is the binomial coefficient function. In particular  $P(n) \in \mathbb{Z}$  for all  $n \in \mathbb{Z}$ .

2) If  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is any function, and if there exists a numerical polynomial  $Q$  such that the difference function  $\delta f : n \mapsto \delta f(n) = f(n+1) - f(n)$  is equal to  $Q(n)$  for all  $n \gg 0$ , then there exists a numerical polynomial  $P$  such that  $f(n) = P(n)$  for all  $n \gg 0$ .

*Proof.* (a) By induction on the degree of  $P$ . If  $P$  is of degree 0, then it is obviously true. Let  $r \geq 1$  such that the assertion is true for any polynomial  $Q \in \mathbb{Q}[z]$  of degree lesser or equal to  $r-1$ . Let  $P \in \mathbb{Q}[z]$ . We can express  $P$  in the above form, with  $c_0, \dots, c_r \in \mathbb{Q}$ . For any polynomial  $Q$ , we define the difference polynomial  $\Delta Q$  by  $\Delta Q(z) = Q(z+1) - Q(z)$ . Since  $\Delta \binom{z}{r} = \binom{z}{r-1}$ , we obtain

$$\Delta P(z) = c_0 \binom{z}{r-1} + c_1 \binom{z}{r-2} + \dots + c_{r-1}.$$

Since  $\Delta P$  is of degree  $r-1$ , we can apply the induction hypothesis to  $\Delta P$ . Consequently,  $c_0, \dots, c_{r-1} \in \mathbb{Z}$ . Since  $P(n) \in \mathbb{Z}$  for  $n \gg 0$ , we obtain that  $c_r \in \mathbb{Z}$ .

(b) Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be any function and  $Q$  be a numerical polynomial such that the difference function  $\Delta f : n \in \mathbb{Z} \rightarrow f(n+1) - f(n) \in \mathbb{Z}$  is equal to  $Q$  for all  $n \in \mathbb{Z}, n \gg 0$ . By the previous point, there exists  $c_0, \dots, c_r \in \mathbb{Z}$  such that

$$Q(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \dots + c_r.$$

We consider the following polynomial

$$P(z) = c_0 \binom{z}{r+1} + c_1 \binom{z}{r-1} + \dots + c_r \binom{z}{1}.$$

Then we have  $\Delta P = Q$ . Since  $Q(n) = \Delta f(n)$  for all  $n \gg 0$ , we have  $\Delta(f - P)(n) = 0$  for all  $n \gg 0$ . Consequently, there exists a constant  $c_{r+1} \in \mathbb{Z}$  such that  $(f - P)(n) = c_{r+1}$  for all  $n \gg 0$ . That means

$$f(n) = P(n) + c_{r+1}$$

for all  $n \gg 0$ . □

**Definition 3.2.4.** Let  $S$  be a graded ring. A graded  $S$ -module is an  $S$ -module  $M$ , together with a decomposition  $M = \bigoplus_{d \in \mathbb{Z}} M_d$ , such that  $S_d \cdot M_e \subseteq M_{d+e}$ .

**Definition 3.2.5.** Let  $S$  be a graded ring and let  $M$  be a graded  $S$ -module.

For any  $l \in \mathbb{Z}$ , we define the twisted module  $M(l)$  with the decomposition  $M(l) = \bigoplus_{d \in \mathbb{Z}} M(l)_d$  such that  $M(l)_d = M_{l+d}$  for any  $d \in \mathbb{Z}$ .

We define the annihilator of  $M$  by  $\text{Ann } M = \{s \in S \mid s \cdot M = 0\}$ .

**Remark:**  $\text{Ann } M$  is a homogeneous ideal of  $S$ .

**Proposition 3.2.6.** *Let  $M$  be a finitely generated graded module over a noetherian graded ring  $S$ . Then there exists a filtration  $0 = M^0 \subseteq M^1 \subseteq \dots \subseteq M^r = M$  by graded submodules, such that for each  $i \in \{1, \dots, r\}$ ,  $M^i/M^{i-1} \simeq (S/\mathfrak{p}_i)(l_i)$ , where  $\mathfrak{p}_i$  is an homogeneous prime ideal of  $S$ , and  $l_i \in \mathbb{Z}$ . The filtration is not unique, but for any such filtration we do have:*

- a) *if  $\mathfrak{p}$  is a homogeneous prime ideal of  $S$ , then  $\mathfrak{p} \supseteq \text{Ann } M \Leftrightarrow \mathfrak{p} \supseteq \mathfrak{p}_i$  for some  $i$ . In particular, the minimal elements of the set  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$  are just the primes which are minimal containing  $\text{Ann } M$ ;*
- b) *for each minimal prime of  $M$ , the number of times which  $\mathfrak{p}$  occurs in the set  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$  is equal to the length of  $M_{\mathfrak{p}}$  over the local ring  $S_{\mathfrak{p}}$  and hence is independent of the filtration.*

*Proof.* • Let us prove first the existence of such a filtration. Let us consider the set  $\mathcal{E}$  of graded submodules of  $M$  which admit such a filtration. Obviously the zero module  $0$  is in  $\mathcal{E}$ , so  $\mathcal{E}$  is nonempty. Since  $M$  is a noetherian module, there is an element  $M'$  in  $\mathcal{E}$  such that  $M'$  is maximal in  $\mathcal{E}$ . Now we're going to show that  $M' = M$ . We consider  $M'' = M/M'$ .

If  $M'' = 0$  then there is nothing to show.

Now, by contradiction, assume that  $M' \subsetneq M$ . Then we consider the set of ideals

$$\mathcal{I} = \{I_m = \text{Ann}(m) \text{ with } m \in M'' \text{ such that } m \neq 0 \text{ is a homogeneous element}\}.$$

Each  $I_m$  is a homogeneous ideal, different of  $S$ . Since  $S$  is a noetherian ring, there exists a nonzero element  $m$  of  $M''$  such that  $I_m$  is a maximal element of the set  $\mathcal{I}$ . We're going to show that  $I_m$  is a prime ideal of  $S$ . Let  $a, b$  in  $S$  such that  $ab$  is in  $I_m$  and  $b$  is not in  $I_m$ . We can decompose  $a$  and  $b$  into homogeneous components so we may assume that  $a$  and  $b$  are homogeneous elements. Since  $b$  is not an element of  $I_m$ , we have that  $bm$  is different from  $0$ . So we have the inclusion  $I_m \subseteq I_{bm}$ . But  $I_m$  is maximal in  $\mathcal{I}$  and  $I_{bm}$  is an element of  $\mathcal{I}$ , so we have  $I_m = I_{bm}$ . Since  $ab \in I_m$  we have  $abm = 0$  and so  $a \in I_{bm} = I_m$ . So  $I_m$  is a homogeneous prime ideal of  $S$ . Let  $l$  be the degree of  $m$ . Then the module  $N \subseteq M''$  generated by  $m$  is isomorphic to  $(S/I_m)(-l)$ . Let  $N' \subseteq M$  be the inverse image of  $N$  in  $M$ . Then  $M' \subseteq N'$  and  $N'/M' \simeq (S/I_m)(-l)$ . That means that  $N'$  also as a filtration of the type required. But  $M'$  was maximal in  $\mathcal{E}$  so we have a contradiction. Consequently  $M' = M$  that shows the existence of the filtration.

- a) Suppose given such a filtration of  $M$ . Let  $\mathfrak{p}$  a homogeneous prime ideal of  $S$ . Then we have  $\text{Ann } M \subseteq \mathfrak{p} \Leftrightarrow \text{Ann}(M^i/M^{i-1}) \subseteq \mathfrak{p}$  for some integer  $i$ . But  $M^i/M^{i-1} \simeq (S/\mathfrak{p}_i)(l_i)$  so  $\text{Ann}(M^i/M^{i-1}) = \mathfrak{p}_i$  which concludes the proof of a).
- b) Let  $\mathfrak{p}$  be a minimal prime ideal of  $M$ . Let us consider the localization of  $M$  in respect to  $\mathfrak{p}$ . Since  $\mathfrak{p}$  is minimal in the set  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ , after localization we will have  $M_{\mathfrak{p}}^i = M_{\mathfrak{p}}^{i-1}$  except when  $\mathfrak{p}_i = \mathfrak{p}$ . In

these cases, we have  $M_{\mathfrak{p}}^i/M_{\mathfrak{p}}^{i-1} \simeq (S/\mathfrak{p})_{\mathfrak{p}} = k(\mathfrak{p})$ , the quotient field of  $S/\mathfrak{p}$ . So  $M_{\mathfrak{p}}$  is an  $S_{\mathfrak{p}}$ -module of finite length equal to the number of times  $\mathfrak{p}$  occurs in the set  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ .  $\square$

**Definition 3.2.7.** Let  $M$  be a graded module over the polynomial ring  $k[x_0, \dots, x_n]$ . We define the Hilbert function  $\phi_M$  of  $M$  by

$$\phi_M : \begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ l & \longmapsto & \dim_k M_l. \end{array}$$

**Theorem 3.2.8** (Hilbert-Serre). Let  $M$  be a finitely generated graded  $k[x_0, \dots, x_n]$ -module. Then there is a unique polynomial  $P_M \in \mathbb{Q}[t]$  such that  $\phi_M(l) = P_M(l)$  for all  $l \gg 0$ .

Moreover we have  $\deg P_M = \dim Z(\text{Ann } M)$ , where  $Z$  denotes the zero set in  $\mathbb{P}^n$  of a homogeneous ideal.

*Proof.* First we notice that if  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence, then we have  $\phi_M = \phi_{M'} + \phi_{M''}$  and  $Z(\text{Ann } M) = Z(\text{Ann } M') \cup Z(\text{Ann } M'')$  so if the theorem is true for  $M'$  and  $M''$ , then it is also true for  $M$ . By hypothesis,  $M$  is a finitely generated graded  $S$ -module where  $S$  is a noetherian graded ring so by the previous proposition, there exists a filtration of  $M$  with quotients of the form  $(S/\mathfrak{p})(l)$  where  $\mathfrak{p}$  is a homogeneous prime ideal of  $S$  and  $l \in \mathbb{Z}$ . So we can reduce to  $M \simeq (S/\mathfrak{p})(l)$ . Since the shift  $l$  corresponds to a change of variables of the form  $z \mapsto z + l$ , it is enough to consider the case  $M = S/\mathfrak{p}$ .

We are going to show the result by induction on  $\dim Z(\text{Ann } M)$ :

If  $\mathfrak{p} = (x_0, \dots, x_n)$  then  $M \simeq k$  so  $\phi_M(l) = 0$  for  $l > 0$  and we have that  $P_M = 0$ , the zero polynomial, is the corresponding polynomial. Making the convention that the zero polynomial has degree  $-1$  and the empty set has dimension  $-1$ , we have  $\deg P_M = \dim Z(\text{Ann } \mathfrak{p})$ .

Now assume that  $\mathfrak{p} \neq (x_0, \dots, x_n)$ . There exists  $0 \leq i \leq n$  such that  $x_i \notin \mathfrak{p}$ . We consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{x_i} M \longrightarrow M'' \longrightarrow 0$$

where  $M'' = M/x_i M$ . Then we have  $\phi_M(l-1) = \phi_M(l) + \phi_{M''}(l)$  and we can rewrite this on the form  $\phi_{M''}(l) = \Delta \phi_M(l-1)$ .

Moreover, we have  $Z(\text{Ann } M'') = Z(\mathfrak{p}, x_i) = Z(\mathfrak{p}) \cup Z(x_i) = Z(\mathfrak{p}) \cap \{x_i = 0\}$ . Since  $x_i \notin \mathfrak{p}$  we have  $Z(\mathfrak{p}) \not\subseteq \{x_i = 0\}$ . So by the Projective Dimension Theorem, we have  $\dim(\text{Ann } M'') = \dim Z(\mathfrak{p}) - \text{codim}(\{x_i = 0\}) = \dim Z(\mathfrak{p}) - 1$ . So, using the induction hypothesis on  $M''$ , we may assume that  $\phi_{M''}$  is a polynomial function, corresponding to a polynomial  $P_{M''}$  of degree  $\dim Z(\text{Ann } M'')$ . From the Proposition 3.2.3., it follows that  $\phi_M$  is a polynomial function, corresponding to a polynomial of degree  $\dim Z(\mathfrak{p})$ , that concludes the proof of the existence.

For the unicity, assume that there exists two such polynomials  $P, Q$  in  $\mathbb{Q}[z]$ . Then for all integer  $l \gg 0$  we have  $P(l) = Q(l)$  so the polynomial  $(P - Q)$  is the zero polynomial that shows  $P = Q$ .  $\square$

**Definition 3.2.9.** The polynomial  $P_M$  of the previous theorem is the Hilbert polynomial of  $M$ .

**Definition 3.2.10.** Let  $Y \subseteq \mathbb{P}^n$  an algebraic set and let  $r$  be the dimension of  $Y$ . We define the Hilbert polynomial of  $Y$  to be the Hilbert polynomial  $P_Y$  of its coordinate ring  $S(Y) = S/I(Y)$ .

**Example:** If  $Y = \mathbb{P}^1$  then the Hilbert polynomial of  $Y$  is  $P_Y(t) = t + 1$ .

**Definition 3.2.11.** Let  $Y$  be an algebraic set of dimension  $r$  in  $\mathbb{P}^n$  with Hilbert polynomial  $P_Y$ . The arithmetic genus of  $Y$  is defined to be  $p_a(Y) = (-1)^r (P_Y(0) - 1)$ .

**Example:** The arithmetic genus of  $\mathbb{P}^1$  is  $p_a(\mathbb{P}^1) = 0$ .

### 3.2.2 Arithmetic genus of a projective scheme

**Definition 3.2.12.** Let  $X$  be a projective scheme over a field  $k$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . We define the Euler characteristic of  $\mathcal{F}$  by

$$\chi(\mathcal{F}) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{F}).$$

**Definition 3.2.13.** Let  $X$  be a projective scheme of dimension  $r$  over a field  $k$ . We define the arithmetic genus  $p_a$  of  $X$  by

$$p_a(X) = (-1)^r (\chi(\mathcal{O}_X) - 1).$$

**Remark:** If  $X$  is an algebraic set of dimension  $r$  in  $\mathbb{P}^n$  then the two definitions of the arithmetic genus coincide.

We will show later, for curves and surfaces, that the arithmetic genus is a birational invariant, but it is still true for any variety.

## 4 The Serre duality theorem

### 4.1 Ext Groups and Sheaves

Let  $(X, \mathcal{O}_X)$  be a ringed topological space. We denote by  $\mathfrak{Mod}(X)$  the category of sheaves of  $\mathcal{O}_X$ -modules on  $(X, \mathcal{O}_X)$ . This is an abelian category.

**Definition 4.1.1.** Let  $\mathcal{F}$  and  $\mathcal{G}$  be sheaves of  $\mathcal{O}_X$ -modules. We define  $\text{Hom}(\mathcal{F}, \mathcal{G})$  the group of  $\mathcal{O}_X$ -modules and  $\mathcal{H}om(\mathcal{F}, \mathcal{G})$  the sheaf defined by: for any open subset  $U \subseteq X$ ,  $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ .

**Proposition 4.1.2.**  $\mathfrak{Mod}(X)$  has enough injectives.

**Definition 4.1.3.** Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. We define the functors  $\text{Ext}^i(\mathcal{F}, \cdot)$  as the right derived functors of  $\text{Hom}(\mathcal{F}, \cdot)$ , and  $\mathcal{E}xt^i(\mathcal{F}, \cdot)$  as the right derived functors of  $\mathcal{H}om(\mathcal{F}, \cdot)$ .

**Proposition 4.1.4.** Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module.

i)  $\text{Ext}^0 = \text{Hom}$ ,

ii) If  $0 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow 0$  is a short exact sequence in  $\mathfrak{Mod}(X)$  then we have a long exact sequence

$$\begin{array}{ccccccc}
& \text{Ext}^0(\mathcal{F}, \mathcal{G}') & \longrightarrow & \text{Ext}^0(\mathcal{F}, \mathcal{G}) & \longrightarrow & \text{Ext}^0(\mathcal{F}, \mathcal{G}'') & \\
& & & & & & \delta^0 \\
& \hookrightarrow & \text{Ext}^1(\mathcal{F}, \mathcal{G}') & \longrightarrow & \text{Ext}^1(\mathcal{F}, \mathcal{G}) & \longrightarrow & \text{Ext}^1(\mathcal{F}, \mathcal{G}'') \\
& & & & & & \delta^1 \\
& \hookrightarrow & \text{Ext}^2(\mathcal{F}, \mathcal{G}') & \longrightarrow & \dots & & 
\end{array}$$

iii) If  $\mathcal{G}$  is an injective object of  $\mathfrak{Mod}(X)$  then  $\text{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$  for any  $i > 0$ .

The same properties hold for the  $\mathcal{E}xt^i(\mathcal{F}, \cdot)$  functors.

**Lemma 4.1.5.** If  $\mathcal{I}$  is an injective object of  $\mathfrak{Mod}(X)$ , then for any open subset  $U \subseteq X$ , we have that  $\mathcal{I}|_U$  is an injective object of  $\mathfrak{Mod}(U)$ .

**Proposition 4.1.6.** For any open subset  $U \subseteq X$  we have

$$\mathcal{E}xt_X^i(\mathcal{F}, \mathcal{G}) \cong \mathcal{E}xt_U^i(\mathcal{F}|_U, \mathcal{G}|_U).$$

**Proposition 4.1.7.** For any  $\mathcal{G} \in \mathfrak{Mod}(X)$  we have:

- i)  $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$ ,
- ii)  $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0$  for any  $i > 0$ ,
- iii)  $\text{Ext}^i(\mathcal{O}_X, \mathcal{G}) \cong H^i(X, \mathcal{G})$  for all  $i \geq 0$ .

*Proof.* The functor  $\mathcal{H}om(\mathcal{O}_X, \cdot)$  is the identity functor, so its derived functors are 0 for  $i > 0$ , so we have i) and ii). The functors  $\text{Hom}(\mathcal{O}_X, \cdot)$  and  $\Gamma(X, \cdot)$  are equal, so their derived functors are the same.  $\square$

**Proposition 4.1.8.** If  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  is a short exact sequence in  $\mathfrak{Mod}(X)$ , then for any  $\mathcal{G}$  we have a long exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ext}^0(\mathcal{F}', \mathcal{G}) & \longrightarrow & \text{Ext}^0(\mathcal{F}, \mathcal{G}) & \longrightarrow & \text{Ext}^0(\mathcal{F}'', \mathcal{G}) \\
& & & & & & \delta^0 \\
& \hookrightarrow & \text{Ext}^1(\mathcal{F}', \mathcal{G}) & \longrightarrow & \text{Ext}^1(\mathcal{F}, \mathcal{G}) & \longrightarrow & \text{Ext}^1(\mathcal{F}'', \mathcal{G}) \\
& & & & & & \delta^1 \\
& \hookrightarrow & \text{Ext}^2(\mathcal{F}', \mathcal{G}) & \longrightarrow & \dots & & 
\end{array}$$

and similarly for the  $\mathcal{E}xt$  sheaves.

