



# Uniformity in Mordell Lang and the Gao-Habegger Inequality

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# Uniformity in Mordell Lang and The Gao-Habegger Inequality



ALGANT Master's Thesis



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# Chapter 1

## Preface

”There is no dark side of the moon really, as a matter of fact it’s all dark. The only thing that makes it lit is the sun.”- Pink Floyd

### 1.1 The problem of Uniformity: A short history

Let  $k$  be a field of characteristic 0. A smooth curve is a geometrically irreducible smooth projective variety of dimension 1 over  $k$ . By the choice of a  $k$ -rational point  $P_0$  (by possibly increasing  $k$  to a finite extension), we can embed  $C$  into its Jacobian  $\text{Jac}(C)$  via the Abel-Jacobi map

$$P \mapsto [P] - [P_0]$$

The genus of the curve is defined as  $g = \dim_k(H^0(C, \Omega_C))$  where  $\Omega_C$  is the sheaf of Kahler differentials of  $C$  and this dimension is finite due to projectivity of  $C$  by Serre’s theorem. We suppose  $C$  has genus  $g \geq 2$ . Then the following question was asked by Mazur in [Maz86] which we now state as a theorem

**Theorem 1.1.1.** *Let  $C$  be a smooth curve defined over  $k$  of genus  $g \geq 2$ . Then there exists a constant  $c(g) \geq 1$  depending only on  $g$  such that*

$$\#((C(k) - P_0) \cap \Gamma) \leq c(g)^{1+r}$$

where  $\Gamma$  is a subgroup of  $\text{Jac}(C)(k)$  of finite rank and  $r = \text{rk}(\Gamma)$

A specialisation argument as in [Mas89] allows us to reduce the above problem to the case when  $k = \overline{\mathbb{Q}}$ . With this reduction, this problem is completely solved by the works in the paper [DGH20] by Dimitrov, Gao and Habegger and the paper [Küh21] by Lars Kuhne. More precisely in [DGH20] this theorem is proved in combination of a height inequality by Gao and Habegger originating in [GH19] together with a result on non-degeneracy of the image of the Faltings-Zhang map due to Gao in [Gao20]. The curves are looked at as fibers of a smooth, flat family over the moduli space of smooth, projective curves embedded inside an abelian scheme which is the relative Jacobian. However this tackles the problem only for fibers over a point in the parameter space which have height above a fixed absolute constant. Finally in [Küh21] the cases for curves over points below this fixed modular height is tackled, hence finally giving the uniform bound for all curves. Apriori, the bound proven in [DGH20] has the constant  $c$  depending both on  $g$  and the degree of the number field over which  $C$  is defined and in the form stated in Theorem 1.1.1 there is no dependence on the degree. However this bound is most certainly dependent on the degree and hence this dependence must be encoded in the exponent  $r$ . In the next paragraph we give a brief history of the evolution of the finiteness problem of rational points on smooth curves.

We divide the evolution of the whole problem into three parts:

**Finiteness:** Mordell conjectured that for a number field  $k$  and a smooth curve  $C$  defined over  $k$  of genus  $g \geq 2$ ,  $C$  has only finitely many  $k$ -rational points *i.e.*  $\#C(k) < \infty$ . Faltings proved this conjecture famously in [Fal83] for which he was awarded the Fields medal. A new proof using linear combination of divisors and Roth’s theorem was given by Vojta in [Voj91] which was simplified by Faltings in [Fal91] and further simplified by Bombieri in [Bom90].

**Bounds:** Once the question of finiteness of  $C(k)$  had been resolved, a natural question was to bound its cardinality. The first effective bounds on this cardinality followed from Bombieri’s simplified proof in [Bom90]. Silverman proved an uniform bound of the cardinality when the curves vary over a family of twists of a given curve in [Sil93]. The Bogomolov conjecture proved by Ullmo [Ull98] and Zhang [Zha98] gives uniform bounds for  $\#((C(k) - P_0) \cap \Gamma)$  when  $k = \overline{\mathbb{Q}}$  but the bound is dependent on  $C$  and hence not ”uniform”. An explicit bound for  $\#((C(k) - P_0) \cap \Gamma)$  when  $k = \overline{\mathbb{Q}}$  was obtained by Rémond in [Rémo00]. However apart from  $\text{rk}(\Gamma)$  and  $g$ , his bound also depended on a suitable height of the Jacobian of  $C$ .