*Proof.* Let  $0 \rightarrow \mathcal{G} \rightarrow I^*$  be an injective resolution of  $\mathcal{G}$ . For any injective sheaf  $\mathcal{I}$ , the functor  $\text{Hom}(\cdot, \mathcal{I})$  is exact, so we get a short exact sequence of complexes

$$0 \longrightarrow \text{Hom}(\mathcal{F}'', \mathcal{I}^*) \longrightarrow \text{Hom}(\mathcal{F}, \mathcal{I}^*) \longrightarrow \text{Hom}(\mathcal{F}', \mathcal{I}^*) \longrightarrow 0.$$

Taking the associated long exact sequence of cohomology groups gives the sequence of  $\text{Ext}^i$ . Similarly, the functor  $\mathcal{H}om(\cdot, \mathcal{I})$  is an exact functor from  $\mathfrak{Mod}(X)$  to  $\mathfrak{Mod}(X)$ . Then the same argument gives the exact sequence of  $\mathcal{E}xt^i$ .  $\square$

**Lemma 4.1.9.** *If  $\mathcal{L} \in \mathfrak{Mod}(X)$  is locally free of finite rank, and  $\mathcal{F} \in \mathfrak{Mod}(X)$  is injective, then  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$  is also injective.*

**Proposition 4.1.10.** *Let  $\mathcal{L}$  be a locally free sheaf of finite rank, and let  $\check{\mathcal{L}} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$  be its dual. Then for any  $\mathcal{F}, \mathcal{G}$  in  $\mathfrak{Mod}(X)$ , we have*

$$\text{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \text{Ext}^i(\mathcal{F}, \check{\mathcal{L}} \otimes \mathcal{G}),$$

and for the sheaf  $\mathcal{E}xt$  we have

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \simeq \mathcal{E}xt^i(\mathcal{F}, \check{\mathcal{L}} \otimes \mathcal{G}) \simeq \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \check{\mathcal{L}}.$$

**Definition 4.1.11.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be abelian categories. A cohomological functor, or an exact  $\delta$ -functor, from  $\mathcal{C}$  to  $\mathcal{D}$  is*

- 1) A family of additive functors  $(T^n)_{n \geq 0}$ ,  $T^n : \mathcal{C} \rightarrow \mathcal{D}$
- 2) For all  $n \geq 0$ , for all exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  in  $\mathcal{C}$ , a morphism  $\delta^n : T^n(C) \rightarrow T^{n+1}(A)$  in  $\mathcal{D}$  such that
  - i)  $T^0(A) \rightarrow T^0(B) \rightarrow T^0(C) \xrightarrow{\delta^0} T^1(A) \rightarrow \dots$  is exact in  $\mathcal{D}$ ,
  - ii) If we have a morphism of exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

then for all  $n \geq 0$ , the following diagram commutes

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \downarrow & & \downarrow \\ T^n(C') & \xrightarrow{\delta^n} & T^{n+1}(A). \end{array}$$

**Definition 4.1.12.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be abelian categories. A cohomological functor  $T = (T^n, \delta^n)_{n \geq 0}$  from  $\mathcal{C}$  to  $\mathcal{D}$  is said to be universal if for all cohomological functor  $T' = (T'^n, \delta'^n)_{n \geq 0}$  from  $\mathcal{C}$  to  $\mathcal{D}$  and for all morphism  $g : T^0 \rightarrow T'^0$  in  $\mathcal{C}$ , there exists a unique morphism of cohomological functors  $f : T \rightarrow T'$  such that  $f^0 = g$ .*

**Lemma 4.1.13.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an additive functor between abelian categories. There exists at most one, up to canonical isomorphism, universal cohomological functor  $T$  from  $\mathcal{C}$  to  $\mathcal{D}$ .*

**Theorem 4.1.14.** *Let  $\mathcal{C}, \mathcal{D}$  be abelian categories. Let  $T = (T^n, \delta^n)_{n \geq 0}$  be a cohomological functor from  $\mathcal{C}$  to  $\mathcal{D}$ . If  $T^n$  is effaceable for all  $n > 0$ , then  $T$  is a universal cohomological functor.*

## 4.2 The Serre duality theorem

**Theorem 4.2.1.** *Let  $A$  be a noetherian ring, and let  $X = \mathbb{P}_A^r$  with  $r \geq 1$ .*

a) *The natural map  $S = A[x_1, \dots, x_r] \rightarrow \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$  is an isomorphism of graded  $S$ -modules.*

b)  *$H^i(X, \mathcal{O}_X(n)) = 0$  for  $0 < i < r$  and all  $n \in \mathbb{Z}$ .*

c)  *$H^r(X, \mathcal{O}_X(-r-1)) \simeq A$ .*

d) *The natural map*

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \rightarrow H^r(X, \mathcal{O}_X(-r-1)) \simeq A$$

*is a perfect pairing of finitely generated free  $A$ -modules, for each  $n \in \mathbb{Z}$ .*

**Definition 4.2.2.** *Let  $X$  be a scheme, and let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_X$ -modules. We say that  $\mathcal{F}$  is generated by global sections if there is a family of global sections  $\{s_i\}_{i \in I}$ , such that for each  $x \in X$ , the images of  $s_i$  in the stalk  $\mathcal{F}_x$  generate that stalk as an  $\mathcal{O}_X$ -module.*

**Theorem 4.2.3.** *Let  $X$  be a projective scheme over a noetherian ring  $A$ , let  $\mathcal{O}(1)$  be a very ample invertible sheaf on  $X$ , and let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Then there is an integer  $n_0$  such that for all  $n \geq n_0$ , the sheaf  $\mathcal{F}(n)$  can be generated by a finite number of global sections.*

**Corollary 4.2.4.** *Let  $X$  be projective over a noetherian ring  $A$ . Then any coherent sheaf  $\mathcal{F}$  on  $X$  can be written as a quotient of a sheaf  $\mathcal{E}$ , where  $\mathcal{E}$  is a finite direct sum of twisted structure sheaves  $\mathcal{O}(n_i)$  for various integers  $n_i$ .*

**Theorem 4.2.5.** *Let  $X$  be a projective scheme over a noetherian ring  $A$ , and let  $\mathcal{O}_X(1)$  be a very ample invertible sheaf on  $X$  over  $\text{Spec}(A)$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Then there exists an integer  $n_0$ , depending on  $\mathcal{F}$ , such that for each  $i > 0$  and  $n \geq n_0$ ,  $H^i(X, \mathcal{F}(n)) = 0$ .*

Let  $k$  be an algebraically closed field, let  $X = \mathbb{P}_k^n$  be the  $n$ -dimensional projective space over  $k$ , and let  $\omega_X = \wedge^n \Omega_{X/k}$  be the canonical sheaf on  $X$ .

**Theorem 4.2.6** (Duality for  $\mathbb{P}_k^n$ ). *Let  $X = \mathbb{P}_k^n$  over a field  $k$ . Then:*

i)  *$H^n(X, \omega_X) \cong k$ . Fix one such isomorphism,*

ii) *for any coherent sheaf  $\mathcal{F}$  on  $X$ , the natural pairing*

$$\text{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \rightarrow H^n(X, \omega_X) \cong k$$

*is a perfect pairing of finite dimensional vector spaces over  $k$ ,*

iii) *for every  $i \geq 0$  there is a natural functorial isomorphism*

$$\text{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})',$$

*where  $'$  denotes the dual vector space, which for  $i = 0$  is the one induced by the pairing of ii).*



*Proof.* i) It follows from the Theorem 3.1.23 that  $\omega_X \simeq \mathcal{O}_X(-n-1)$ , thus we have  $H^n(X, \omega_X) \simeq H^n(X, \mathcal{O}_X(-n-1)) \simeq k$  by the Theorem 4.2.1.

ii) A homomorphism of  $\mathcal{F}$  to  $\omega_X$  induces a map of cohomology groups  $H^n(X, \mathcal{F}) \rightarrow H^n(X, \omega_X)$ , which gives the natural pairing. If  $\mathcal{F} \simeq \mathcal{O}(q)$  for some  $q \in \mathbb{Z}$ , then  $\text{Hom}(\mathcal{F}, \omega_X) \simeq H^0(X, \omega_X(-q))$ , so the result follows from the Theorem 4.2.1. The result holds also for a finite direct sum of sheaves of the form  $\mathcal{O}(q_i)$ . Now assume that  $\mathcal{F}$  is an arbitrary coherent sheaf on  $X$ . Since  $\mathcal{F}$  is coherent,  $\mathcal{F}$  can be written as a cokernel of a map of sheaves  $\mathcal{E}_1 \rightarrow \mathcal{E}_2$  where each  $\mathcal{E}_i$  is a direct sum of sheaves  $\mathcal{O}(q_i)$ . Hence we have an exact sequence

$$\mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{F} \rightarrow 0.$$

Since the functors  $\text{Hom}(\cdot, \omega_X)$  and  $H^n(X, \cdot)'$  are left-exact and contravariant, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(\mathcal{F}, \omega_X) & \longrightarrow & \text{Hom}(\mathcal{E}_2, \omega_X) & \longrightarrow & \text{Hom}(\mathcal{E}_1, \omega_X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^n(X, \mathcal{F})' & \longrightarrow & H^n(X, \mathcal{E}_2)' & \longrightarrow & H^n(X, \mathcal{E}_1)' \end{array}$$

where the horizontal arrows are exact. The first vertical arrow is obviously an isomorphism. Since  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are direct sums of sheaves  $\mathcal{O}(q_i)$ , both arrows  $\text{Hom}(\mathcal{E}_i, \omega_X) \rightarrow H^n(X, \mathcal{E}_i)'$  are isomorphisms and so  $\text{Hom}(\mathcal{E}_1, \omega_X) \rightarrow H^n(X, \mathcal{E}_1)'$  is a monomorphism. Consequently,  $\text{Hom}(\mathcal{F}, \omega_X) \rightarrow H^n(X, \mathcal{F})'$  is an isomorphism.

iii) Let us denote  $\mathbf{Coh}(X)$  the abelian category of coherent sheaves on  $X$ . Then  $(\text{Ext}^i(\cdot, \omega_X))_{i \geq 0}$  and  $(H^i(X, \cdot))'_{i \geq 0}$  both are cohomological functors from  $\mathbf{Coh}(X)$  to  $\mathbf{Coh}(X)$  since they are the  $i$ -th right derived functor of  $\text{Hom}(\cdot, \omega_X)$  and  $\Gamma(X, \cdot)$  respectively. By the previous point, we have that  $\text{Ext}^0(\cdot, \omega_X) \simeq H^0(X, \cdot)$ . Hence if we show that for  $i > 0$ ,  $\text{Ext}^i(\cdot, \omega_X)$  and  $H^{n-i}(X, \cdot)$  are coexact, then by the Theorem 4.1.14, these two cohomological functors will be universal cohomological functors from  $\mathbf{Coh}(X)$  to  $\mathbf{Coh}(X)$  and by the Lemma 4.1.13, we will have that these two universal cohomological functors from  $\mathbf{Coh}(X)$  to  $\mathbf{Coh}(X)$  are isomorphic. Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . Then we can write  $\mathcal{F}$  as the quotient of a sheaf  $\mathcal{E} = \bigoplus_{i=1}^N \mathcal{O}(-q)$ , with  $q \gg 0$ . Then, by the Theorem 4.2.1, we have for  $i > 0$

$$\text{Ext}^i(\mathcal{E}, \omega_X) = \bigoplus H^i(X, \mathcal{O}(-q)) = 0.$$

Hence  $\text{Ext}^i(\cdot, \omega_X)$  is coexact for  $i > 0$ . Furthermore we have  $H^{n-i}(X, \mathcal{E})' = \bigoplus H^{n-i}(X, \mathcal{O}(-q))'$ , which is 0 for  $i > 0$ ,  $q > 0$ . Hence these functors are coexact for  $i > 0$ . Consequently, these two cohomological functors from  $\mathbf{Coh}(X)$  to  $\mathbf{Coh}(X)$  are universal, hence isomorphic.  $\square$

**Definition 4.2.7** (Dualizing sheaf). *Let  $X$  be a proper scheme of dimension  $n$  over a field  $k$ . A dualizing sheaf for  $X$  is a coherent sheaf  $\omega_X^\circ$  on  $X$ , together with a trace morphism*

$$t : H^n(X, \omega_X^\circ) \longrightarrow k$$

*such that for all coherent sheaves  $\mathcal{F}$ , the natural pairing*

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \rightarrow H^n(X, \omega_X^\circ)$$

*followed by  $t$  gives an isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \xrightarrow{\sim} H^n(X, \mathcal{F})'.$$

**Proposition 4.2.8.** *Let  $X$  be a proper scheme over  $k$ . Then a dualizing sheaf for  $X$ , if it exists, is unique up to a unique isomorphism of sheaves. More precisely, if  $\omega^\circ$  is a dualizing sheaf for  $X$ , with its trace map  $t$ , and if  $\omega'^\circ$ , with  $t'$ , is another dualizing sheaf for  $X$ , then there exists a unique isomorphism  $\phi : \omega^\circ \longrightarrow \omega'^\circ$  such that  $t = t' \circ H^n(\phi)$ .*

*Proof.* Taking  $\mathcal{F} = \omega'^\circ$  in the definition of a dualizing sheaf on  $X$ , we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\omega'^\circ, \omega^\circ) \simeq H^n(X, \omega'^\circ)'.$$

Then there exists a unique morphism  $\phi : \omega'^\circ \longrightarrow \omega^\circ$  such that  $t \circ H^n(\phi) = t'$ . By symmetry, we have a unique morphism  $\psi : \omega^\circ \longrightarrow \omega'^\circ$  such that  $t' \circ H^n(\psi) = t$ . Then  $\phi \circ \psi : \omega^\circ \longrightarrow \omega^\circ$  verifies  $t = t \circ H^n(\phi \circ \psi)$ . By the uniqueness, this implies that  $\phi \circ \psi$  is the identity map of  $\omega^\circ$ . By symmetry of the problem, we have that  $\psi \circ \phi : \omega'^\circ \longrightarrow \omega'^\circ$  is the identity map of  $\omega'^\circ$ . Hence  $\psi$  is an isomorphism.  $\square$

We will prove that any projective scheme over  $k$  admits a dualizing sheaf.

**Lemma 4.2.9.** *Let  $X$  be a closed subscheme of codimension  $r$  of  $\mathbb{P} = \mathbb{P}_k^n$ . Then  $\mathcal{E}xt_{\mathbb{P}}^i(\mathcal{O}_X, \omega_{\mathbb{P}}) = 0$  for all  $i < r$ .*

**Lemma 4.2.10.** *Let  $X$  be a closed subscheme of codimension  $r$  of  $\mathbb{P} = \mathbb{P}_k^n$  and let  $\omega_X^\circ = \mathcal{E}xt_{\mathbb{P}}^r(\mathcal{O}_X, \omega_{\mathbb{P}})$ . Then for any  $\mathcal{O}_X$ -module  $\mathcal{F}$ , there is a functorial isomorphism*

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \simeq \mathrm{Ext}_{\mathbb{P}}^r(\mathcal{F}, \omega_{\mathbb{P}}).$$

*Proof.* Let  $0 \rightarrow \omega_{\mathbb{P}} \rightarrow \mathcal{I}^*$  be an injective resolution of  $\omega_{\mathbb{P}}$  in  $\mathfrak{Mod}(\mathbb{P})$ . We are going to calculate the cohomology groups  $\mathrm{Ext}_{\mathbb{P}}^i(\mathcal{F}, \omega_{\mathbb{P}})$  of the complex  $\mathrm{Hom}_{\mathbb{P}}(\mathcal{F}, \mathcal{I}^*)$ . Since  $\mathcal{F}$  is an  $\mathcal{O}_X$ -module, any morphism  $\mathcal{F} \rightarrow \mathcal{I}^i$  factors through  $\mathcal{G}^i = \mathcal{H}om_{\mathbb{P}}(\mathcal{O}_X, \mathcal{I}^i)$ . Thus we have  $\mathrm{Ext}_{\mathbb{P}}^i(\mathcal{F}, \omega_{\mathbb{P}}) = H^i(\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}^*))$ . For  $\mathcal{F} \in \mathfrak{Mod}(X)$ , we have  $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}^i) = \mathrm{Hom}_{\mathbb{P}}(\mathcal{F}, \mathcal{I}^i)$  so  $\mathrm{Hom}_X(\cdot, \mathcal{G}^i)$  is an exact functor, hence  $\mathcal{G}^i$  is an injective  $\mathcal{O}_X$ -module. Furthermore, by the previous Lemma, we have  $H^i(\mathcal{G}^*) = 0$  for  $i < r$ , so the complex  $\mathcal{G}^*$  is exact up to the  $r$ -th step,

and it is split exact up to the  $r$ -th step. So we can write the complex as a direct sum  $\mathcal{G}^* = \mathcal{G}_1^* \oplus \mathcal{G}_2^*$  where  $\mathcal{G}_1^*$  is an injective complex in degrees  $0 \leq i \leq r$  and exact, and  $\mathcal{G}_2^*$  is in degrees  $i \geq r$ . It follows that  $\omega_X^\circ = \ker(d^r : \mathcal{G}_2^r \rightarrow \mathcal{G}_2^{r+1})$ , and that for any  $\mathcal{O}_X$ -module  $\mathcal{F}$ ,  $\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \simeq \mathrm{Ext}_{\mathbb{P}}^r(\mathcal{F}, \omega_{\mathbb{P}})$ .  $\square$

**Proposition 4.2.11.** *Let  $X$  be a projective scheme over a field  $k$ . Then  $X$  has a dualizing sheaf.*

*Proof.* We embed  $X$  in a closed subscheme of  $\mathbb{P} = \mathbb{P}_k^n$  for some  $n$ . Let  $r$  be its codimension and let  $\omega_X^\circ = \mathcal{E}xt_{\mathbb{P}}^r(\mathcal{O}_X, \omega_{\mathbb{P}})$ . Then, by the previous Lemma, we have an isomorphism for any  $\mathcal{O}_X$ -module  $\mathcal{F}$ :  $\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \simeq \mathrm{Ext}_{\mathbb{P}}^r(\mathcal{F}, \omega_{\mathbb{P}})$ . Furthermore, by the Theorem 4.2.6, we have for  $\mathcal{F}$  coherent on  $X$ ,  $\mathrm{Ext}_{\mathbb{P}}^r(\mathcal{F}, \omega_{\mathbb{P}}) \simeq H^{n-r}(\mathbb{P}, \mathcal{F})'$ . Since  $n - r = \dim(X) = N$ , we obtain a functorial isomorphism, for  $\mathcal{F} \in \mathbf{Coh}(X)$ ,  $\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \simeq H^N(X, \mathcal{F})'$ . Taking  $\mathcal{F} = \omega_X^\circ$ , we define our trace map as the image of the element  $1 \in \mathrm{Hom}(\omega_X^\circ, \omega_X^\circ)$  by this isomorphism, which gives us a map  $t : H^N(X, \omega_X^\circ) \rightarrow k$ . Then, by functoriality,  $(\omega_X^\circ, t)$  is a dualizing sheaf on  $X$ .  $\square$

**Definition 4.2.12.** *Let  $A$  be a ring and let  $M$  be an  $A$ -module. A sequence  $x_1, \dots, x_r$  of elements of  $A$  is called a regular sequence for  $M$  if  $x_1$  is not a zero divisor in  $M$  and for all  $2 \leq i \leq r$ ,  $x_i$  is not a zero divisor in  $M/(x_1, \dots, x_{i-1})M$ .*

**Definition 4.2.13.** *Let  $A$  be a local ring with maximal ideal  $\mathfrak{m}$  and let  $M$  be an  $A$ -module. The depth of  $M$  is the maximum length of a regular sequence  $x_1, \dots, x_r$  for  $M$  with all  $x_i \in \mathfrak{m}$ .*

**Definition 4.2.14.** *Let  $A$  be a local ring. We say that  $A$  is Cohen-Macaulay if  $\mathrm{depth} A = \dim A$ .*

**Definition 4.2.15.** *We say that a scheme is Cohen-Macaulay if all of its local rings are Cohen-Macaulay.*

**Theorem 4.2.16** (Duality for a projective scheme). *Let  $X$  be a projective scheme of dimension  $n$  over an algebraically closed field  $k$ . Let  $\omega_X^\circ$  be a dualizing sheaf on  $X$ , and let  $\mathcal{O}(1)$  be a very ample sheaf on  $X$ . Then:*

a) *for all  $i \geq 0$  and  $\mathcal{F}$  coherent on  $X$ , there are natural functorial maps*

$$\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \rightarrow H^{n-1}(X, \mathcal{F})'$$

*such that  $\theta^0$  is the map given in the definition of dualizing sheaf above ;*

b) *the following conditions are equivalent:*

- i)  *$X$  is Cohen-Macaulay and all its irreducible components have the same dimension;*
- ii) *for any  $\mathcal{F}$  locally free on  $X$ , we have  $H^i(X, \mathcal{F}(-q)) = 0$  for  $i < n$  and  $q \gg 0$ ;*
- iii) *the maps  $\theta^i$  of a) are isomorphisms for all  $i \geq 0$  and for all coherent sheaf  $\mathcal{F}$  on  $X$ .*

**Corollary 4.2.17.** *Let  $X$  be a closed subscheme of  $P = \mathbb{P}_k^n$  which is a local complete intersection of codimension  $r$ . Let  $\mathcal{I}$  be the ideal sheaf of  $X$ .*

*Then  $\omega_X^\circ \simeq \omega_P \otimes \wedge^r (\mathcal{I}/\mathcal{I}^2)^\vee$ .*

**Definition 4.2.18.** *Let  $A$  be a ring, and let  $f_1, \dots, f_r \in A$ . We define the Koszul complex  $K. (f_1, \dots, f_r)$  as follows:  $K_1$  is a free  $A$ -module of rank  $r$  with basis  $e_1, \dots, e_r$ . For each  $p = 0, \dots, r$ ,  $K_p = \wedge^p K_1$ . We define the boundary map  $d : K_p \rightarrow K_{p-1}$  by its action on the basis vectors:*

$$d(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum (-1)^{j-1} f_{i_j} e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \wedge \dots \wedge e_{i_p}.$$

*Thus  $K. (f_1, \dots, f_r)$  is a complex of  $A$ -modules. If  $M$  is any  $A$ -module, then we define  $K. (f_1, \dots, f_r; M) := K. (f_1, \dots, f_r) \otimes_A M$ .*

**Theorem 4.2.19.** *Let  $X$  be a closed subscheme of  $P = \mathbb{P}_k^N$  which is a local complete intersection of codimension  $r$ . Let  $\mathcal{I}$  be the ideal sheaf of  $X$ . Then  $\omega_X^\circ \simeq \omega_P \otimes \wedge^r (\mathcal{I}/\mathcal{I}^2)^\vee$ . In particular,  $\omega_X^\circ$  is an invertible sheaf on  $X$ .*

**Proposition 4.2.20.** *If  $X$  is a projective nonsingular variety over an algebraically closed field, then the dualizing sheaf  $\omega_X^\circ$  is isomorphic to the canonical sheaf  $\omega_X$ .*

**Corollary 4.2.21.** *If  $X$  is a projective nonsingular variety of dimension  $n$ , then we have an isomorphism  $H^n(X, \omega_X) \simeq k$ .*

**Corollary 4.2.22.** *If  $X$  is a projective nonsingular curve, then we have  $p_g(X) = p_a(X)$ .*

## 5 Divisors

**Definition 5.0.1.** *A scheme  $X$  is said regular in codimension one, or nonsingular in codimension one, if every local ring  $\mathcal{O}_x$  of  $X$  of dimension one is regular.*

In this section,  $X$  will be a noetherian integral separated scheme which is regular in codimension one.

**Definition 5.0.2.** *A prime divisor on  $X$  is a closed integral subscheme  $Y$  of codimension one. A Weil divisor is an element of the free abelian group  $\text{Div}(X)$  generated by the prime divisors.*

We will write a Weil divisor as  $D = \sum n_i Y_i$ , where the  $Y_i$  are the prime divisors, the  $n_i$  are integers, and only finitely many  $n_i$  are different from zero.

**Definition 5.0.3.** *Let  $D = \sum n_i Y_i \in \text{Div}(X)$ . If all the  $n_i$  are greater or equal to zero, then we say that  $D$  is effective.*

**Definition 5.0.4.** *Let  $Y$  is a prime divisor on  $X$  and let  $\eta \in Y$  be its generic point. Then the local ring  $\mathcal{O}_{\eta, X}$  is a discrete valuation ring with quotient field  $K$ , the function field of  $X$ . We call the corresponding valuation  $\nu_Y$  the valuation of  $Y$ .*

**Remark:** With the notations of the previous definition, let  $f \in K^*$  be any nonzero rational function on  $X$ . Then  $\nu_Y(f)$  is an integer. If it is positive, then we say that  $f$  has a zero along  $Y$ , of that order. If it is negative, then we say that  $f$  has a pole along  $Y$ , of order  $-\nu_Y(f)$ .

**Lemma 5.0.5.** *Let  $f \in K^*$  be a nonzero function on  $X$ . Then  $\nu_Y(f) = 0$  for all except finitely many prime divisors  $Y$ .*

*Proof.* Let  $U = \text{Spec}(A)$  be an affine open subset of  $X$  on which  $f$  is regular. We define  $Z = X \setminus U$ , which is a proper closed subset of  $X$ . Since  $X$  is noetherian,  $Z$  can contain at most finitely many prime divisor of  $X$ . Hence, to prove the Lemma, we have to show that there are only finitely many prime divisors  $Y$  of  $U$  for which  $\nu_Y(f)$  is a nonzero integer. We have  $\nu_Y(f) \geq 0$  because  $f$  is regular and  $\nu_Y(f) > 0$  if  $Y$  is contained in the closed subset of  $U$  defined by the ideal  $fA$  of  $A$ . Since  $f$  is nonzero, this closed subset of  $U$  is a proper closed subset, and so contains only finitely many closed irreducible subsets of codimension one of  $U$ .  $\square$

**Definition 5.0.6.** *Let  $f \in K^*$ . We define the divisor of  $f$ , denoted  $(f)$ , by*

$$(f) = \sum \nu_Y(f) \cdot Y$$

*where the sum is taken over all prime divisors of  $X$ . Such a divisor is called a principal divisor.*

**Remark:** The subset of  $\text{Div}(X)$  formed by the principal divisors is a subgroup of  $\text{Div}(X)$ .

**Definition 5.0.7.** *Two divisors  $D$  and  $D'$  are linearly equivalent, denoted by  $D \sim D'$ , if  $D - D'$  is a principal divisor. The group  $\text{Div}(X)$  of all divisors divided by the subgroup of principal divisors is called the divisor class group of  $X$ , and is denoted by  $\text{Cl}(X)$ .*

**Proposition 5.0.8.** *Let  $A$  be a noetherian domain. Then  $A$  is a Unique Factorisation Domain if and only if  $X = \text{Spec}(A)$  is normal and  $\text{Cl}(X) = 0$ .*

## 5.1 Divisors on curve

A curve over  $k$  is an integral separated scheme  $X$  of finite type over  $k$ , of dimension one. If all the local rings of  $X$  are regular local rings, we say that  $X$  is nonsingular.

**Proposition 5.1.1.** *Let  $X$  be a nonsingular curve over  $k$  with function field  $K$ . Then the following assertions are equivalent:*

- (i)  $X$  is projective,
- (ii)  $X$  is proper over  $k$ ,
- (iii)  $X \simeq t(C_K)$  where  $C_k$  is an abstract nonsingular curve and  $t$  is functor from varieties to schemes.

**Definition 5.1.2.** Let  $f : X \rightarrow Y$  be a finite morphism of curves. We define the degree of  $f$  as the degree  $[K(X) : K(Y)]$ .

If  $X$  is a nonsingular curve then  $X$  is a noetherian integral separated scheme which is regular in codimension one so the divisors on  $X$  are well-defined. From now,  $X$  will be a nonsingular curve. In this case, a prime divisor is a closed point so we can write, for a divisor,  $D = \sum n_i P_i$  where the  $P_i$  are the closed points and  $n_i \in \mathbb{Z}$ .

**Definition 5.1.3.** Let  $D = \sum n_i P_i$  be a divisor on  $X$ . We define the degree of  $D$  to be  $\deg(D) = \sum n_i$ . It is a finite integer since all the  $n_i$  except for finitely many are zero.

**Proposition 5.1.4.** Let  $X$  be a proper nonsingular curve over  $k$ , let  $Y$  be a nonsingular curve and let  $f : X \rightarrow Y$  be a morphism. Then either:

- (1)  $f(X) = \text{a point}$ , or
- (2)  $f(X) = Y$ .

In case (2),  $K(X)$  is a finite extension field of  $K(Y)$ ,  $f$  is a finite morphism and also  $Y$  is proper over  $k$ .

*Proof.*  $X$  and  $Y$  are  $S = \text{Spec}(k)$ -schemes such that  $X \rightarrow S$  is proper and  $Y \rightarrow S$  is separated. So  $f : X \rightarrow Y$  is closed and even proper because we have the following commutative diagram

$$\begin{array}{ccccc} \Gamma_f & \xrightarrow{\Delta'} & X \times_S Y & \xrightarrow{q} & Y \\ & \searrow & \downarrow p & \nearrow f & \\ & & X & & \end{array}$$

where  $q = \pi_Y$  is proper,  $\Gamma_f = (X \times_S Y) \times_{Y \times_S Y} Y$  is the graph of  $f$  and  $\Delta'$  is obtained by the closed immersion  $\Delta_{Y/S} : Y \rightarrow Y \times_S Y$  base changed to  $X \times_S Y \rightarrow Y \times_S Y$ . Moreover  $f(X)$  is irreducible so either  $f(X) = \text{a point}$  (1) or  $f(X) = Y$  (2), and in (2)  $Y$  is also proper over  $k$ .

Assume that  $f(X) = Y$ . Then  $f$  is dominant so it induces an inclusion  $K(Y) \subseteq K(X)$ . Both fields are finitely generated extension fields of transcendental degree 1 of  $k$ , so  $K(X)$  is a finite algebraic extension of  $K(Y)$ . To show that  $f$  is a finite morphism. Let  $V = \text{Spec}(B)$  be any open affine subset of  $Y$  and let  $A$  be the integral closure of  $B$  in  $K(X)$ . Since  $K(X)$  is a finite algebraic extension of  $K(Y)$ ,  $A$  is a finite  $B$ -module and  $\text{Spec}(A)$  is isomorphic to an open subset  $U$  of  $X$ . Hence we have  $U = f^{-1}(V)$ , so this shows that  $f$  is a finite morphism.  $\square$

**Definition 5.1.5.** If  $f : X \rightarrow Y$  is a finite morphism of nonsingular curves, we define a homomorphism  $f^* : \text{Div}(Y) \rightarrow \text{Div}(X)$  as follows. For any point  $Q \in Y$ , let  $t \in \mathcal{O}_Q$  be an element of  $K(Y)$  with  $\nu_Q(t) = 1$  where  $\nu_Q$  is the valuation corresponding to the discrete valuation ring  $\mathcal{O}_Q$ . A such  $t \in \mathcal{O}_Q$  is called a local parameter at  $Q$ . We define  $f^*Q = \sum_{f(P)=Q} \nu_P(t) \cdot P$ . It is a finite sum, since  $f$  is finite, so it defines a divisor on  $X$ .

**Proposition 5.1.6.** *With the notations of the previous Definition,  $f^*Q$  doesn't depend of the choice of the local parameter.*

*Proof.* Let  $t, t' \in \mathcal{O}_Q$  be two such local parameters. The ring  $\mathcal{O}_Q$  is a regular local ring and  $\nu_Q(t) = \nu_Q(t')$  so there exists a unit  $u \in \mathcal{O}_Q$  such that  $t' = ut$ . For any point  $P$  such that  $f(P) = Q$ ,  $u$  will be a unit in  $\mathcal{O}_P$  so we have  $\nu_P(t) = \nu_P(t')$ .  $\square$

**Remark:**  $f^* : \text{Div}(Y) \longrightarrow \text{Div}(X)$  induces an homomorphism  $f^* : \text{Cl}(Y) \longrightarrow \text{Cl}(X)$ .

**Proposition 5.1.7.** *Let  $f : X \longrightarrow Y$  be a finite morphism of nonsingular curves. Then for any divisor  $D$  on  $Y$ , we have  $\deg(f^*D) = \deg(f) \cdot \deg(D)$ .*

*Proof.* By definition of the degree, if  $D, D' \in \text{Div}(X)$  then we have  $\deg(D + D') = \deg(D) + \deg(D')$  so it is enough to show that for any closed point  $Q \in Y$ , we have  $\deg(f^*Q) = \deg(f)$  since we have  $\deg(Q) = 1$ . Let  $V = \text{Spec}(A)$  be an open affine subset of  $Y$  containing  $Q$ . Let  $B$  be the integral closure of  $A$  in  $K[X]$ . Then we have that  $U = \text{Spec}(B) = f^{-1}(V)$  and is an open subset of  $X$ . Let  $\mathfrak{m}_Q$  be the maximal ideal (since  $Q$  is a closed point of  $V$ ) of  $Q$  in  $B$ . Localizing  $B$  and  $A$  with respect to  $S = A \setminus \mathfrak{m}_Q$ , we obtain a ring extension  $\mathcal{O}_Q \hookrightarrow B'$  where  $B'$  is a finitely generated  $\mathcal{O}_Q$ -module.  $B'$  is torsion-free and has rank equal to  $n = [K(X) : K(Y)]$  so  $B'$  is a free  $\mathcal{O}_Q$ -module of rank  $n = \deg(f)$ . It follows that if  $t$  is a local parameter at  $Q$  then  $B'/tB'$  is a  $k$ -vector space of dimension  $n$ . In addition, the points  $P_i \in X$  such that  $f(P_i) = Q$  are in one to one correspondance with the maximal ideals  $\mathfrak{m}_i$  of  $B'$ , and for each  $i$ ,  $B'_{\mathfrak{m}_i} = \mathcal{O}_{P_i}$ . Then we have  $tB' = \bigcap_i (tB'_{\mathfrak{m}_i} \cap B')$  so it follows that  $B'/tB' = B' / \left( \bigcap_i (tB'_{\mathfrak{m}_i} \cap B') \right)$  so by the Chinese remainder theorem, we have  $B'/tB' = \bigoplus_i B' / (tB'_{\mathfrak{m}_i} \cap B')$  and we obtain that

$$\dim_k B'/tB' = \sum_i \dim_k B' / (tB'_{\mathfrak{m}_i} \cap B').$$

But we have  $B' / (tB'_{\mathfrak{m}_i} \cap B') \simeq B'_{\mathfrak{m}_i} / tB'_{\mathfrak{m}_i} = \mathcal{O}_{P_i} / t\mathcal{O}_{P_i}$  so  $\dim_k B' / (tB'_{\mathfrak{m}_i} \cap B') = \nu_{P_i}(t)$ . So we have that  $f^*Q = \sum \nu_{P_i}(t) \cdot P_i$ , that means that  $\deg(f^*Q) = \deg(f)$ .  $\square$

**Corollary 5.1.8.** *A principal divisor on a proper nonsingular curve  $X$  has degree zero. Consequently the degree function induces a surjective homomorphism*

$$\deg : \text{Cl}(X) \longrightarrow \mathbb{Z}.$$

## 5.2 Cartier divisors

**Definition 5.2.1.** *Let  $X$  be a scheme. For each open affine subset  $U = \text{Spec}(A)$ , let  $S$  be the set of the elements of  $A$  which are not zero divisors, and let  $K(U)$  be the localization of  $A$  by the multiplicative system  $S$ . We call  $K(U)$  the total quotient ring of  $A$ . Since  $U$  varies, the rings  $K(U)$  form a presheaf. We consider the sheaf of rings  $\mathcal{K}$  which is*

the sheaf associated to the presheaf form by the  $K(U)$ . Then we call  $\mathcal{K}$  the sheaf of total quotient rings of  $\mathcal{O}$ .

We denote by  $\mathcal{K}^*$  the sheaf, of multiplicative groups, of invertible elements in the sheaf of rings  $\mathcal{K}$ .

Similarly  $\mathcal{O}^*$  is the sheaf of invertible elements in  $\mathcal{O}$ .

**Remark:** The sheaf  $\mathcal{K}$  replaces the concept of function field of an integral scheme.

**Definition 5.2.2.** A Cartier divisor on a scheme  $X$  is a global section of the sheaf  $\mathcal{K}^*/\mathcal{O}^*$ .

A Cartier divisor is principal if it is in the image of the natural map  $\Gamma(X, \mathcal{K}^*) \rightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$ .

Two Cartier divisors  $D$  and  $D'$  are linearly equivalent if  $D - D'$  is a principal divisor. We then write  $D \sim D'$ .

A Cartier divisor  $D$  is called effective if it is in the image of the canonical map  $\Gamma(X, \mathcal{O}_X \cap \mathcal{K}_X^*) \rightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ . We then write  $D \geq 0$ .

**Remark:** Although the group operation on  $\mathcal{K}^*/\mathcal{O}^*$  is multiplication, we will use the language of additive groups when speaking of Cartier divisors, so as to preserve the analogy with Weil divisors.

We can represent a Cartier divisor  $D$  by a system  $\{(U_i, f_i)_i\}$  where the  $U_i$  are open subsets of  $X$  forming a covering of  $X$ ,  $f_i$  is the quotient of two regular elements of  $\mathcal{O}_X(U_i)$ , and  $f_i|_{U_i \cap U_j} \in f_j|_{U_i \cap U_j} \mathcal{O}_X(U_i \cap U_j)^*$  for every  $i, j$ .

**Proposition 5.2.3.** Let  $X$  be an integral, separated noetherian scheme such that all of whose local rings are Unique Factorisation Domain. Then the group  $\text{Div}(X)$  of Weil divisors on  $X$  is isomorphic to the group of Cartier divisors  $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$ , and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.

*Proof.* Since a Unique Factorisation Domain is integrally closed, we have that  $\mathcal{O}_{X,x}$  is integrally closed for any  $x \in X$ , so  $X$  is a normal scheme. Hence  $X$  is a noetherian integral separated scheme which is regular in codimension one, so the Weil divisors on  $X$  are well-defined. Since  $X$  is an integral scheme, then for any open affine subset  $U \subseteq X$ ,  $\mathcal{O}_X(U)$  is an integral domain, so the elements of  $\mathcal{O}_X(U)$  which are not zero divisors are the elements which are not zero. Hence  $K(U) = \text{Frac}(\mathcal{O}_X(U)) \simeq K(X) = K$ , where  $K(X)$  is the field of fractions on  $X$ , since  $X$  is an integral scheme. So  $\mathcal{K}$  is the constant sheaf corresponding to the function field of  $X$ .

Now let a Cartier divisor be represented by  $\{(U_i, f_i)_i\}$  where  $\{U_i\}$  is an open cover of  $X$ , and  $f_i \in \Gamma(U_i, \mathcal{K}^*) = K^*$ . Let us define the associated Weil divisor as follows. For each prime divisor  $Y$ , we take the coefficient of  $Y$  to be  $\nu_Y(f_i)$ , where  $i$  is any index such that  $f_i \in Y \cap U_i \neq \emptyset$ . If  $j$  is another such index, then  $f_i|_{U_i \cap U_j} \in f_j|_{U_i \cap U_j} \mathcal{O}_X(U_i \cap U_j)^*$  so  $f_i/f_j$  is invertible on  $\mathcal{O}_X(U_i \cap U_j)$  and we get  $\nu_Y(f_i/f_j) = 0$  so  $\nu_Y(f_i) = \nu_Y(f_j)$ . Therefore we obtain a well-defined Weil divisor  $D = \sum \nu_Y(f_i) Y$  on  $X$ .



Conversely, if  $D$  is a Weil divisor on  $X$ , let  $x \in X$  be any point. Then  $D$  induces a Weil divisor  $D_x$  on the local scheme  $\text{Spec}(\mathcal{O}_{X,x})$ . Since  $\mathcal{O}_{X,x}$  is a Unique Factorisation Domain, we have, by the Proposition 5.0.8,  $\text{Cl}(\text{Spec}(\mathcal{O}_{X,x})) = 0$  so  $D_x$  is a principal divisor. Let  $f_x \in K$  such that  $D_x = (f_x)$ . Then the principal divisor  $(f_x)$  on  $X$  has the same restriction to  $\text{Spec}(\mathcal{O}_{X,x})$  as  $D$ , hence they differ only at the prime divisor which do not pass through  $x$ . There are only finitely many of these prime divisors which have a non-zero coefficient in  $D$  or  $(f_x)$  so we can find an open neighborhood  $x \in U_x$  such that  $D|_{U_x} = (f_x)|_{U_x}$ . Hence we can find a system  $\{(U_x, f_x)_x\}$  where the  $U_x$  are open subsets of  $X$  a covering of  $X$ , and  $f_x \in K$  such that  $D|_{U_x} = (f_x)|_{U_x}$  and we have  $(f_x)|_{U_x \cap U_y} = D|_{U_x \cap U_y} = (f_y)|_{U_x \cap U_y}$  so  $f_x/f_y \in \mathcal{O}_X(U_x \cap U_y)^*$ . Thus we have a well-defined Cartier divisor. These two constructions are inverse to each other so the group of Weil divisor and the group of Cartier divisor are isomorphic. Furthermore, the principal divisors correspond to each other.  $\square$

**Theorem 5.2.4** (Auslander-Buschbaum). *A regular local ring is a Unique Factorization Domain.*

**Corollary 5.2.5.** *If  $X$  is a nonsingular variety, then we have  $\text{Div}(X) \simeq \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$ .*

**Proposition 5.2.6.** *Let  $\mathcal{L}$  and  $\mathcal{M}$  be invertible sheaves on a ringed topological space  $(X, \mathcal{O}_X)$ . Then  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$  is an invertible sheaf and there exists an invertible sheaf  $\mathcal{L}^{-1}$  on  $X$  such that  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \simeq \mathcal{O}_X$ .*

**Definition 5.2.7.** *For any ringed topological space  $(X, \mathcal{O}_X)$  we define the Picard group of  $X$ , denoted  $\text{Pic}(X)$ , to be the group of isomorphism classes of invertible sheaves on  $X$ , under the operation  $\otimes_{\mathcal{O}_X}$ .*

**Definition 5.2.8.** *Let  $D$  be a Cartier divisor on a scheme  $X$ , represented by  $\{(U_i, f_i)\}$  as above. We define a subsheaf  $\mathcal{L}(D)$  of the sheaf of total quotient rings  $\mathcal{K}$  by taking  $\mathcal{L}(D)$  to be the sub- $\mathcal{O}_X$ -module of  $\mathcal{K}$  generated by  $f_i^{-1}$  on  $U_i$ . Since  $f_i/f_j$  is invertible on  $\mathcal{O}_X(U_i \cap U_j)$ ,  $f_i^{-1}$  and  $f_j^{-1}$  generate the same  $\mathcal{O}_X$ -module so it is well-defined. We call  $\mathcal{L}(D)$  the sheaf associated to  $D$ .*

**Proposition 5.2.9.** *Let  $X$  be a scheme.*

- (a) *For any Cartier divisor  $D$ ,  $\mathcal{L}(D)$  is an invertible sheaf on  $X$ . The map  $D \mapsto \mathcal{L}(D)$  gives a one to one correspondance between Cartier divisors on  $X$  and invertible subsheaves of  $\mathcal{K}$ .*
- (b)  $\mathcal{L}(D - D') \simeq \mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{L}(D')^{-1}$ .
- (c)  $D \sim D'$  if and only if  $\mathcal{L}(D) \simeq \mathcal{L}(D')$  as abstract invertible sheaves.