**Results towards uniformity;** Let us turn to previous results towards Theorem 1.1.1 in the direction of rational points, *i.e.* the bound  $\#C(k) \leq c(g, [k : \mathbb{Q}])^{1+\text{rkJac}(C)(k)}$  for  $k$  a number field. Based on the method of Vojta, David–Philippon [DP07] proved this bound if  $\text{Jac}(C)$  is contained in a power of an elliptic curve, and David–Nakamaye–Philippon proved this bound for some families of curves [Paz15]. More recently, Alpoge [Alp18] proved that the average number of rational points on a curve of genus 2 with a marked Weierstrass point is bounded. Pazuki [Paz21] showed that a suitable version of the far-reaching Lang–Silverman conjecture implies the desired bound. For more developements, we refer to the complete and well written introduction by Gao in [Gao20]

## 1.2 Structure of the thesis

We give a short overview of how the thesis is structured. Our main references for the work are the three papers [DGH19], [DGH20] and [GH19]. In the first chapter we treat the Silverman-Tate height inequality. In earlier works of Silverman, he proved this inequality under a flatness assumption of the compactified structure morphism of the abelian scheme. In the appendix of the paper [DGH20], the authors get rid of this flatness assumption by replacing Weil divisors by Cartier divisors. However, their use of Hironaka’s resolution of singularities makes the proof effective only for characteristic 0 global fields. In our work we replace Hironaka by blow-ups to make our proof effective for any global field (also positive characteristics) and also without the initial flatness assumption.

In the second chapter, we treat the central height inequality due to Gao and Habegger (Theorem 3.3.1) which was obtained by the above authors in [GH19]. The proof roughly consists of proving two bounds of opposite nature on intersection numbers of certain appropriately chosen line-bundles and then using Theorem of Siu to assert bigness of the difference of these bundles. The inequality then follows outside the support of a chosen non-zero global section of this difference. The proof we try to present here is a mixture of the proofs found in [DGH20] and [GH19].

In the final chapter, we pass to the case where our family is just a pencil *i.e.*  $S$  is a curve which is the setting of the earlier work [DGH19]. After this we combine the Silverman-Tate height inequality and the Gao-Habegger inequality to obtain a lower bound on the fiberwise Neron-Tate height in terms of the height of the parameter space outside naturally defined ’bad parts’ of the Faltings-Zhang image of the product of the family of curves. This together with certain elementary lemmas, a bit of Euclidean geometry and Mumford-Vojta’s work on bounding large points finally give us the desired bound as in Theorem 4.1.1.

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## Chapter 2

# Silverman Tate Height Inequality

### 2.1 Introduction

#### 2.1.1 A short history

Let  $\mathcal{A}$  be a family of abelian varieties parametrized by a normal, irreducible and quasi-projective variety  $S$  over a field  $k$  with Product Formula. Consider the compactified family  $\overline{\mathcal{A}} \rightarrow \overline{S}$ . There are three kinds of height functions associated to such a family: the Weil heights  $h_{\overline{\mathcal{A}}}$  and  $h_{\overline{S}}$  on the family  $\overline{\mathcal{A}}$  and on the base  $\overline{S}$  respectively, and the Néron-Tate height  $h_s$  on each fiber of the family  $\mathcal{A} \rightarrow S$ . It is natural to ask how these heights are related.

In 1983, Silverman gave an answer to this question: he established and proved an inequality [Sil83, Theorem A], which will be called the Height Inequality in this paper, in the situation where the family  $\overline{\mathcal{A}} \rightarrow \overline{S}$  is flat and  $\overline{S}$  is smooth over a field satisfying the Product formula. In the same year Tate improved a consequence of the Height Inequality, namely [Sil83, Theorem B] concerning the limit behaviour of the heights involved in the situation where  $\overline{\mathcal{A}} \rightarrow \overline{S}$  is an elliptic surface ([Sil83, §7]) and the parameter space is a curve.

When the base space is a smooth curve, the compactified structure morphism is always dominant and hence flat. However the same is not true for parameter spaces of arbitrary dimension. In the recent paper [DGH20, Appendix A] the authors proved the Height Inequality without flatness assumption on the family  $\overline{\mathcal{A}} \rightarrow \overline{S}$  but  $S$  is assumed to be regular. They use it (combined with other more sophisticated methods) to prove an old conjecture of Mazur on uniformity in the Mordell-Lang conjecture. In their proof without flatness, the authors use Hironaka's famous theorem on resolution of singularities. As Hironaka's theorem is unknown in fields of arbitrary characteristic, this makes the proof in [DGH20] only true for fields of characteristic 0.

In this article we replace the arguments involving Hironaka's theorem with simple blow-up techniques and slightly modifying the arguments, we obtain a different proof. So in totality we are able to completely get rid of the flatness assumption over fields of arbitrary characteristic satisfying the Product formula and the base space  $S$  can be assumed to be only normal.