*Proof.* (a) Let  $D$  be a Cartier divisor on  $X$  represented by  $\{(U_i, f_i)\}$ . For each  $f_i$ , the map  $\mathcal{O}_{U_i} \rightarrow \mathcal{L}(D)|_{U_i}$ , defined by  $1 \mapsto f_i^{-1}$  is an isomorphism. So for every point  $x \in X$ , we have  $x \in U_{i_0}$  for some  $i_0$  and  $\mathcal{L}|_{U_{i_0}} \simeq \mathcal{O}_X|_{U_{i_0}}$  as  $\mathcal{O}_{U_{i_0}}$ -modules. Then  $\mathcal{L}$  is an invertible sheaf on  $X$ . We can recover  $D$  from  $\mathcal{L}(D)$  together with its

embedding in  $\mathcal{K}$ , by taking  $f_i$  on  $\mathcal{O}_X(U_i)$  to be the inverse of a local generator of  $\mathcal{L}(D)$ . For any invertible subsheaf of  $\mathcal{K}$ , this construction gives a Cartier divisor, so we have a one to one correspondance.

- (b) If  $D$  is locally defined by  $f_i$  and  $D'$  is locally defined by  $g_i$ , then  $\mathcal{L}(D - D')$  is locally generated by  $f_i^{-1}g_i$  so  $\mathcal{L}(D - D') = \mathcal{L}(D) \cdot \mathcal{L}(D')^{-1}$  as subsheaves of  $\mathcal{K}$ , and this product is isomorphic to the abstract tensor product  $\mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{L}(D')^{-1}$ .
- (c) By the previous point, we have to show that  $D_1 = D - D'$  is principal if and only if  $\mathcal{L}(D_1) = \mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{L}(D')^{-1} \simeq \mathcal{O}_X$ . If  $D_1$  is principal, then there exists  $f \in \Gamma(X, \mathcal{K}^*)$  such that the image of  $f$  by the natural map  $\Gamma(X, \mathcal{K}^*) \rightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$  is  $D_1$ . So  $\mathcal{L}(D_1)$  is globally generated by  $f^{-1}$ , so sending  $1 \mapsto f^{-1}$  gives an isomorphism  $\mathcal{O}_X \simeq \mathcal{L}(D_1)$ . Conversely, given such an isomorphism, the image of  $1$  by this isomorphism gives us an element of  $\Gamma(X, \mathcal{K}^*)$  whose inverse will define  $D$  as a principal divisor.

□

**Definition 5.2.10.** Let  $D$  be an effective Cartier divisor on  $X$ , represented by  $\{(U_i, f_i)\}$  with  $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$ . We define the associated subscheme of codimension 1  $Y$  to be the closed subscheme defined by the sheaf of ideals  $\mathcal{I}$  which is locally generated by  $f_i$ .

**Proposition 5.2.11.** Let  $D$  be an effective Cartier divisor on a scheme  $X$ , and let  $Y$  be the associated locally principal closed subscheme. Then  $\mathcal{I}_Y \simeq \mathcal{L}(-D)$ .

*Proof.*  $\mathcal{L}(-D)$  is the subsheaf of  $\mathcal{K}$  generated locally by  $f_i$ . Since  $D$  is effective, this is a subsheaf of  $\mathcal{O}_X$ , which is the ideal sheaf  $\mathcal{I}_Y$  of  $Y$ . □

**Definition 5.2.12.** Let  $X$  be a nonsingular projective variety over an algebraically closed field  $k$ , and let  $\mathcal{L}$  be an invertible sheaf on  $X$ . Let  $s \in \Gamma(X, \mathcal{L})$  be a nonzero section of  $\mathcal{L}$ . We define an effective divisor  $D = (s)_0$ , the divisor of zeros of  $s$ , as follows. Over any open set  $I \subseteq X$  where  $\mathcal{L}$  is trivial, let  $\phi: \mathcal{L}|_I \rightarrow \mathcal{O}_I$  be an isomorphism. Then  $\phi(s) \in \Gamma(I, \mathcal{O}_I)$ . As  $I$  ranges over a covering of  $X$ , the collection  $\{(I, \phi(s))\}$  determines, up to multiplication by an element of  $\Gamma(I, \mathcal{O}_I^*)$ , an effective Cartier divisor  $D$  on  $X$ .

**Proposition 5.2.13.** Let  $X$  be a non singular projective variety over the algebraically closed field  $k$ . Let  $D_0$  be a divisor on  $X$  and let  $\mathcal{L} \simeq \mathcal{L}(D_0)$  be the corresponding invertible sheaf.

- (a) For each nonzero  $s \in \Gamma(X, \mathcal{L})$ , the divisor of zeros  $(s)_0$  is an effective divisor linearly equivalent to  $D_0$ .
- (b) Every effective divisor linearly equivalent to  $D_0$  is  $(s)_0$  for some  $s \in \Gamma(X, \mathcal{L})$ .
- (c) Two sections  $s$  and  $s'$  of  $\Gamma(X, \mathcal{L})$  have the same divisor of zeros if and only if there is a  $\lambda \in k^*$  such that  $s' = \lambda s$ .

**Definition 5.2.14.** A complete linear system on a nonsingular projective variety is defined as the set of all effective divisors linearly equivalent to some given divisor  $D_0$ . It is denoted by  $|D_0|$ .

**Remark:** The previous Proposition tells us that we have a one to one correspondance between the set  $|D_0|$  and the set  $(\Gamma(X, \mathcal{L}) \setminus \{0\})/k^*$ . This gives  $|D_0|$  a structure of the set of closed points of a projective space over  $k$ .

**Theorem 5.2.15.** Let  $k$  be a field, let  $A$  be a finitely generated  $k$ -algebra, let  $X$  be a projective scheme over  $A$ , and let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Then  $\Gamma(X, \mathcal{F})$  is a finitely generated  $A$ -module. In particular, if  $A = k$ ,  $\Gamma(X, \mathcal{F})$  is a finite-dimensional  $k$ -vector space.

**Definition 5.2.16.** A linear system  $\mathfrak{d}$  on  $X$  is a subset of a complete linear system  $|D_0|$  which is a linear subspace for the projective space structure of  $|D_0|$ . Thus  $\mathfrak{d}$  corresponds to a sub-vector space  $V \subseteq \Gamma(X, \mathcal{L})$ , where  $V = \{s \in \Gamma(X, \mathcal{L}), (s)_0 \in \mathfrak{d}\} \cup \{0\}$ . The dimension of the linear system  $\mathfrak{d}$  is its dimension as a linear projective variety. Hence  $\dim(\mathfrak{d}) = \dim(V) - 1$ .

**Remark:** Since  $\Gamma(X, \mathcal{L})$  is a finite-dimensional vector space by the previous Theorem, the dimension of  $V$  is also finite.

**Definition 5.2.17.** If  $|D|$  is a complete linear system, we denote by  $l(D)$  the dimension of the  $k$ -vector space  $\Gamma(X, \mathcal{L}(D))$ .

**Lemma 5.2.18.** Let  $D$  be a divisor on a curve  $X$ . Then if  $l(D) \neq 0$ , we must have  $\deg(D) \geq 0$ . Furthermore, if  $l(D) \neq 0$  and  $\deg(D) = 0$  then we have  $D \sim 0$  hence  $\mathcal{L}(D) \simeq \mathcal{O}_X$ .

*Proof.* If  $l(D) \neq 0$  then  $|D|$  is nonempty. Let  $D'$  be an effective divisor linearly equivalent to  $D$ . Then we have  $\deg(D) = \deg(D') \geq 0$  since  $D'$  is effective. Now assume that  $\deg(D) = 0$ . Then  $D \sim D'$  with  $\deg(D') = 0$  so  $D' = 0$ .

□

## 6 The Luroth's theorem

The Luroth's problem in dimension one is equivalent to the Luroth's theorem, that we will prove in this section. Recall that if  $X$  is a curve, then  $p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$ . In this section, we will assume that  $X$  is a curve, so we will denote  $g = p_g(X) = p_a(X)$ . Moreover, every field  $k$  in this section is algebraically closed.

### 6.1 The Riemann-Roch's theorem

**Definition 6.1.1.** Let  $X$  be a curve over a field  $k$ . Any Cartier divisor  $K$  on  $X$  such that  $\mathcal{L}(K) \simeq \omega_X$  is called a canonical divisor on  $X$ . It is well-defined up to a linear equivalence.

**Theorem 6.1.2** (Riemann-Roch). *Let  $D$  be a divisor on a curve of genus  $g$ . Then we have*

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

**Corollary 6.1.3.** *If  $X$  is a curve of genus  $g$ , then the canonical divisor  $K$  has degree  $2g - 2$ .*

*Proof.* Applying the Riemann-Roch theorem with  $D = K$ , we obtain

$$l(K) - l(0) = \deg(K) + 1 - g.$$

Since  $l(K) = g$ , we obtain the equality

$$\deg(K) = 2g - 2.$$

□

**Proposition 6.1.4.** *A curve  $X$  is rational if and only if  $g(X) = 0$ .*

*Proof.* Since  $g(\mathbb{P}_k^1) = 0$  and the genus is a birational invariant, we have that if  $g(X) > 0$  then  $X$  is not rational. By contraposition,  $X$  is rational implies  $g(X) = 0$ .

Now assume that  $g(X) = 0$  and let us show that  $X$  is rational. Let  $P$  and  $Q$  be two distinct rational points of  $X$ . Let us denote  $D = P - Q \in \text{Div}(X)$ . Then we have  $l(K - D) = 0$ . Indeed, if  $l(K - D) > 0$  then we would have an effective divisor  $D'$  such that  $K - D \sim D'$  but  $\deg(K - D) = \deg(K) - \deg(D) = -2$  and  $\deg(D') \geq 0$  which is a contradiction. Hence  $l(K - D) = 0$ . Then, applying the Riemann-Roch Theorem to the divisor  $D$ , we obtain that  $l(D) = 1$ . Since  $\deg(D) = 0$ , by Lemma 5.2.18, we have  $D \sim 0$ . Then there exists a rational function  $f \in K(X)$  such that  $(f) = P - Q$ . The inclusion morphism  $k(f) \rightarrow k(X)$  induces a finite morphism  $\phi : X \rightarrow \mathbb{P}_k^1$  and  $(f) = \phi^*(\{0\} - \{\infty\})$  hence we have  $\phi^*(\{0\}) = P$ . Consequently,  $\phi$  must be a morphism of degree 1 and so  $K(X) = K(\mathbb{P}^1)$ . Hence  $\phi$  is birational and  $X$  is a rational curve. □

## 6.2 The Hurwitz formula and Luroth's theorem

**Definition 6.2.1.** *Let  $f : X \rightarrow Y$  be a finite morphism of curves. Let  $P \in X$  and  $Q = f(P)$ .*

*Let  $t \in \mathcal{O}_{Q,Y}$  be a local parameter at  $Q$ , that is  $t$  is an element of  $K(Y)$  with  $v_Q(t) = 1$  and  $v_Q$  is the valuation associated to the valuation ring  $\mathcal{O}_{Q,Y}$ . We define the ramification index  $e_P$  at  $P$  with*

$$e_P = v_P(f^\#(t)).$$

**Definition 6.2.2.** *With the notation of the previous definition, we say that:*

- $f$  is ramified at  $P$ , and that  $Q$  is a branch point of  $f$  if  $e_P > 1$ ,
- $f$  is unramified at  $P$  if  $e_P = 1$ .

**Definition 6.2.3.** We say that the morphism  $f : X \rightarrow Y$  is separable if  $K(X)$  is a separable field extension of  $K(Y)$ .

**Lemma 6.2.4.** If  $K/k$  is a finite algebraic extension, then  $\Omega_{K/k} = 0$  if and only if  $K/k$  is separable.

**Proposition 6.2.5.** Let  $f : X \rightarrow Y$  be a finite separable morphism of curves. There is an exact sequence of sheaves on  $X$

$$0 \longrightarrow f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

*Proof.* By the Proposition 3.1.22, we just have to prove that  $f^*\Omega_Y \rightarrow \Omega_X$  is injective. Since both of these sheaves are invertible sheaves on  $X$ , it is enough to show that the map is nonzero at the generic point  $\eta$  of  $X$ . Since  $K(X)$  is separable over  $K(Y)$  and is a finite algebraic extension of  $K(Y)$ , we have  $\Omega_{X/Y} = 0$ . Since at  $\eta$  we have  $\text{Im}(f^*\Omega_Y \rightarrow \Omega_X) = \text{Ker}(\Omega_X \rightarrow \Omega_{X/Y}) = \Omega_X$ , the map  $f^*\Omega_Y \rightarrow \Omega_X$  is surjective at the generic point of  $X$ , hence is nonzero.  $\square$

**Definition 6.2.6.** Let  $f : X \rightarrow Y$  be a finite separable morphism of curves. Let  $P$  be a point of  $X$  and let  $Q = f(P)$ . Let  $t$  be a local parameter at  $Q$  and let  $u$  be a local parameter at  $P$ . Then  $dt$  is a generator of the free  $\mathcal{O}_Q$ -module  $\Omega_{Y,Q}$ , and  $du$  is a generator of the free  $\mathcal{O}_P$ -module  $\Omega_{X,P}$ . In particular, there is a unique element  $g \in \mathcal{O}_P$  such that  $f^*dt = g \cdot du$ . We denote this element  $g$  by  $dt/du$ .

**Proposition 6.2.7.** Let  $f : X \rightarrow Y$  be a finite, separable morphism of curves.

- (a)  $\Omega_{X/Y}$  is a torsion sheaf on  $X$ , with support equal to the set of ramification points of  $f$ . In particular,  $f$  is ramified at only finitely many points.
- (b) For each  $P \in X$ , the stalk  $(\Omega_{X/Y})_P$  is a principal  $\mathcal{O}_P$ -module of finite length equal to  $\nu_P(dt/du)$ .
- (c) If  $f$  is tamely ramified at  $P \in X$  then  $\text{length}(\Omega_{X/Y})_P = e_P - 1$ .

*Proof.* (a) Since  $f^*\Omega_Y$  and  $\Omega_X$  are both invertible sheaves on  $X$ , the previous proposition tells us that  $\Omega_{X/Y}$  is a torsion sheaf on  $X$ .

- (b) Let us consider the exact sequence from the previous Proposition. Taking the stalks at  $P$ , where  $f(P) = Q$ , we obtain that  $(\Omega_{X/Y})_P \simeq \Omega_{X,P}/f^*\Omega_{Y,Q}$ , which is isomorphic as an  $\mathcal{O}_P$ -module to  $\mathcal{O}_P(dt/du)$ .
- (c) If  $f$  has ramification index  $e = e_P$ , then there exists a unit  $a \in \mathcal{O}_P$  such that  $t = au^e$  and so we have  $dt = aeu^{e-1}(du) + u^e(da)$ . If the ramification is tame, then  $e$  is a nonzero element of  $k$  and we have  $\nu_P(de/du) = e - 1$ .

$\square$

**Definition 6.2.8.** Let  $f : X \rightarrow Y$  be a finite, separable morphism of curves. We define the ramification divisor of  $f$  to be

$$R = \sum_{P \in X} \text{length}(\Omega_{X/Y})_P \cdot P.$$

**Proposition 6.2.9.** Let  $f : X \rightarrow Y$  be a finite separable morphism of curves. Let  $K_X$  and  $K_Y$  be the canonical divisors of  $X$  and  $Y$ , respectively. Then  $K_X \sim f^*K_Y + R$ .

**Theorem 6.2.10** (Hurwitz formula). Let  $f : X \rightarrow Y$  be a finite separable morphism of curves. Let  $n = \deg f$ . Then

$$2g(X) - 2 = n(2g(Y) - 2) + \deg R$$

Moreover if  $f$  has only tame ramification, then

$$\deg R = \sum_{P \in X} (e_P - 1).$$

*Proof.* Taking the degrees in the previous proposition, we obtain that  $\deg(K_X) = \deg(f) \deg(K_Y) + \deg(R)$ . By the Corollary 6.1.3, we have that  $\deg(K_X) = 2g(X) - 2$  and  $\deg(K_Y) = 2g(Y) - 2$ . So we have

$$2g(X) - 2 = n \cdot (g(Y) - 2) + \deg(R).$$

Furthermore, if  $f$  has only tame ramification then  $\deg(\Omega_{X/Y}) = e_P - 1$  for any  $P \in X$  by the Proposition 6.2.7. Hence we obtain that  $\deg(R) = \sum_{P \in X} (e_P - 1)$ .  $\square$

**Definition 6.2.11.** Let  $X$  be a scheme such that all of whose local rings are of characteristic  $p$ . We define the Frobenius morphism  $F : X \rightarrow X$  as follows. As a map of topological spaces,  $F = \text{id}_X$  and  $F^\sharp : \mathcal{O}_X \rightarrow \mathcal{O}_X$  is the  $p$ -th power map. This defines a morphism of schemes since  $X$  has all its local rings of characteristic  $p$ .

**Proposition 6.2.12.** Let  $f : X \rightarrow Y$  be a finite morphism of curves, and suppose that  $K(X)$  is a purely inseparable field extension of  $K(Y)$ . Then  $X$  and  $Y$  are isomorphic as abstract schemes, and  $f$  is a composition of  $k$ -linear Frobenius morphisms. In particular,  $g(X) = g(Y)$ .

*Proof.* Let  $p^r$  be the degree of  $f$ . Then we have  $K(X)^{p^r} \subseteq K(Y)$ . We consider the following sequence of schemes morphisms

$$Y_{p^r} \longrightarrow Y_{p^{r-1}} \longrightarrow \dots \longrightarrow Y_p \longrightarrow Y$$

where  $Y_p$  is defined to be the same  $k$ -scheme than  $Y$  but with the structural morphism  $F \circ \pi$ , with  $\pi : Y \rightarrow k$ , and  $Y_{p^i} = (Y_{p^{i-1}})_p$  for  $1 \leq i \leq r$ . The composition of these morphisms is a morphism  $f' : X \rightarrow Y$  which has also degree  $p^r$ . Then we have  $K(X) \subseteq K(Y)^{1/p^r}$  and  $K(Y_{p^r}) = K(Y)^{1/p^r}$  and both  $K(X)$  and  $K(Y_{p^r})$  have the same degree so  $k(X) = K(Y_{p^r})$ . Since the function field of a curve determines the curve, we have an isomorphism  $X \simeq Y_{p^r}$  as schemes and  $f = f'$ . So  $X \simeq Y$  as schemes and so  $g(X) = g(Y)$ .  $\square$

**Corollary 6.2.13.** *Let  $f : X \rightarrow Y$  be a finite separable morphism of nonsingular projective curves over  $k$ . Then  $g(X) \geq g(Y)$ .*

*Proof.* We have  $K(X) = k(T, s)$  with  $T$  transcendent over  $k$  and  $s$  separable algebraic over  $k(T)$  so we can factor the field extension  $K(Y) \subseteq K(X)$  into a separable extension followed by a purely inseparable extension. By the previous Proposition, the genus doesn't change for a purely inseparable extension so we can reduce to the case  $f$  is separable. If  $g(Y) = 0$  then there is nothing to prove since the geometric genus is a non-negative integer. Assume that  $g(Y) \geq 1$ . Applying the Hurwitz formula we obtain

$$g(X) = g(Y) + (n - 1) \cdot (g(Y) - 1) + \deg(R) / 2.$$

Since  $g(Y) \geq 1$  and  $\deg(R) \geq 0$ , we obtain the inequality claimed.  $\square$

**Theorem 6.2.14** (Luroth's theorem). *Let  $k$  be a field, and let  $L$  be a subfield of a purely transcendental extension  $k(t)$ . Then either  $L = k$ , or  $L$  is purely transcendental over  $k$ .*

*Proof.* Let us suppose  $L \neq k$ . Then  $L$  is of transcendence degree 1 over  $k$ . Then  $L$  is a function field of a curve  $Y$ . Since  $L \subseteq k(t)$ , we obtain a finite morphism  $f : \mathbb{P}_k^1 \rightarrow Y$ . By the previous Corollary, we obtain that  $0 = g(\mathbb{P}_k^1) \geq g(Y) \geq 0$  so  $g(Y) = 0$ . Hence by the Proposition 6.1.4.,  $Y$  is birational to  $\mathbb{P}_k^1$  so  $L \simeq k(u)$  for some  $u$ .  $\square$

Consequently, if  $X$  is a curve over an algebraically closed field  $k$ , then we have that  $X$  is unirational if and only if  $X$  is rational. If  $k$  is not algebraically closed, then the result still holds.

## 7 The Luroth's problem in dimension 2

We will say that  $X$  is a surface if  $X$  is a nonsingular projective variety of dimension 2 over an algebraically closed field  $k$ .

### 7.1 Divisors on a surface

**Definition 7.1.1.** *Let  $(X, \mathcal{O}_X)$  be a ringed topological space. We define the Picard group of  $X$ , denoted by  $\text{Pic}(X)$ , to be the group of isomorphism classes of invertible sheaves on  $X$ , under the operation  $\otimes_{\mathcal{O}_X}$ .*

**Proposition 7.1.2.** *If  $X$  is an integral scheme, we have an isomorphism  $\text{CaCl}(X) \rightarrow \text{Pic}(X)$ , where  $\text{CaCl}(X)$  is the group of Cartiers divisors on  $X$  modulo linear equivalence.*

*Proof.* The map  $D \rightarrow \mathcal{L}(D)$  gives us an injective homomorphism from  $\text{CaCl}(X)$  to  $\text{Pic}(X)$  by the Proposition 5.2.9. It remains to show that every invertible sheaf is isomorphic to a subsheaf of  $\mathcal{K}$ . First note that in this case,  $\mathcal{K}$  is just the constant sheaf  $K = K(X)$ , the function field of  $X$ . Let  $\mathcal{L}$  be any invertible sheaf, and consider the sheaf  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$ . On any open set  $U$  where  $\mathcal{L} \simeq \mathcal{O}_X$ , we have  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K} \simeq \mathcal{K}$ , so it is

a constant sheaf on any such  $U$ . Since  $\mathcal{L}$  is an invertible sheaf, we have an open covering of  $X$  such that the restriction to each of these open subset of  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$  is isomorphic to the constant sheaf  $\mathcal{K}$ . Since  $X$  is irreducible, that implies that  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$  is isomorphic to the constant sheaf  $\mathcal{K}$ , and the natural map  $\mathcal{L} \rightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$  express  $\mathcal{L}$  as a subsheaf of  $\mathcal{K}$ .  $\square$

**Corollary 7.1.3.** *If  $X$  is a noetherian, integral, separated locally factorial scheme, then there is a natural isomorphism  $Cl(X) \simeq Pic(X)$ .*

**Definition 7.1.4.** *An invertible sheaf  $\mathcal{L}$  is said to be ample if for every coherent sheaf  $\mathcal{F}$  on  $X$ , there is an integer  $n_0(\mathcal{F}) > 0$  such that for every  $n \geq n_0(\mathcal{F})$ , the sheaf  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$  is generated by its global sections, where  $\mathcal{L}^n = \mathcal{L}^{\otimes n}$  denotes the  $n$ fold tensor power of  $\mathcal{L}$  with itself.*

*If  $X$  is any scheme over  $Y$ , an invertible sheaf  $\mathcal{L}$  on  $X$  is very ample relative to  $Y$  if there is an immersion  $i : X \rightarrow \mathbb{P}_Y^r$  for some  $r$ , such that  $i^*(\mathcal{O}(1)) \simeq \mathcal{L}$ .*

*A Cartier divisor on  $X$  is called ample, resp very ample, if  $\mathcal{L}(D)$  is an ample sheaf, resp very ample sheaf, on  $X$ .*

We can't define the notion of degree for a divisor on a surface, unlike the degree of a divisor on a curve. However, we can define the intersection of two divisors on a surface. Let  $X$  be a surface over  $k$ . A curve on a surface will mean any effective divisor on the surface. A point will mean a closed point, unless otherwise specified.

**Definition 7.1.5.** *Let  $C, D$  be curves on  $X$ . Let  $P \in C \cap D$  be a point of intersection of  $C$  and  $D$ . We say that  $C$  and  $D$  meet transversally at  $P$  if the local equations  $f, g$  of  $C, D$  at  $P$  generate the maximal ideal  $\mathfrak{m}_P$  of  $\mathcal{O}_{X,P}$ .*

Our goal is to define the "number of intersection" of  $C$  and  $D$ .

**Lemma 7.1.6.** *Let  $C_1, \dots, C_r$  be irreducible curves on the surface  $X$  and let  $D$  be a very ample divisor. Then almost all curves  $D'$  in the complete linear system  $|D|$  are irreducible, nonsingular, and meet each of the  $C_i$  transversally.*

**Lemma 7.1.7.** *Let  $C$  be an irreducible nonsingular curve on  $X$ , and let  $D$  be any curve meeting  $C$  transversally. Then*

$$\#(C \cap D) = \deg_C(\mathcal{L}(D) \otimes_{\mathcal{O}_X} \mathcal{O}_C).$$

*Proof.* Since  $\mathcal{L}(-D)$  is the ideal sheaf of  $D$  on  $X$ , we obtain, tensoring with  $\mathcal{O}_C$ , the following exact sequence

$$0 \longrightarrow \mathcal{L}(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0$$

where  $C \cap D$  denotes the scheme intersection. We have that  $\mathcal{L}(-D)$  is the invertible sheaf on  $C$  corresponding to the divisor  $C \cap D$ , and since the intersection is transversal, the degree of this divisor is the number of points  $\#(C \cap D)$ .  $\square$



**Theorem 7.1.8.** *There is a unique pairing  $\text{Div}(X) \times \text{Div}(X) \rightarrow \mathbb{Z}$ , denoted by  $(C \cdot D)$  for any two divisors  $C, D$  on  $X$ , such that*

- (i) *If  $C$  and  $D$  are nonsingular curves meeting transversally, then  $(C \cdot D) = \sharp(C \cap D)$ , the number of points of  $C \cap D$ ,*
- (ii) *for any  $C$  and  $D$  divisors on  $X$  we have  $(C \cdot D) = (D \cdot C)$ ,*
- (iii) *for any  $C, C'$  and  $D$  divisors on  $X$ ,  $((C + C') \cdot D) = (C \cdot D) + (C' \cdot D)$ ,*
- (iv) *if  $C \sim C'$  then  $(C \cdot D) = (C' \cdot D)$ .*

**Definition 7.1.9.** *Let  $C$  and  $D$  be curves on  $X$ . If  $C$  and  $D$  have no common irreducible components, and if  $P \in C \cap D$ , then we define the intersection multiplicity  $(C \cdot D)_P$  of  $C$  and  $D$  at  $P$  to be the dimension of the  $k$ -vector space  $\mathcal{O}_{X,P}/(f, g)$ , where  $f, g$  are local equations of  $C, D$  at  $P$ .*

**Proposition 7.1.10.** *If  $C$  and  $D$  are curves on  $X$  having no common irreducible component, then*

$$(C \cdot D) = \sum_{P \in C \cap D} (C \cdot D)_P.$$

*Proof.* Let  $\mathcal{L}$  be the invertible sheaf on  $X$  corresponding to the divisor  $D$  on  $X$ . We have the following exact sequence

$$0 \longrightarrow \mathcal{L}(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0$$

where we consider  $C \cap D$  as a scheme. The scheme has support at the points of  $C \cap D$  and for any  $P \in C \cap D$  its structure sheaf is the  $k$ -algebra  $\mathcal{O}_{X,P}/(f, g)$ . Hence we have

$$\dim_k(\Gamma(X, \mathcal{O}_{C \cap D})) = \sum_{P \in C \cap D} (C \cdot D)_P.$$

Moreover, the previous exact sequence tells us that  $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_{C \cap D}) + \chi(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_C)$ , where  $\chi$  is the Euler characteristic. Hence the expression  $\sum_{P \in C \cap D} (C \cdot D)_P$  only depends of the linear equivalence class of  $D$  and, by symmetry, of the linear equivalence class of  $C$ . Consequently, replacing  $C$  and  $D$  by differences of nonsingular curves, all transversal to each other, we have that this sum is equal to the intersection number  $(C \cdot D)$ .  $\square$

**Definition 7.1.11.** *If  $D$  is any divisor on the surface  $X$ , we can define the self-intersection number  $(D \cdot D)$ , denoted by  $(D^2)$ .*

**Remark:** We cannot use the previous Proposition in order to calculate the self-intersection number of any divisor  $D$  on the surface  $X$ . To determine it, we have to use the fact this number only depends of the linear equivalence class of  $D$ .

Recall that any divisor  $K$  in the linear equivalence class corresponding to  $\omega_X = \wedge^2 \Omega_{X/k}$  is called a canonical divisor on  $X$ . Then  $(K^2)$ , the self-intersection number of the canonical divisor, is a number depending only on  $X$ , since the self-intersection number of a divisor doesn't depend on the choice of a representative  $K$ .

**Proposition 7.1.12.** *If  $C$  is a nonsingular curve of genus  $g$  on the surface  $X$ , and if  $K$  is the canonical divisor on  $X$ , then  $2g - 2 = (C \cdot (C + K))$ .*

*Proof.* We have  $\omega_C \simeq \omega_X \otimes \mathcal{L}(C) \otimes \mathcal{O}_C$ . We see that the canonical divisor on a curve of genus  $g$  has degree  $2g - 2$  so  $\deg(\omega_C) = 2g - 2$ . Furthermore, by the Lemma 7.1.7, we have that  $\deg(\omega_X \otimes \mathcal{L}(C) \otimes \mathcal{O}_C) = (C \cdot (C + K))$ .  $\square$

**Theorem 7.1.13** (Riemann-Roch). *If  $D$  is any divisor on the surface  $X$ , then*

$$l(D) - s(D) + l(K - D) = \frac{1}{2}(D \cdot (D + K)) + 1 - p_a,$$

where  $p_a = p_a(X) = \chi(X) - 1$  is the arithmetic genus of  $X$ ,  $l(D) = \dim_k(H^0(X, \mathcal{L}(D))) = \dim |D| + 1$  and  $s(D) = \dim(H^1(X, \mathcal{L}(D)))$  is the superabundance of  $D$ .

*Proof.* By the Serre Duality Theorem, we have that  $l(K - D) = \dim(H^0(X, \mathcal{L}(D)^\vee \otimes \omega_X)) = \dim(H^2(X, \mathcal{L}(D)))$ . So we can rewrite the left-hand side in the equality of the theorem as

$$l(D) - s(D) + l(K - D) = \dim(H^0(X, \mathcal{L}(D))) - \dim(H^1(X, \mathcal{L}(D))) + \dim(H^2(X, \mathcal{L}(D)))$$

Hence we have to show that

$$\chi(\mathcal{L}(D)) = \frac{1}{2}(D \cdot (D - K)) + 1 + p_a.$$

First notice that both sides only depend of the linear equivalence class of  $D$ . Hence we can write  $D \sim C - E$  where  $C$  and  $E$  are two nonsingular curves. So we have the following exact sequences

$$0 \longrightarrow \mathcal{L}(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{L}(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Tensoring these exact sequences by  $\mathcal{L}(C)$ , we obtain the exact sequences

$$0 \longrightarrow \mathcal{L}(C - C) \simeq \mathcal{O}_X \longrightarrow \mathcal{L}(C) \longrightarrow \mathcal{L}(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{L}(C - E) \longrightarrow \mathcal{L}(C) \longrightarrow \mathcal{L}(C) \otimes_{\mathcal{O}_X} \mathcal{O}_E \longrightarrow 0.$$

Consequently, taking the Euler characteristic in these exact sequences, we obtain that

$$\chi(\mathcal{L}(C - E)) = \chi(\mathcal{O}_X) + \chi(\mathcal{L}(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C) - \chi(\mathcal{L}(C) \otimes_{\mathcal{O}_X} \mathcal{O}_E).$$

Let us explain the terms of the right-hand side. By definition of the arithmetic genus of a variety, we have  $\chi(\mathcal{O}_X) = p_a - 1$ . Since  $C$  is a nonsingular curve we have  $\chi(\mathcal{L}(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C) = \deg_C(\mathcal{L}(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C) + 1 - g_C$  by the Riemann-Roch theorem for the curves and so, applying the

Lemma 7.1.7 we obtain  $\chi(\mathcal{L}(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C) = C^2 + 1 - g_C$ . By a similar reasonment, we have  $\chi(\mathcal{L}(C) \otimes_{\mathcal{O}_X} \mathcal{O}_E) = C \cdot E + 1 - p_E$ . Then the previous Proposition gives us

$$g_C = \frac{1}{2}(C \cdot (C + K)) + 1, \quad g_E = \frac{1}{2}(E \cdot (E + K)) + 1.$$

Finally, we obtain

$$\chi(D) = \chi(\mathcal{L}(C - E)) = \frac{1}{2}((C - E) \cdot (C - E - K)) + 1 + p_a = \frac{1}{2}(D \cdot (D - K)) + 1 + p_a.$$

□

**Theorem 7.1.14** ( The Nakai-Moishezon Criterion). *A divisor  $D$  on the surface  $X$  is ample if and only if  $D^2 > 0$  and  $(D \cdot C) > 0$  for all irreducible curves  $C$  in  $X$ .*

## 7.2 Albanese variety

**Definition 7.2.1.** *Let  $S$  be a scheme. A group scheme over  $S$  is an  $S$ -scheme  $G$  endowed with the following morphisms of  $S$ -schemes*

- (multiplication)  $m : G \times_S G \longrightarrow G$
- (unit section)  $\epsilon : S \longrightarrow G$
- (inverse)  $inv : G \longrightarrow G$

such that we have the following commutative diagrams

$$\begin{array}{ccc}
 G \times_S G \times_S G & \xrightarrow{m \times Id_G} & G \times_S G \\
 \text{(associativity)} \quad \downarrow Id_G \times m & & \downarrow m \\
 G \times_S G & \xrightarrow{m} & G
 \end{array} ,$$

$$\begin{array}{ccc}
 G = G \times_S S & \xrightarrow{Id_G \times \epsilon} & G \times_S G \\
 \text{(right-identity)} \quad \searrow Id_G & & \downarrow m \\
 & & G
 \end{array} ,$$

$$\begin{array}{ccc}
 G & \xrightarrow{\Delta_{G/S}} & G \times_S G & \xrightarrow{Id_G \times inv} & G \times_S G \\
 \text{(right-inverse)} \quad \downarrow & & & & \downarrow m \\
 S & \xrightarrow{\epsilon} & & & G
 \end{array} .$$

**Definition 7.2.2.** *A group scheme  $G$  over a field  $k$  is called an algebraic group over  $k$  if  $G$  is moreover of finite type over  $k$ .*

**Definition 7.2.3.** *Let  $k$  be a field. An Abelian variety over  $k$  is defined to be an algebraic group that is geometrically integral and proper over  $k$ .*

**Definition 7.2.4.** Let  $X$  be a nonsingular projective variety and let  $x_0 \in X$  be a fixed closed point. A pair  $(A, \alpha)$  consisting of an Abelian variety  $A$  and a morphism  $\alpha : X \rightarrow A$  such that  $\alpha(x_0) = 0_A$  (the zero element of  $A$ ), is called the Albanese variety of the variety  $X$  if it verifies the following universal property:

for every morphism  $f : X \rightarrow B$  such that  $B$  is an Abelian variety and  $f(x_0) = 0$ , there exists a unique homomorphism  $g : A \rightarrow B$  of Abelian varieties such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ & \searrow \alpha & \nearrow g \\ & & A \end{array}$$

commutes.

**Theorem 7.2.5.** If  $X$  is a nonsingular projective variety, then the Albanese variety  $(\text{Alb}(X), \alpha)$  exists, is unique up to a unique isomorphism, and  $\text{Alb}(X)$  coincides with the dual of the Picard variety  $P(X)$ . In particular, if  $X$  is a surface, then the dimension of the variety  $\text{Alb}(X)$  is equal to  $\dim(P(X))$ , and therefore  $\dim(\text{Alb}(X)) \leq q$ , where  $q = \dim(H^1(X, \mathcal{O}_X))$ , and the equality holds if  $p_g = 0$ .

### 7.3 Blowing-up and monoidal transformations

**Definition 7.3.1.** Let  $X = \text{Spec}(A)$  be an affine noetherian scheme. Let  $I$  be an ideal of  $A$ . We consider the graded  $A$ -algebra  $\tilde{A} = \bigoplus_{n \geq 0} I^n$  where  $I^0 := A$ . Let  $f_1, \dots, f_n$  be a system of generators of  $I$  and let  $t_i \in I = \tilde{A}_1$  denote the element  $f_i$  considered as a homogeneous element of degree 1. We have a surjective homomorphism of graded  $A$ -algebras  $\phi : A[T_1, \dots, T_n] \rightarrow \tilde{A}$  defined by  $\phi(T_i) = t_i$ . Hence  $\tilde{A}$  is a homogeneous  $A$ -algebra. We let  $\tilde{X} = \text{Proj}(\tilde{A})$ , and the canonical morphism  $\tilde{X} \rightarrow X$  is called the blowing-up of  $X$  with center, or along,  $V(I)$ .

**Proposition 7.3.2.** Let  $A$  be a noetherian ring and let  $I$  be an ideal of  $A$ .

- a) If  $I$  is generated by a regular element, then  $\tilde{A} \simeq A[T]$ , that is  $\text{Proj}(\tilde{A}) \rightarrow \text{Spec}(A)$  is an isomorphism.
- b)  $\text{Proj}(\tilde{A}) = \emptyset$  if and only if  $I$  is nilpotent.
- c) The ring  $\tilde{A}$  is integral, resp reduced, if and only if  $A$  is integral, resp reduced.
- d) Let  $B$  be a flat  $A$ -algebra, and let  $\tilde{B}$  be the graded  $B$ -algebra associated to the ideal  $IB$ . Then we have a canonical isomorphism  $\tilde{B} \simeq B \otimes_A \tilde{A}$ .
- e) Let  $S_i = T_i/T_1 \in \mathcal{O}(D_+(T_1))$ . Then  $(\text{Ker}(\phi))_{(T_1)}$  is equal to

$$J_1 = \left\{ P(S) \in A[S_2, \dots, S_n], \exists d \geq 0 f_1^d P \in (f_1 S_2 - f_2, \dots, f_1 S_n - f_n) \right\},$$

and we can identify  $\tilde{A}_{(t_1)}$  with the sub- $A$ -algebra of  $A_{f_1}$  generated by the elements  $f_i/f_1$ ,  $2 \leq i \leq n$ .

*Proof.* a) Let  $f$  be a generator of  $I$ , and let  $\phi : A[T] \rightarrow \tilde{A}$  defined by  $\phi(T) = t$  where  $t$  denote the element  $T$  considered as an homogeneous element. Then  $\phi$  is well-defined and is an isomorphism.

b) With the notations of the previous Definition, we have  $\text{Proj}(\tilde{A}) = \emptyset$  if and only if the  $t_i$  are nilpotents. Since  $t_i$  denote the element  $f_i$  considered as an homogeneous element,  $\text{Proj}(\tilde{A}) = \emptyset$  if and only if the  $T_i$  are nilpotents, and so if and only if  $I$  is nilpotent.

c) Assume that  $A$  is integral. Let  $a, b \in \tilde{A}$  be such that  $ab = 0$  and that  $a \neq 0$ . Let  $a_n, b_m \in A$  be their respective homogeneous components of highest degree. Then  $a_n b_m = 0$  and so  $b_m = 0$ . Consequently,  $b = 0$  and  $\tilde{A}$  is integral. Conversely, if  $\tilde{A}$  is integral, then  $A$  is also integral since it is a subring of  $\tilde{A}$ . We show the assertion for reduced rings with the same way.

d) We have a canonical isomorphism  $B \otimes_A I \simeq IB$ . Moreover we have  $B \otimes_A I^d \simeq (IB)^d$ . Hence  $\tilde{B} \simeq \bigoplus_{n \geq 0} (B \otimes_A I^n) \simeq B \otimes_A \left( \bigoplus_{n \geq 0} I^n \right) \simeq B \otimes_A \tilde{A}$ .

e) Let  $P(S) \in A[S_2, \dots, S_n]$ . After successive Euclidean divisions, we obtain

$$f_1^d P(S) = \sum_{2 \leq i \leq n} Q_i(S) (f_1 S_i - f_i) + a, \quad a \in A.$$

Assume that  $P \in (\text{Ker}(\phi))_{(T_1)}$ . Then the image of  $a$  in  $\tilde{A}_{(t_1)}$  is zero, so there exists  $r \geq 0$  such that  $at_1^r = 0$ . Hence  $af_1^r = 0$ . Replacing  $d$  by  $d+r$ , we can assume that  $a = 0$ . Hence  $(\text{Ker}(\phi))_{(T_1)} \subseteq J_1$ .

Conversely, let  $P \in J_1$ . Then  $f_1^d P \in (f_1 S_2 - f_2, \dots, f_1 S_n - f_n)$  for some  $d \geq 0$ . Let  $Q$  be a homogeneous polynomial of degree  $r$  such that  $P(S) = Q(T)/T_1^r$ . Then  $f_1^d T_1^m Q(T) \in (f_1 S_2 - f_2, \dots, f_1 S_n - f_n)$  for some  $m \geq 0$ . Hence  $f_1^d t_1^m Q(t) = 0$ , which implies that  $t_1^{d+m} Q(t) = 0$ . Since  $P(S) = (Q(T) T_1^{d+m}) / (T_1^{d+m+r})$  we have  $J_1 \subseteq (\text{Ker}(\phi))_{(T_1)}$ .

Let us consider the  $A$ -algebra homomorphism  $\psi : A[S_2, \dots, S_n] \rightarrow A_{f_1}$ , which sends  $S_i$  to  $f_i/f_1$ . Then we have  $\text{Ker}(\psi) = J_1$ . Therefore we can identify  $\tilde{A}_{(t_1)}$  with its image in  $A_{f_1}$ . □

**Definition 7.3.3.** Let  $X$  be a scheme. A graded  $\mathcal{O}_X$ -algebra  $\mathcal{B}$  is a quasi-coherent sheaf of  $\mathcal{O}_X$ -algebras with a grading  $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}_n$ , where the  $\mathcal{B}_n$  are quasi-coherent sub- $\mathcal{O}_X$ -modules. We say that  $\mathcal{B}$  is a homogeneous graded  $\mathcal{O}_X$ -algebra if, moreover  $\mathcal{B}_1$  is finitely generated, and if  $(\mathcal{B}_1)^n = \mathcal{B}_n$  for every  $n \geq 1$ . For any affine open subset  $U$  of  $X$ ,  $\mathcal{B}(U)$  is then a homogeneous  $\mathcal{O}_X(U)$ -algebra.

**Lemma 7.3.4.** Let  $X$  be a scheme and  $\mathcal{B}$  a graded  $\mathcal{O}_X$ -algebra. Then there exists a unique  $X$ -scheme  $f : \text{Proj}(\mathcal{B}) \rightarrow X$  such that for any

affine open subscheme  $U$  of  $X$ , we have an isomorphism of  $U$ -schemes  $h_U : f^{-1}(U) \simeq \text{Proj}(\mathcal{B}(U))$  that is compatible with the restriction to any affine open subscheme  $V \subseteq U$ .

**Definition 7.3.5.** Let  $\mathcal{I}$  be a coherent sheaf of ideals on a locally noetherian scheme  $X$ . The  $X$ -scheme  $\text{Proj}\left(\bigoplus_{n \geq 0} \mathcal{I}^n\right) \rightarrow X$  is called the blowing-up of  $X$  with center, or along,  $V(\mathcal{I})$ . This scheme depends on the closed subscheme structure of  $V(\mathcal{I})$ . We will denote the scheme  $\text{Proj}\left(\bigoplus_{n \geq 0} \mathcal{I}^n\right)$  by  $\tilde{X}$ .

Let  $\pi : Y \rightarrow X$  be a morphism of schemes, and let  $\mathcal{I}$  be a quasi-coherent sheaf of ideals on  $X$ . We have a canonical homomorphism  $\pi^* \mathcal{I} \rightarrow \pi^* \mathcal{O}_X = \mathcal{O}_Y$ . Then the image of  $\pi^* \mathcal{I}$  in  $\mathcal{O}_Y$  is a quasi-coherent sheaf of ideals on  $Y$ . We denote it  $(\pi^{-1} \mathcal{I}) \mathcal{O}_Y$ , or abusively  $\mathcal{I} \mathcal{O}_Y$ .

**Proposition 7.3.6.** Let  $X$  be a locally noetherian scheme, and let  $\mathcal{I}$  be a quasi-coherent sheaf of ideals on  $X$ . Let  $\pi : \tilde{X} \rightarrow X$  be the blowing-up of  $X$  with center  $V(\mathcal{I})$ .

- a) The morphism  $\pi$  is an isomorphism if and only if  $\mathcal{I}$  is an invertible sheaf on  $X$ .
- b) The morphism  $\pi$  is proper.
- c) Let  $Z \rightarrow X$  be a flat morphism with  $Z$  locally noetherian. Let  $\tilde{Z} \rightarrow Z$  be the blowing up of  $Z$  with center  $\mathcal{I} \mathcal{O}_Z$ . Then  $\tilde{Z} \simeq \tilde{X} \times_X Z$ .
- d) The morphism  $\pi$  induces an isomorphism  $\pi^{-1}(X \setminus V(\mathcal{I})) \rightarrow X \setminus V(\mathcal{I})$ . If  $X$  is integral and if  $\mathcal{I} \neq 0$ , then  $\tilde{X}$  is integral, and  $\pi$  is a birational morphism.

*Proof.* a) If  $\mathcal{I}$  is invertible, then by Proposition 7.3.2 a),  $\pi$  is an isomorphism. The converse follows from the e) of the same Proposition.

b) If  $X$  is affine, then  $\pi$  is projective by construction, and so is proper since projective morphisms are proper morphisms. Hence  $\pi$  is a proper morphism.

c) It is a consequence of Proposition 7.3.2 d).

d) Let  $U = X \setminus V(\mathcal{I})$ . Then  $\mathcal{I}|_U = \mathcal{O}_U$ . So, applying a) and c) to the flat morphism  $U \rightarrow X$ ,  $\pi^{-1}(U) \rightarrow U$  is an isomorphism. □

**Proposition 7.3.7.** Let  $f : W \rightarrow X$  be a morphism of locally noetherian schemes. Let  $\mathcal{I}$  be a quasi-coherent sheaf of ideals on  $X$ , and  $\mathcal{J} = (f^{-1} \mathcal{I}) \mathcal{O}_W$ . Let  $\pi : \tilde{X} \rightarrow X$  and  $\rho : \tilde{W} \rightarrow W$  denote the blowing-ups of  $X$  and of  $W$  with respective centers  $\mathcal{I}$  and  $\mathcal{J}$ . Then there exists a unique morphism  $\tilde{f} : \tilde{W} \rightarrow \tilde{X}$  which makes the following diagram commutative:

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow \rho & & \downarrow \pi \\ W & \xrightarrow{f} & X \end{array}$$

**Corollary 7.3.8.** *Let  $\pi : \tilde{X} \rightarrow X$  be the blowing-up with center  $\mathcal{I}$  of a locally noetherian scheme. This morphism has the following universal property: for any morphism  $f : W \rightarrow X$  such that  $(f^{-1}\mathcal{I})\mathcal{O}_W$  is an invertible sheaf of ideals on  $W$ , there exists a unique morphism  $g : W \rightarrow \tilde{X}$  making the following diagram commutative :*

$$\begin{array}{ccc} W & \xrightarrow{g} & \tilde{X} \\ & \searrow f & \downarrow \pi \\ & & X \end{array}$$

**Corollary 7.3.9.** *With the hypothesis of Proposition 7.3.7. Let us suppose that  $W \rightarrow X$  is a closed immersion and that the image of  $W$  is not contained in the center  $V(\mathcal{I})$ . Then the morphism  $\tilde{W} \rightarrow \tilde{X}$  is also a closed immersion.*

**Definition 7.3.10.** *Let  $X$  be a locally noetherian scheme and  $\pi : \tilde{X} \rightarrow X$  the blowing-up of  $X$  along a closed subscheme  $V(\mathcal{I})$ . Let  $W$  be a closed subscheme of  $X$  not contained in the center  $V(\mathcal{I})$ . We call the closed subscheme  $\tilde{W} \subseteq \tilde{X}$  the strict transform of  $W$ .*

**Definition 7.3.11.** *Let  $X$  be a surface over  $k$ . We define a monoidal transformation of  $X$  with center  $P$ , and denote it by  $\pi : \tilde{X} \rightarrow X$  the blowing-up of  $X$  along a closed point  $P$ .*

**Proposition 7.3.12.** *Let  $X$  be a surface,  $P$  be a point of  $X$  and let  $\pi : \tilde{X} \rightarrow X$  be the monoidal transformation of  $X$  with center  $P$ . Let  $E$  be the inverse image of  $P$ . The natural maps  $\pi^* : \text{Pic}(X) \rightarrow \text{Pic}(\tilde{X})$  and  $\mathbb{Z} \rightarrow \text{Pic}(\tilde{X})$  defined by  $1 \rightarrow 1 \cdot E$  give rise to an isomorphism  $\text{Pic}(\tilde{X}) \simeq \text{Pic}(X) \oplus \mathbb{Z}$  and the intersection theory on  $\tilde{X}$  is determined by the rules:*

- a) *If  $C, D \in \text{Pic}(X)$ , then  $((\pi^*C) \cdot (\pi^*D)) = (C \cdot D)$ ,*
- b) *If  $C \in \text{Pic}(X)$ , then  $((\pi^*C) \cdot E) = 0$ ,*
- c)  *$(E^2) = -1$ .*

**Proposition 7.3.13.** *We have  $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ , and  $H^i(X, \mathcal{O}_X) \simeq H^i(\tilde{X}, \mathcal{O}_{\tilde{X}})$  for all  $i \geq 0$ .*

**Corollary 7.3.14.** *Let  $\pi : \tilde{X} \rightarrow X$  be a monoidal transformation. Then  $p_a(X) = p_a(\tilde{X})$ .*

*Proof.* We have  $p_a(X) = \dim(H^2(X, \mathcal{O}_X)) - \dim(H^1(X, \mathcal{O}_X)) = \dim(H^2(\tilde{X}, \mathcal{O}_{\tilde{X}})) - \dim(H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})) = p_a(\tilde{X})$ .  $\square$

**Proposition 7.3.15.** *Let  $C$  be an effective divisor on  $X$ , and let  $P$  be a point of multiplicity  $r$  on  $C$ , and let  $\pi : \tilde{X} \rightarrow X$  be the monoidal transformation with center  $P$ . Then  $\pi^*C = \tilde{C} + rE$  where  $E$  is the inverse image of  $P$  and we have  $(\tilde{C} \cdot E) = r$  and  $p_a(\tilde{C}) = p_a(C) - r(r-1)/2$ .*

**Definition 7.3.16.** Let  $X$  and  $Y$  be varieties and let  $T : X \dashrightarrow Y$  a birational map from  $X$  to  $Y$ . There is a largest open subset  $U \subseteq X$  such that  $T$  is represented by a morphism  $\phi : U \rightarrow Y$ . We say that  $T$  is defined at the points of  $U$  and we call the points of  $X \setminus U$  the fundamental points of  $T$ .

**Definition 7.3.17.** Let  $T : X \dashrightarrow Y$  be a birational map between the surfaces  $X$  and  $Y$ . Let  $S = \text{Spec}(k)$ . Let  $Z \subset X \times_S Y$  be the graph of  $T$ , and  $p_1 : Z \rightarrow X$ ,  $p_2 : Z \rightarrow Y$  the projections. For any closed point  $x \in X$ , we call the closed subset  $p_2(p_1^{-1}(x))$  of  $Y$  the total transform of  $x$  by  $T$ . If  $T$  is defined at  $x$ , then the total transform of  $x$  is none other than the point  $T(x)$ . In the general case, we still let  $T(x)$  denote the total transform of  $x$  by  $T$ .

**Lemma 7.3.18.** If  $T : X \dashrightarrow Y$  is a birational transformation of projective varieties, and if  $X$  is normal, then the fundamental points of  $T$  form a closed subset of codimension at least 2.

*Proof.* If  $P \in X$  is a point of codimension 1, then  $\mathcal{O}_{X,P}$  is a discrete valuation ring. Since  $T$  is defined at the generic point of  $X$ , and  $Y$  is projective, hence proper, it follows from the Valuative Criterion of Properness that  $T$  is also defined at  $P$ .  $\square$

**Corollary 7.3.19.** Let  $X$  be a surface and let  $\pi : \tilde{X} \rightarrow X$  be the monoidal transformation with center  $P$ . Then  $\pi$  is defined everywhere and its inverse  $\pi^{-1}$  is a birational transformation having  $P$  as a fundamental point.

**Theorem 7.3.20** (Zariski's Main Theorem). Let  $T : X \dashrightarrow Y$  be a birational transformation of projective varieties, and assume that  $X$  is normal. If  $P$  is a fundamental point of  $T$ , then the total transform  $T(P)$  is connected and of dimension at least 1.

**Definition 7.3.21.** Let  $X$  be a surface and  $S = \text{Spec}(k)$ . A prime divisor  $E$  on  $X$  is called an exceptional divisor if there exists a surface  $Y$  over  $k$  and a morphism  $f : X \rightarrow Y$  of  $S$ -schemes such that  $f(E)$  is reduced to a point, and that  $f : X \setminus E \rightarrow Y \setminus f(E)$  is an isomorphism.

**Proposition 7.3.22.** Let  $f : X' \rightarrow X$  be a birational morphism of surfaces. Let  $P$  be a fundamental point of  $f^{-1}$ . Then  $f$  factors through the monoidal transformation  $\pi : \tilde{X} \rightarrow X$  with center  $P$ .

*Proof.* Let  $T$  be the birational transformation of  $X'$  to  $\tilde{X}$  defined as  $\pi^{-1} \circ f$ . Let us show by contradiction that  $T$  is a morphism. Assume that it is not. Then it has a fundamental closed point  $P'$ . So we have  $f(P') = P$  and, by the Zariski's Main Theorem,  $T(P')$  has dimension at least 1 in  $\tilde{X}$ . Thus  $T(P')$  must be the exceptional divisor  $E$  of  $\pi$ . On the other hand, by Lemma 7.3.18,  $T^{-1}$  is defined at all except finitely many points of  $\tilde{X}$ , so we can find a closed point  $Q \in E$  where  $T^{-1}$  is defined. Hence we have  $T^{-1}(Q) = P'$ . We choose local coordinates  $x, y$  at  $P$  on  $X$ . Then there is an open neighborhood  $V$  of  $P$  such that  $\pi^{-1}(V)$  is defined by the equation  $ty - ux$  in  $\mathbb{P}_V^1$ . By a linear change of variables in



$x, y$  and  $t, u$ , we may assume that  $Q$  is the point with local coordinates  $t = 0, u = 1$  in  $E$ . Then  $t, y$  form local coordinates at  $Q$  on  $\tilde{X}$  and the local equation of  $E$  is  $y = 0$  and  $x = ty$ . Since  $P$  is a fundamental point of  $f^{-1}$ , the Zariski's Main Theorem tells us that we have  $f^{-1}(P)$  connected of dimension at least 1. So there is an irreducible curve  $C$  in  $f^{-1}(P)$  containing  $P'$ . Let  $z = 0$  be a local equation for  $C$  at  $P'$ . Let us consider  $\mathcal{O}_P, \mathcal{O}_{P'}$  and  $\mathcal{O}_Q$  as subrings of  $K = K(X) \simeq K(X')$ .  $f^{-1}(P)$  is defined by the equations  $x = y = 0$  so the images of  $x, y$ , in  $\mathcal{O}_{P'}$  are in the ideal generated by  $z$ , hence we can write  $x = az$  and  $y = bz$  for  $a, b \in \mathcal{O}_{P'}$ . Since  $\mathcal{O}_Q$  dominates  $\mathcal{O}_{P'}$  and  $t, y$  are local coordinates at  $Q$ , so  $y \notin \mathfrak{m}_Q^2$ , and we have  $y \notin \mathfrak{m}_{P'}^2$  in  $\mathcal{O}_{P'}$ . Consequently,  $b$  is a unit in  $\mathcal{O}_{P'}$  and we get that  $t = x/y = a/b$  is in the local ring  $\mathcal{O}_{P'}$ . Since  $t \in \mathfrak{m}_Q$ , we need to have  $t \in \mathfrak{m}_{P'}$ . But  $T(P') = E$  so for any  $s \in \mathfrak{m}_{P'}$  the image of  $s$  in  $\mathcal{O}_Q$  must be contained in the ideal generated by  $y$  because  $y$  is the local equation for  $E$ . In particular, taking  $s = t$ , we obtain that  $t \in (y)$ , which contradicts the fact that  $t$  and  $y$  are local coordinates at  $Q$ . Hence  $T$  is a morphism.  $\square$

**Corollary 7.3.23.** *Let  $f : X' \rightarrow X$  be a birational morphism of surfaces. Let  $n(f)$  be the number of irreducible curves  $C' \subseteq X'$  such that  $f(C')$  is a point. Then  $n(f)$  is finite and  $f$  can be factored into a composition of exactly  $n(f)$  monoidal transformations.*

*Proof.* If  $f(C')$  is a point  $P$ , then  $P$  is a fundamental point of  $f^{-1}$ . By Lemma 7.3.18, the fundamental points of  $f^{-1}$  form a finite set, and for each one, its inverse image  $f^{-1}(P)$  is a closed subset of  $X'$ , having only finitely many irreducible components. So the set of curves  $C'$  which are mapped to a point is finite. Now let  $P$  be a fundamental point of  $f^{-1}$ . Then by the previous Proposition  $f$  factors through the monoidal transformation  $\pi : \tilde{X} \rightarrow X$  with center  $P$ , that is  $f = \pi \circ f_1$  for some morphism  $f_1 : X' \rightarrow \tilde{X}$ . Let us show that  $n(f_1) = n(f) - 1$ . Indeed, if  $f_1(C')$  is a point, then certainly  $f(C')$  is a point. Conversely, if  $f(C')$  is a point, then either  $f_1(C')$  is a point, or  $f_1(C') = E$ , the exceptional curve of  $\pi$ . Furthermore, since  $f_1^{-1}$  is a morphism except at finitely many points, there is a unique irreducible curve  $E'$  in  $X'$  with  $f_1(E') = E$ . Thus  $n(f_1) = n(f) - 1$ .  $\square$

**Theorem 7.3.24.** *Let  $T : X \dashrightarrow X'$  be a birational transformation of surfaces. Then it is possible to factor  $T$  into a finite sequence of monoidal transformations and their inverses.*

*Proof.* By the previous Corollary, it will be sufficient to show that there is a surface  $X''$  and birational morphisms  $f : X'' \rightarrow X$  and  $g : X'' \rightarrow X'$  such that  $T = g \circ f^{-1}$ . Let us construct  $X''$ . Let  $H'$  be a very ample on  $X'$  and let  $C'$  be an irreducible nonsingular curve in the linear system  $|2H'|$ , which does not pass through any of the fundamental points of  $T^{-1}$ . In other words,  $C'$  is entirely contained in the largest open set  $U' \subseteq X'$  on which  $T^{-1}$  is represented by a morphism  $\phi : U' \rightarrow X$ . Let  $C = \phi(C')$  be the image of  $C'$  by  $\phi$  in  $X$ . We define an integer  $m$  by  $m(C) = p_a(C) - p_a(C')$ . Since we have a finite birational morphism of  $C'$  to  $C$ , we see that  $m \geq 0$ , and  $m = 0$  if and only if  $C'$  is isomorphic to  $C$ . Note also that if we replace  $C'$  by a linearly equivalent curve

$C'_1$ , also missing the fundamental points of  $T^{-1}$ , then  $C_1 = \phi(C'_1)$  is linearly equivalent to  $C$ . In fact, if  $C' - C'_1 = (f)$  for some rational function  $f$  on  $X$ , then  $C - C_1 = (f)$  on  $X$ . Since the arithmetic genus of a curve depends only on  $T$  and  $H'$ , and not on the particular curve  $C' \in |2H'|$  chosen. Now fix  $C'$  temporarily. If  $m > 0$ , then  $C$  must be singular. Let  $P$  be a singular point of  $C$ , and let  $\pi : \tilde{X} \rightarrow X$  be the monoidal transformation with center  $P$ , and let  $\tilde{C}$  be the strict transform of  $C$ . Then  $p_a(\tilde{C}) < p_a(C)$ . Thus if  $\tilde{T} = T \circ \pi$  then  $m(\tilde{T}) < m(T)$ . Continuing in this fashion, we have a morphism  $f : X'' \rightarrow X$ , obtained by a finite number of monoidal transformations, such that if  $T' = T \circ f$  then  $m(\tilde{T}) < m(T)$ . Now we're going to prove by contradiction that  $T'$  is a morphism. Assume that  $T'$  is not a morphism. Then  $T'$  has a fundamental point  $P$ . So, by the Zariski's Main Theorem,  $T'(P)$  contains an irreducible curve  $E' \subseteq X'$ . Since  $H'$  is very ample,  $(E' \cdot H') > 0$  and so  $(C' \cdot E') \geq 2$  for any  $C' \in |2H'|$ . Let  $C'$  such that  $C'$  doesn't contain any fundamental point of  $T'^{-1}$  and  $C'$  meets  $E'$  transversally. Then  $C'$  meets  $E'$  in at least two distinct points, so the corresponding curve  $C$  in  $X''$  has at least a double point at  $P$ , which contradicts the fact that  $m(T') = 0$ . Hence  $T'$  is a morphism of  $X''$  to  $X'$  so the previous Corollary completes the proof.  $\square$

**Corollary 7.3.25.** *The arithmetic genus of a nonsingular projective surface is a birational invariant.*

*Proof.* The arithmetic genus of a nonsingular projective surface is invariant by a monoidal transformation hence the result follows from the last Theorem.  $\square$

**Proposition 7.3.26.** *Let  $X$  be a surface over  $k$ . Let  $K$  be a canonical divisor on  $X$  and  $E \subset X_s$  a vertical prime divisor on  $X$ . Then the divisor  $E$  is exceptional if and only if  $(K \cdot E) < 0$  and  $(E^2) < 0$ .*

**Definition 7.3.27.** *We say that a surface is relatively minimal if every birational morphism is an isomorphism.*

**Theorem 7.3.28** (Contraction Criterion of Castelnuovo-Enriques). *If  $Y$  is a curve on a surface, with  $Y \simeq \mathbb{P}^1$  and  $(Y^2) = -1$ , then there exists a morphism  $f : X \rightarrow X_0$  to a surface  $X_0$ , and a point  $P \in X_0$  such that  $X$  is isomorphic via  $f$  to the monoidal transformation of  $X_0$  with center  $P$ , and  $Y$  is the exceptional divisor.*

**Proposition 7.3.29.** *A surface  $X$  is relatively minimal if and only if  $X$  does not contain any exceptional divisor.*

*Proof.* If  $X$  is relatively minimal then every birational morphism is an isomorphism. According to Corollary 7.3.23, we can factor every birational morphism into a finite sequence of monoidal transformations. By the Castelnuovo-Enriques Contraction Criterion, this is possible only if  $X$  contains a curve  $Y$  such that  $Y \simeq \mathbb{P}^1$  and  $(Y^2) = -1$ . Therefore,  $X$  does not contain any exceptional divisor. Conversely, if  $X$  does not contain any exceptional divisor, then it is clear that  $X$  is relatively minimal.  $\square$

**Proposition 7.3.30.** *Let  $X$  be a surface over an algebraically closed field  $k$ . There exists a group homomorphism  $c : \text{Pic}(X) \rightarrow H^1(X, \Omega_X)$  such that we have a nondegenerate bilinear map  $B : H^1(X, \Omega) \times H^1(X, \Omega) \rightarrow k$  defined by, for any two divisors  $C$  and  $D$  on  $X$ ,  $B(c(C), c(D)) = (C \cdot D) \cdot 1$ .*

**Theorem 7.3.31.** *Every surface admits a birational morphism to a relatively minimal surface.*

*Proof.* Let  $X$  be a surface. If  $X$  is relatively minimal, then there is nothing to do. If not, then let  $E$  be an exceptional divisor on  $X$ . By the Contraction Criterion of Castelnuovo-Enriques, there exists a morphism  $X \rightarrow X_1$  contracting  $E$ . We continue in this manner, contracting exceptional divisors, whenever one exists. Hence we obtain a sequence of birational morphisms

$$X_0 = X \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow \dots$$

We must show that the sequence is necessarily finite. Suppose that we have a sequence of  $n$  contractions

$$X = X_0 \rightarrow \cdots \rightarrow X_n$$

as above. For each  $i = 1, \dots, n$ , let us denote  $E'_i \subseteq X_{i-1}$  be the exceptional curve of the contraction  $X_{i-1} \rightarrow X_i$ . Let  $E_i$  be its total transform on  $X$ . With the notations of the previous Proposition, let us denote  $e_i = c(E_i)$ . Then we have  $B(e_i, e_i) = (E_i^2) = -1$  and  $B(e_i, e_j) = 0$  for  $i \neq j$ . Then  $e_1, \dots, e_n$  are linearly independent elements of the vector space  $H^1(X, \Omega_X)$ . Hence we have  $n \leq \dim(H^1(X, \Omega_X))$ . Since it is a finite dimensional vector space,  $n$  is bounded, and so the contraction process must terminate.  $\square$

## 7.4 Castelnuovo's theorem

**Definition 7.4.1.** *If  $X$  is a projective nonsingular surface, then we define  $q = q(X)$  the irregularity genus of  $X$  by  $q = p_g(X) - p_a(X)$ .*

**Remark:** If  $X$  is a projective nonsingular surface, then we have  $p_g(X) = \dim(H^2(X, \mathcal{O}_X))$  and  $p_a(X) = \dim(H^2(X, \mathcal{O}_X)) - \dim H^1(H^1(X, \mathcal{O}_X))$  so  $q = \dim(H^1(X, \mathcal{O}_X))$ .

**Definition 7.4.2.** *Let  $X$  be a nonsingular projective variety and let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. To  $\mathcal{L}$  we can associate the graded  $k$ -algebra  $\Gamma_*(\mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^n)$ , where the homogeneous part of degree  $n$  is  $H^0(X, \mathcal{L}^n)$  and the product of  $s \in H^0(X, \mathcal{L}^n)$  and  $s' \in H^0(X, \mathcal{L}^m)$  is given by  $s \cdot s' = s \otimes s' \in H^0(X, \mathcal{L}^{n+m})$ . Since  $\Gamma_*(\mathcal{L})$  is an integral domain and contains  $k$ , we can define the transcendence degree  $\lambda(\mathcal{L})$  of  $\Gamma_*(\mathcal{L})$  over  $k$ .*

**Definition 7.4.3.** *If  $X$  is a nonsingular projective variety of dimension  $d \geq 1$ , then the ring  $\Gamma_*(\omega_X)$  is called the canonical ring of  $X$  and will be denoted by  $R(X)$ .*

**Definition 7.4.4.** The number  $\kappa(X) = \lambda(\omega_X) - 1$  if  $\lambda(\omega_X) > 0$ , resp  $\kappa(X) = -\infty$  if  $\lambda(\omega_X) = 0$ , is called the canonical dimension of  $X$ , or the Kodaira dimension of  $X$  and  $p_n = p_n(X) = \dim(H^0(X, \omega_X^n))$ , for  $n \geq 1$ , is called the  $n$ -plurigenus of  $X$ .

**Proposition 7.4.5.** If  $X$  and  $Y$  are two surfaces and  $u : X \rightarrow Y$  is a birational map, then  $u$  induces an isomorphism of graded  $k$ -algebras  $u^* : R(Y) \rightarrow R(X)$ . In particular, for each  $n \geq 1$ , the genus  $p_n$  is a birational numerical invariant.

**Proposition 7.4.6.** If  $X$  is a relatively minimal surface, then the following conditions are equivalent:

- a) There exists an integral curve  $C$  on  $X$  such that  $(K \cdot C) < 0$ ,
- b) Adjunction terminates on  $X$ , that is for every divisor  $D \in \text{Div}(X)$  there exists an integer  $n_D$  such that  $|D + nK| = \emptyset$  for all  $n \geq n_D$ ,
- c)  $\kappa(X) = -\infty$ , that is  $p_n = 0$  for all  $n \geq 1$ ,
- d)  $p_{12} = 0$ .

**Proposition 7.4.7.** Let  $X$  be a surface,  $x_1, \dots, x_n$   $n$  distinct closed points on  $X$ , and  $D \in \text{Div}(X)$  a divisor such that  $\dim |D| \geq 3n$ . Then there exists an effective divisor  $D' \in |D|$  such that every  $x_i$  is on  $D'$  and is a multiple point of  $D'$ .

**Lemma 7.4.8.** If  $X$  is a relatively minimal surface with  $\kappa(X) = -\infty$ , then there exists a hyperplane section  $H$  on  $X$  such that  $(K \cdot H) < 0$ .

*Proof.* Since we have  $\kappa(X) = -\infty$ , Proposition 7.4.6 assures the existence of an integral curve  $C$  on  $X$ , such that  $(K \cdot C) < 0$ . As  $X$  is relatively minimal, we have  $(C^2) \geq 0$ , otherwise  $X$  would have an exceptional divisor  $C$ . Let  $H_1$  be a hyperplane section on  $X$ . For every  $n \geq 0$ , we have

$$(nC + H_1)^2 = n^2(C^2) + 2n(C \cdot H_1) + (H_1^2) \geq (H_1^2) > 0,$$

and  $(nC + H_1 \cdot D) = n(C \cdot D) + (H_1 \cdot D) \geq (H_1 \cdot D) > 0$  for every integral curve  $D$  on  $X$ . So, by the Nakai-Moishezon Criterion, the divisor  $D = nC + H_1$  is ample. Furthermore we have  $(K \cdot D) = n(K \cdot C) + (K \cdot H_1) < 0$  for all  $n$  enough large.  $\square$

**Proposition 7.4.9.** Let  $X$  be a relatively minimal surface with  $\kappa(X) = -\infty$  and  $(K^2) \leq 0$ . Then for every  $n \geq 0$  there exists a divisor  $D \in \text{Div}(X)$  such that  $|K + D| = \emptyset$ ,  $(D \cdot K) < 0$  and  $\dim |D| \geq n$ .

*Proof.* By the previous Lemma, there is a hyperplane section  $H$  on  $X$  such that  $(K \cdot H) \leq -2n - 2q$ . Since  $\kappa(X) = -\infty$ , adjunction on  $X$  terminates. Then there exists  $m \geq 0$  such that  $|H + mK| \neq \emptyset$  and  $|H + (m+1)K| = \emptyset$ . Let  $D' \in |H + mK|$ . If  $D' = 0$  then  $H \sim -mK$  and therefore  $(H^2) = m^2(K^2) \leq 0$ , which is a contradiction, therefore we have  $D' \neq 0$ . Let  $D$  consists of the components  $E$  of  $D'$  such that  $(K \cdot E) < 0$ , and  $D''$  consists of the components  $E$  of  $D'$  such that  $(K \cdot E) \geq 0$ . By construction, we have  $|K + D| \subseteq |K + D'| = |H +$

$(m+1)K| = \emptyset$ . Since  $|K - D| \subseteq |K| = \emptyset$ , we have, by the Riemann-Roch Theorem:

$$\begin{aligned}
\dim |D| &\geq (D^2)/2 - (D \cdot K)/2 - q \\
&\geq (D^2)/2 - (D \cdot K)/2 - (D'' \cdot K)/2 - q \\
&= (D^2)/2 - (D' \cdot K)/2 - q \\
&= (D^2)/2 - (H \cdot K)/2 - m(K^2)/2 - q \\
&\geq -(H \cdot K)/2 - q \\
&\geq (n+q) - q = n.
\end{aligned}$$

Since  $D \approx 0$ , we have  $(K \cdot D) < 0$ . Also,  $|K + D| \subseteq |K + D'| = |H + (m+1)K| = \emptyset$ , that ends the proof.  $\square$

**Proposition 7.4.10.** *Let  $X$  be a relatively minimal surface and let  $D = \sum_{i=1}^r n_i E_i$  be an effective divisor on  $X$  such that  $|K + D| = \emptyset$ , where  $n_i \geq 1$  for all  $i$  and the  $E_i$  are mutually distinct integral curves on  $X$ . Then every  $E_i$  is a nonsingular curve and  $\sum_{i=1}^r p_a(E_i) \leq q$ .*

**Proposition 7.4.11.** *Let  $X$  be a relatively minimal surface with  $q = 0$ , and let  $E$  be a rational curve on  $X$  such that  $(K \cdot E) < 0$  and  $|K + E| = \emptyset$ . Then  $X$  is either a geometrically ruled surface over  $\mathbb{P}^1$  or isomorphic to  $\mathbb{P}^2$ . Furthermore, in the geometrically ruled case the ruling is given by a morphism  $\phi_{|C|}$  for a (nonsingular) rational curve  $C$  on  $X$  such that  $(K \cdot C) < 0$  and  $|K + C| = \emptyset$ .*

*Proof.* By hypothesis  $X$  is a relatively minimal surface and we have  $(K \cdot E) < 0$  so  $(E^2) \geq 0$ . Let us denote  $\mathcal{E}$  the set

$$\mathcal{E} := \left\{ (C^2), C \text{ is an integral rational curve with } (C^2) \geq 0 \text{ and } |K + C| = \emptyset \right\},$$

and let  $a = \min(\mathcal{E})$ . Let  $\mathcal{C}$  the set of all rational curves  $C$  with  $(C^2) = a$  and  $|K + C| = \emptyset$ . Let  $H$  be a hyperplane section on  $X$ . We put  $b = \min\{(C \cdot H), C \in \mathcal{C}\}$ . We fix a curve  $C \in \mathcal{C}$  with  $(C \cdot H) = b$ . Let us prove that every element of  $|C|$  is a nonsingular rational curve. Let  $D = \sum_{i=1}^r n_i E_i \in |C|$ , with  $n_i \geq 1$  and  $E_i$  mutually distinct integral curves on  $X$ . Since  $(K \cdot D) = (K \cdot C) < 0$ , then we have  $(K \cdot E_i) < 0$  for at least one  $i$ . We fix a such index  $i$ . Since  $X$  is relatively minimal and  $(K \cdot E_i) \geq 0$ . Since  $|K + D| = |K + C| = \emptyset$ , Proposition 7.4.10 implies that  $E_i$  is a nonsingular curve and  $|K + E_i| = \emptyset$ . Therefore  $(E_i^2) \geq a$ , since  $a = \min(\mathcal{E})$ . Moreover, we have  $a = (D^2) = \sum_{j=1}^r n_j (C \cdot E_j)$ . We claim that  $(C \cdot E_j) \geq 0$  for any  $1 \leq j \leq r$ . Indeed  $C$  and  $E_j$  have no common irreducible component, otherwise we would have  $C = E_1$  and so  $(n_1 - 1)C + \sum_{j=2}^r n_j E_j \sim 0$  hence we have  $r = 1$  and  $D = E_1 = C$  because  $n_j \geq 1$  for  $1 \leq j \leq r$ . Applying this to the previous inequality, we obtain that  $a \geq n_i (C \cdot E_i)$ . Then we have that  $a \geq n_i \sum_{h=1}^r n_h (E_h \cdot E_i) \geq n_i^2 (E_i^2)$ . Since  $n_i \geq 1$ , we deduce that  $a \geq (E_i^2)$ , hence  $a = (E_i^2)$  by definition

of  $a$ . Consequently  $E_i \in \mathcal{C}$ . From that, we have by definition of  $b$ ,  $(H \cdot E_i) \geq b = (H \cdot C) = \sum_{h=1}^r n_h (H \cdot E_h)$ . As  $(H \cdot E_h) > 0$  for all  $h$ , we get  $r = 1$  and  $n_i = 1$ . So, by Proposition 7.4.10, we have  $p_a(E_i) = 0$  so  $E_i$  is rational. We have  $D = E_i$  and  $E_i$  is a nonsingular rational curve hence every  $D \in |C|$  is a nonsingular rational curve. In particular, from Proposition 7.4.7, we have  $\dim(|C|) \leq 2$ . From the exact sequence  $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_X(C) \otimes \mathcal{O}_C \rightarrow 0$  we get the cohomology exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_X(C)) \longrightarrow H^0(\mathcal{O}_X(C) \otimes \mathcal{O}_C) \longrightarrow H^1(\mathcal{O}_X) = 0.$$

Since  $\deg_C(\mathcal{O}_X(C) \otimes \mathcal{O}_C) = (C^2)$  and  $C \simeq \mathbb{P}^1$ , we have  $\dim(H^0(\mathcal{O}_X(C) \otimes \mathcal{O}_C)) = 1 + (C^2)$ , and therefore  $\dim(H^0(\mathcal{O}_X(C))) = 2 + (C^2)$ , or  $\dim(|C|) = 1 + (C^2)$ . Since  $\dim(|C|) \leq 2$ , we need  $(C^2) = 0$  or  $(C^2) = 1$ . If  $(C^2) = 0$  then  $\dim(|C|) = 1$ , hence the morphism  $\phi_{|C|}$  maps  $X$  to  $\mathbb{P}^1$ . The fibers of this morphism are the elements of  $|C|$ , which are all nonsingular rational curves, so that  $X$  is a geometrically ruled surface with base  $\mathbb{P}^1$ . If  $(C^2) = 1$ , and so  $\dim(|C|) = 2$ , let  $D$  be an arbitrary integral curve on  $X$ , and let  $x \in D$  be a closed point. Since  $\dim(|C|) = 2$ , there exists a curve  $C' \in |C|$  such that  $x \in C'$ . If  $C' = D$ , then  $(C \cdot D) = (C \cdot C') = (C^2) = 1 > 0$ . If  $C' \neq D$  then also  $(C \cdot D) = (C' \cdot D) > 0$ . Therefore the morphism  $\phi_{|C|} : X \rightarrow \mathbb{P}(H^0(\mathcal{O}_X(C))) \simeq \mathbb{P}^2$  is a finite surjective morphism. Since  $C$  is the inverse image by  $\phi_{|C|}$  of a straight line in  $\mathbb{P}^2$  and  $(C^2) = 1$ ,  $\phi_{|C|}$  is an isomorphism.  $\square$

**Proposition 7.4.12.** *Let  $X$  be a relatively minimal surface with  $p_2 = q = 0$ . Then  $\kappa(X) = -\infty$ . If in addition  $(K^2) \leq 0$ , then there exists a nonsingular rational curve  $C$  on  $X$  with  $(K \cdot C) < 0$  and  $|K + C| = \emptyset$ ; in particular in this case, the surface  $X$  is rational.*

*Proof.* First we're going to prove that if  $(K^2) < 0$  then  $\kappa(X) = -\infty$ . By contradiction, assume that there exists  $D \in |nK|$  with  $n \geq 1$ . Then  $(K \cdot D) = n(K^2) < 0$ . Then we have an integral curve  $D$  on  $X$  such that  $(K \cdot C) < 0$ . Hence by the Proposition 7.4.6, we have that  $\kappa(X) = -\infty$  which is a contradiction.

Now assume that  $p_2 = q = 0$  and  $(K^2) \geq 0$ . Applying the Riemann-Roch Theorem to the divisor  $2K$  on  $X$ , we obtain the inequality

$$\dim(|2K|) + \dim(|-K|) \geq (K^2) - 1.$$

Since  $p_2 = 0 = \dim(H^0(X, \omega_X^2))$ , we obtain  $l(2K) = \dim(H^0(X, \mathcal{L}(2K))) = 0$ , hence  $\dim(|2K|) = -1$  and  $\dim(|-K|) \geq (K^2) \geq 0$ . Thus there exists an effective divisor  $D \in |-K|$ . Furthermore we cannot have  $D = 0$  otherwise we would have  $K \sim 0$  and so  $p_g = 1$  which is a contradiction. Therefore  $D \neq 0$ . Then for every divisor  $D' \in \text{Div}(X)$  there exists an integer  $n_{D'} \geq 1$  such that  $|D' + nK| = |D' - nD| = \emptyset$  for all  $n \geq n_{D'}$ . Therefore, adjunction on  $X$  terminates, so that  $\kappa(X) = -\infty$  by the Proposition 7.4.6. Finally, assume that  $(K^2) \leq 0$ . By the Proposition 7.4.9, there exists an effective divisor  $E$  on  $X$  such that  $(K \cdot E) < 0$  and  $|K + E| = \emptyset$ . Then some irreducible component  $C$  of  $E$  still has the properties  $(K \cdot C) < 0$  and  $|K + C| = \emptyset$ . Then we have  $p_a(C) \leq q = 0$  so

that  $C$  is a nonsingular rational curve. The last statement follows from the Proposition 7.4.11.  $\square$

**Remark:** The last Proposition proves the Castelnuovo's Theorem in the case  $(K^2) \leq 0$ .

**Lemma 7.4.13.** *If  $X$  is a relatively minimal surface with  $p_2 = q = 0$ ,  $(K^2) > 0$  and  $(K \cdot D) \geq 0$  for every effective divisor  $D$  on  $X$  such that  $|K + D| = \emptyset$ , then*

- a) *Pic( $X$ ) is generated by  $\omega_X = \mathcal{O}_X(K)$ , and  $\mathcal{O}_X(-K)$  is ample. In particular,  $X$  does not contain any nonsingular rational curves.*
- b) *Every  $D \in |-K|$  is an integral curve with  $p_a(D) = 1$ .*
- c)  *$(K^2) \leq 5$  and  $b_2 \geq 5$ .*

*Proof.* To prove this Lemma, we will first prove that b) is true, then, using b) we will prove a) and finally c).

- b) Let us begin to prove b). Let  $D \in |-K|$ . To show that  $D$  is an irreducible and reduced curve, we are going to proceed by contradiction: assume that  $D$  is reducible. We have  $D > 0$  and  $(K \cdot D) = -(K^2) < 0$ . Consequently, at least one component  $E_i$  of  $D$  verifies  $(K \cdot E_i) < 0$ . Hence we have  $|K + E_i| = \emptyset$ . Indeed, if  $|K + E_i| \neq \emptyset$  then we would have  $C \in |K + E_i|$  such that  $C \sim 0$  and so  $D - E_i + C \sim -K - E_i + K + E_i = 0$ , which is a contradiction since  $D - E_i + C \sim 0$ . So  $|K + E_i| = \emptyset$ . But by hypothesis we would have  $(K + D) \geq 0$  so we have a contradiction. Hence  $D$  is irreducible. The same argument shows that every  $D \in |-K|$  is reduced. So every  $D \in |-K|$  is an integral curve. Moreover, for every  $D \in |-K|$  we have  $2p_a(D) - 2 = (D \cdot (D + K))$ , so using the fact that  $D + K \sim 0$  we obtain that  $p_a(D) = 1$ .
- a) Since  $X$  is a relatively minimal surface with  $p_2 = q = 0$ , by Proposition 7.4.12, we have  $\kappa(X) = -\infty$ , so, by Proposition 7.4.6, adjunction terminates on  $X$ . Let  $E$  be an effective divisor on  $X$ . Since adjunction terminate on  $X$ , there exists an integer  $n \geq 0$  such that  $|E + nK| \neq \emptyset$ , and  $|E + (n + 1)K| = \emptyset$ . Let  $D = E + nK$ . Then  $D$  is an effective divisor on  $X$  such that  $|D + K| = \emptyset$ . By contradiction, assume that  $D > 0$ . Let  $x \in D$  be a point. Since  $\dim(|-K|) \geq (K^2) \geq 1$ , there exists  $C \in |-K|$  passing through  $x$ . Since  $C \in |-K|$ , b) tells us that  $C$  is an integral curve on  $X$ . Then  $C$  cannot be one of the components of  $D$ , otherwise we would have  $|K + C| \subseteq |K + D|$  and so  $|0| \subseteq \emptyset$  which is absurd. Since  $D$  and  $C$  meet at  $x$ , we have  $D \cap C \neq \emptyset$  and  $D$  and  $C$  have no common irreducible component, hence  $(C \cdot D) = \sharp(C \cap D) > 0$ . But it means that  $(K \cdot D) = (-C \cdot D) = -(C \cdot D) < 0$ , which contradicts the fact that  $(K \cdot D) \geq 0$  for every effective divisor  $D$  on  $X$ . So  $D$  is the zero divisor. Hence we have  $E \sim -nK$ . Since every divisor on  $X$  is the difference of two effective divisor on  $X$ , that shows that  $\text{Pic}(X)$  is cyclic and generated by  $\mathcal{O}_X(K) = \omega_X$ .

Moreover, if  $H$  is a hyperplane section on  $X$ , then  $H \sim -nK$  with  $n \geq 0$ , and, if  $H \approx 0$  then  $n > 0$ , so  $\mathcal{O}_X(-K)$  is ample.

It remains to show that  $X$  does not contain any nonsingular rational curves. Let  $C$  be an integral curve on  $X$ . Then we have  $C \sim -nK$  for some  $n > 0$ . Then we have

$$p_a(C) = p_a(-nK) = \frac{1}{2}(-nK \cdot (-nK + K)) + 1 = \frac{1}{2}n(n-1)(K^2) + 1 \geq 1.$$

Consequently,  $C$  is not a nonsingular rational curve.

- c) By contradiction, assume that  $(K^2) \geq 6$ . Then  $\dim(|-K|) \geq (K^2) \geq 6$ . Let  $x$  and  $y$  be two distinct points on  $X$ . Then there exists  $C \in |-K|$  such that  $x$  and  $y$  are multiple points of  $C$ . We have a contradiction since  $C$  is an integral curve and  $p_a(C) = 1$  by b), and so  $C$  have at most one singular point. Hence  $(K^2) \leq 5$ .

□

**Lemma 7.4.14.** *If  $k$  is an algebraically closed field of characteristic zero, then there are no surfaces  $X$  with  $q = 0$  satisfying conditions a) and c) of Lemma 7.4.13.*

**Lemma 7.4.15.** *Let  $X$  be a surface with  $q = 0$  and with properties a), b) and c) in Lemma 7.4.13. Then there exists a nonsingular curve  $D \in |-K|$ .*

**Lemma 7.4.16.** *Let  $X$  be a surface with  $q = 0$  and with properties a), b) and c) in Lemma 7.4.13. Then  $H^2(X, T_X) = 0$ , where  $T_X = (\Omega_{X/k}^1)$  is the sheaf associated to the tangent bundle of  $X$ .*

**Proposition 7.4.17.** *Let  $f : X \rightarrow Y$  be a dominant morphism from an irreducible nonsingular variety  $X$  of dimension at least 2 to an irreducible curve  $Y$ , such that  $k(Y)$  is algebraically closed in  $k(X)$ . Then the fiber  $f^{-1}(y)$  is geometrically integral, that means  $f^{-1}(y)_{k'}$  is integral over  $k'$  for every field extension  $k'$  of  $k$ , for all but a finite number of closed points  $y \in Y$ .*

**Lemma 7.4.18.** *There are no surfaces  $X$  with  $q = 0$  satisfying the conditions a), b) and c) of Lemma 7.4.13.*

**Theorem 7.4.19** (Castelnuovo). *Let  $X$  be a relatively minimal surface. Then  $X$  is a rational surface if and only if  $q = p_2 = 0$ .*

*Proof.* Assume that  $p_2(X) = q(X) = 0$ . Let  $K$  be a canonical divisor on  $X$ . If  $(K^2) \leq 0$ , then by Proposition 7.4.12, the surface  $X$  is rational. Now assume that  $(K^2) > 0$ . By Lemma 7.4.18,  $X$  doesn't satisfy conditions a), b) and c) of Lemma 7.4.13. Hence, there exists an effective divisor  $D$  on  $X$  such that  $(K \cdot D) < 0$  and  $|K + D| = \emptyset$ . So at least one component  $E_i$  of  $D$  verify  $(K \cdot E_i) < 0$  and  $|K + E_i| = \emptyset$ . Then, by Proposition 7.4.10  $p_a(E_i) \leq q = 0$  and  $E_i$  is a nonsingular rational curve. Then, Proposition 7.4.11 tells us that  $X$  is rational. To prove the converse, assume that  $X$  is rational. Since the  $n$ -th plurigenus, the arithmetical genus and the geometric genus of a variety are birational invariants, we have  $q(X) = p_2(X) = 0$ . □



Recall that a surface  $X$  is a unirational variety if there exists a separable and dominant rational map  $\phi : \mathbb{P}^2 \rightarrow X$ .

**Theorem 7.4.20.** *Every unirational surface is rational.*

*Proof.* Since  $X$  is birational to a relatively minimal surface, we may assume that  $X$  is relatively minimal. Let  $\phi : \mathbb{P}^2 \rightarrow X$  be a separable and dominant rational map. After a suitable number of quadratic transformations of  $\mathbb{P}^2$ , we arrive at a separable, surjective regular morphism  $\psi : Y \rightarrow X$ , with  $Y$  a rational surface. For each  $n \geq 1$ , the separable morphism  $\psi$  induces an injective map  $\psi^* : \Gamma(X, \omega_X^n) \rightarrow \Gamma(Y, \omega_Y^n)$ , and since  $p_n(Y) = 0$  for all  $n \geq 1$  (because  $Y$  is a rational surface), we have  $p_n(X) = 0$  for all  $n \geq 1$ . On the other hand, if  $\alpha_X : X \rightarrow \text{Alb}(X)$  is the Albanese morphism of  $X$ , we have  $q(X) \geq \dim(\text{Alb}(X))$ . But  $\text{Alb}(X)$  is an abelian variety generated by  $\alpha_X(X) = (\alpha_X \circ \psi)(Y) = (\text{Alb}(\psi) \circ \alpha_Y)(Y) = 0$ . Since  $p_g(X) = 0$ , we get  $q(X) = \dim(\text{Alb}(X)) = 0$ . In particular, we have  $p_2(X) = q(X) = 0$ , so by the Castelnuovo's Theorem,  $X$  is rational.  $\square$

This result still holds in any characteristic if  $k$  is not algebraically closed but it becomes false in general if we remove the separable hypothesis, as Zariski proved it.

## 8 Unirationality of cubic hypersurface in dimension $n \geq 3$

Let  $k$  be an algebraically closed field of characteristic zero. When it is not confusing, we will denote by  $\mathbb{P}^n = \mathbb{P}_k^n$  the projective space of dimension  $n$  over a field  $k$ . In this section, we're going to see that any smooth cubic hypersurfaces  $X \subset \mathbb{P}^n$ , with  $n \geq 3$  is unirational.

**Definition 8.0.1.** *Let  $V$  be a finite-dimensional linear space over  $k$  and  $W$  be a proper, nonzero linear subspace of  $V$ . Let us denote  $\Lambda = \mathbb{P}(V/W)$ . The canonical map  $V \rightarrow V/W$  induces a closed embedding  $\mathbb{P}(V/W) \hookrightarrow \mathbb{P}(V)$ . The inclusion  $W \hookrightarrow V$  induces a map  $\phi : \mathbb{P}(V) \setminus \Lambda \rightarrow \mathbb{P}(W)$  which is called the linear projection with center  $\Lambda$ .*

**Proposition 8.0.2.** *Let  $V$  be a finite-dimensional linear space over  $k$  and  $W$  be a proper, nonzero linear subspace of  $V$ . Let  $\phi : \mathbb{P}(V) \setminus \mathbb{P}(V/W) \rightarrow \mathbb{P}(W)$  be the linear projection with center  $\mathbb{P}(V/W)$ . Let  $h : \widetilde{\mathbb{P}(V)} \rightarrow \mathbb{P}(V)$  the blow-up of  $\mathbb{P}(V)$  along  $\mathbb{P}(V/W)$  with exceptional divisor  $E$ . Then there is a lift  $g : \widetilde{\mathbb{P}(V)} \rightarrow \mathbb{P}(W)$  of  $\phi$  which realizes  $\widetilde{\mathbb{P}(V)}$  as a projective bundle over  $\mathbb{P}(W)$ .*

**Definition 8.0.3.** *Let  $X$  and  $Y$  be two varieties over  $k$ . A morphism of varieties  $f : X \rightarrow Y$  is a quadric bundle if it can be decomposed as a closed embedding  $i : X \hookrightarrow Y \times \mathbb{P}^n$  followed by the canonical projection*

$p : Y \times \mathbb{P}^n \rightarrow Y$ , with  $n \geq 2$ , that is, we have the following commutative diagram

$$\begin{array}{ccc} & & f \\ & \curvearrowright & \\ X & \xrightarrow{i} Y \times \mathbb{P}^n \xrightarrow{p} & Y \end{array}$$

such that the fiber over each  $s \in S$  is a quadric in  $\mathbb{P}_{k(s)}^{n+1}$ , smooth for general  $s$ .

If  $n = 2$  then we say that  $f$  is a conic bundle.

A rational map  $\phi : X \dashrightarrow Y$  is a rational quadric bundle if there are open subsets  $U \subseteq X$  and  $V \subseteq Y$  such that  $\phi$  corresponds to a morphism  $U \rightarrow V$ , which is a quadric bundle.

**Proposition 8.0.4.** *Given a morphism*

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n \times Y \\ \downarrow f & & \swarrow \\ Y & & \end{array}$$

as in the previous Definition, we have that  $f$  is a rational quadric bundle if and only if the generic fiber

$$\begin{array}{ccc} X_\eta & \hookrightarrow & \text{Spec}(K(Y)) \\ \downarrow f & & \swarrow \\ \text{Spec}(K(Y)) & & \end{array}$$

is a smooth quadric. Moreover, if  $X$  is smooth then  $f$  is smooth.

**Proposition 8.0.5.** *Let  $X$  and  $S$  be two varieties over  $k$  and  $\phi : X \dashrightarrow S$  be a rational quadric bundle and  $Y \hookrightarrow X$  a closed subvariety.*

- i) *If  $S$  is rational and  $\phi$  induces a birational map  $Y \dashrightarrow S$ , then  $X$  is rational.*
- ii) *If  $Y$  is unirational and  $\phi$  induces a dominant rational map  $Y \dashrightarrow S$ , then  $X$  is unirational.*

**Remark:** When  $Y$  has the property that  $\phi$  induces a dominant, generically finite rational map  $Y \dashrightarrow S$ , one sometimes refers to  $Y$  as a multi-section of  $\phi$ .

**Theorem 8.0.6.** *Let  $n \geq 3$  and let  $X \subset \mathbb{P}^n$  be a smooth cubic hypersurface over  $k$ . If  $X$  contains a line, then  $X$  is unirational.*

*Proof.* Assume that there is a line  $L \subset X$ . Let  $\Lambda \subset \mathbb{P}^n$  be a linear subspace of dimension  $(n-2)$  such that  $L \cap \Lambda = \emptyset$ . We consider the projection map  $\phi : \mathbb{P}^n \setminus L \rightarrow \Lambda$ . If  $\pi : \widetilde{\mathbb{P}^n} \rightarrow \mathbb{P}^n$  is the blow-up of  $\mathbb{P}^n$  along  $L$ , with exceptional divisor  $E$ , then the rational map  $\phi \circ \pi$  extends as a morphism  $f : \widetilde{\mathbb{P}^n} \rightarrow \Lambda$ . Let  $\tilde{X}$  denote the strict transform of  $X$  and  $Y = E \cap \tilde{X}$ . Then  $\tilde{X}$  is the blow-up of the smooth variety  $X$  along  $L$

and  $Y$  is the exceptional divisor. So we have the following commutative diagram:

$$\begin{array}{ccccc}
Y & \xrightarrow{\quad} & \tilde{X} & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
& & E & \xrightarrow{\quad} & \tilde{\mathbb{P}}^n \\
\downarrow & & \downarrow & & \downarrow \pi \\
L & \xrightarrow{\quad} & X & & \mathbb{P}^n \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow f \\
L & \xrightarrow{\quad} & L & \xrightarrow{\quad} & \mathbb{P}^n \dashrightarrow \Lambda
\end{array}$$

Moreover, the morphism  $i = (f, \pi) : \tilde{\mathbb{P}}^n \rightarrow \Lambda \times \mathbb{P}^n$  exhibits  $\tilde{\mathbb{P}}^n$  as a projective subbundle over  $\Lambda$ . Then both  $\tilde{X}$  and  $Y$  are smooth and irreducible. Moreover, since  $Y$  is a projective bundle over  $\Lambda$ , we deduce that it is rational. Let us consider now the restriction  $g : \tilde{X} \rightarrow \Lambda$  of  $f$ . We claim that this is a rational conic bundle. For every  $Q \in \Lambda$ , the map  $i$  embeds the fiber  $f^{-1}(Q)$  in  $\mathbb{P}^n$  as the linear span  $\langle L, Q \rangle$  of  $L$  and  $Q$ . Moreover,  $f^{-1}(Q) \cap E$  corresponds to  $L$ . Then we get an induced isomorphism  $g^{-1}(Q) \setminus Y = (f^{-1}(Q) \cap \tilde{X}) \setminus E \simeq (\langle L, Q \rangle \cap X) \setminus L$ . For  $Q$  general, the intersection  $\langle L, Q \rangle \cap X$  is not contained in  $L$ . Therefore, it is a cubic hypersurface in  $\langle L, Q \rangle \simeq \mathbb{P}^2$ . It contains  $L$  and we will shortly show that for  $Q$  general, it contains it with multiplicity 1. On the other hand, since  $\tilde{X}$  is smooth and  $k$  is of characteristic zero, the fiber  $g^{-1}(Q)$  is smooth, and so is connected since this is a smooth plane curve. We thus conclude that for  $Q$  general,  $g^{-1}(Q) = \overline{g^{-1}(Q) \setminus Y} \hookrightarrow \langle L, Q \rangle$ . Since  $X \cap \langle L, Q \rangle$  is a cubic curve containing the line  $L$  that is also embedded in  $\langle L, Q \rangle \simeq \mathbb{P}^2$ , we have  $g^{-1}(Q) \subseteq \langle L, Q \rangle$  is a smooth conic. This shows  $g : \tilde{X} \rightarrow \Lambda$  is a rational conic bundle. We now show that for  $Q \in \Lambda$  general,  $L$  appears with multiplicity 1 in  $\langle L, Q \rangle \cap X$ . We may choose coordinates on  $\mathbb{P}^n$  such that  $L = (x_2 = \cdots = x_n = 0)$  and  $\Lambda = (x_0 = x_1 = 0)$ . If  $f \in k[x_0, \dots, x_n]$  is an equation of  $X$ , then  $f \in (x_2, \dots, x_n)$ , hence we can write  $f = \sum_{i=2}^n x_i f_i$  for some homogeneous polynomials  $f_2, \dots, f_n$  of degree 2. Let  $g_i(x_0, x_1) = f_i(x_0, x_1, 0, \dots, 0)$ . Such  $i$  with  $g_i \neq 0$  exists since  $L$  is not contained in the singular locus of  $X$ . Given  $Q = (\lambda_2, \dots, \lambda_n) \in \Lambda$ , we have coordinates  $a, b, c$  on  $\langle L, Q \rangle$  such that  $L = (c = 0)$  and  $X \cap \langle L, Q \rangle$  is defined by

$$f(a, b, c\lambda_2, \dots, c\lambda_n) = c \cdot \sum_{i=2}^n \lambda_i f_i(a, b, c\lambda_1, \dots, c\lambda_n).$$

If  $Q$  is general, then  $\sum_{i=2}^n \lambda_i g_i \neq 0$ , which implies that  $L$  appears with multiplicity 1 in  $\langle L, Q \rangle$ . Finally,  $Y$  is a multi-section of  $g$ . Indeed, for  $Q$  general,  $g^{-1}(Q) \cap E$  is the intersection of the plane conic  $g^{-1}(Q)$  with the line  $L$ , hence it consists of 2 points, by generic smoothness. Since  $Y$  is rational, hence unirational, Proposition 8.0.5 allows us to conclude that  $\tilde{X}$  is unirational, and so is  $X$ .  $\square$

**Theorem 8.0.7** (Theorem on the Dimension of Fibres). *Let  $f : X \rightarrow Y$  be a regular map between irreducible varieties. Suppose that  $f$  is surjective and that  $\dim(X) = n$ ,  $\dim(Y) = m$ . Then  $m \leq n$ , and*

- i)  $\dim(F) \geq n - m$  for any  $y \in Y$  and for any component  $F$  of the fiber  $f^{-1}(y)$ ;
- ii) There exists a nonempty open subset  $U \subset Y$  such that  $\dim(f^{-1}(y)) = n - m$  for  $y \in U$ .

**Theorem 8.0.8.** *If  $X \subset \mathbb{P}^3$  is a cubic hypersurface over  $k$ , then  $X$  contains at least one line.*

*Proof.* Idea of the proof : Consider  $G = G(2, 4)$  be the Grassmannian parametrizing lines in  $\mathbb{P}^3$  and  $\mathbb{P}$  be the projective space parametrizing cubic hypersurfaces of degree 3. Then  $\mathbb{P}$  is isomorphic to  $\mathbb{P}^{\binom{3+3}{3}-1} = \mathbb{P}^{19}$ . Next, consider  $M = \{(X, L) \in \mathbb{P} \times G, L \subset X\}$ . This is a closed subset of  $\mathbb{P} \times G$ . Let us denote by  $p : M \rightarrow \mathbb{P}$  and  $q : M \rightarrow G$  the projection maps. We have the following result:

The fibers of  $q$  are projective subspaces of  $\mathbb{P}$ , of codimension 4. In particular,  $M$  is irreducible, with  $\dim(M) = 19$ .

Now  $p$  is a morphism between projective varieties of dimension 19. We want to show that  $p$  is surjective. Since  $p$  is proper, hence closed, we only need that it is dominant. If some fiber of  $p$  has dimension 0 then the Theorem on the Dimension of Fibres tells us that  $\dim(p(M)) = 19$ . The fact  $p(M) \subset \mathbb{P}^{19}$  and  $p(M)$  is irreducible will implies that  $p(M) = \mathbb{P}^{19}$ . So it is enough to find such fiber of  $p$ . We claim that there exists such a surface.  $\square$

**Corollary 8.0.9.** *Let  $n \geq 3$  and let  $X \subset \mathbb{P}^n$  be a smooth cubic hypersurface over  $k$ . Then  $X$  is unirational.*

*Proof.* Let  $\Lambda \subset \mathbb{P}^n$  be a linear subspace of dimension 3 which is not contained in  $X$ . Then  $X \cap \Lambda$  is a cubic surface in  $\Lambda \simeq \mathbb{P}^3$ . Hence  $X \cap \Lambda$  contains a line, and so does  $X$ . Consequently, by Theorem 8.0.6,  $X$  is unirational.  $\square$

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