Lastly we would like to talk about two applications of the Silverman-Tate height inequality. As mentioned earlier, this inequality is used together with other sophisticated methods to prove Uniformity in Mordell-Lang in the recent paper [DGH20]. Another application of the Silverman Tate inequality is in the proof of the Geometric Bogomolov Conjecture for the function field of a curve defined over  $\mathbb{Q}$  in the paper [GH19] by Gao and Habegger. This paper just like the previous one uses other sophisticated methods as well which might not generalise to positive characteristic.

### 2.1.2 Description of the Height Inequality

All our varieties (i.e. reduced separated schemes which are of finite type over some field) are defined over some field  $k$  with Product Formula and hence we can define the notions of height (cf. [BG06, §1.5] or [Lan83, Chapter IV]). We consider a normal, irreducible and quasi-projective variety  $S$  as our base space. Over that we have the abelian scheme  $\mathcal{A}$  which is our main object of interest with the structure morphism

$$\pi: \mathcal{A} \rightarrow S.$$

On our abelian scheme, we have the "multiplication by 2" morphism

$$[2]: \mathcal{A} \rightarrow \mathcal{A}$$

over  $S$ . Now we want to "compactify" the base scheme. Let  $S$  be considered to be immersed in some projective space  $\mathbb{P}_k^n$  and let us denote by  $\overline{S}$  the projective closure of  $S$ . Note that we do not have necessarily that  $\overline{S}$  is regular but just a projective, irreducible variety.

Like in [DGH20], Appendix A(p. 31) we will also assume that we are presented with a closed immersion

$$\mathcal{A} \hookrightarrow \mathbb{P}_k^m \times S$$

Then note that as  $S$  is immersed inside  $\mathbb{P}_k^n$ , altogether we have an immersion

$$\mathcal{A} \hookrightarrow \mathbb{P}_k^m \times \mathbb{P}_k^n$$

and let us call by  $\overline{\mathcal{A}}$  the Zariski closure of  $\mathcal{A}$  under this immersion in the multi-projective space.

Next we come to the consideration of the Weil Heights that is our main object of interest. We have the canonical very ample line-bundle  $\mathcal{O}(1, 1)$  on the multiprojective space  $\mathbb{P}_k^{m,n} = \mathbb{P}_k^m \times \mathbb{P}_k^n$ . We denote as

$$\mathcal{L} = \mathcal{O}(1, 1)|_{\overline{\mathcal{A}}}$$

and we chose the representative of the height class  $h_{\mathcal{L}}$  of the line bundle  $\mathcal{L}$  given by its embedding in  $\mathbb{P}_k^{m,n}$ . Less abstractly if we denote a point of  $\overline{\mathcal{A}}(\overline{k}) \subseteq \mathbb{P}_k^{m,n}$  as  $(P, s)$  where  $P \in \mathbb{P}_k^m(\overline{k})$  and  $s \in \overline{S}(\overline{k})$ , then we have

$$h_{\mathcal{L}}(P, s) = h(P) + h_{\overline{S}}(s)$$

where  $h$  is the canonical Weil Height of  $\mathbb{P}_k^m$  and  $h_{\overline{S}}$  is the Weil Height on  $\overline{S}$  coming from its embedding in  $\mathbb{P}_k^n$ . Additionally if  $\eta$  is the generic point of  $S$  and  $\mathcal{A}_{\eta}$  is the generic fiber of  $\mathcal{A}$ , then we can restrict the line bundle  $\mathcal{L}$  to  $\mathcal{A}_{\eta}$  (call it  $\mathcal{L}_{\eta}$ ). We assume that  $\mathcal{L}_{\eta}$  is an even ample line bundle on the abelian variety  $\mathcal{A}_{\eta}$  defined over the function field  $\kappa(S) = \kappa(\eta)$ . Then note that in particular from the theorem of the cube ([BG06, Prop. 8.7.1]), we have that

$$[2]^* \mathcal{L}_{\eta} = \mathcal{L}_{\eta}^{\otimes 4}.$$

With this language we are in a position to state the theorem that we wish to prove:

**Theorem 2.1.1.** *There exists a constant  $c > 0$  such that for all  $P \in \mathcal{A}(\overline{k})$  we have*

$$|h_{\mathcal{L}}([2]P) - 4 \cdot h_{\mathcal{L}}(P)| \leq c \cdot \max\{1, h_{\overline{S}}(\pi(P))\}.$$

Note that this immediately implies the Height Inequality, see Theorem 2.1.2 below.

For the reader's convenience we recall the construction of the fiberwise Néron-Tate height here. Let  $\eta \in S$  be the generic point of  $S$  and consider the line bundle  $\mathcal{L}_{\eta}$  on the generic fiber. This is an ample line bundle, because it gives rise to a closed immersion  $\mathcal{A}_{\eta} \hookrightarrow \mathbb{P}_{\kappa(\eta)}^n$ .

The fiber  $\mathcal{A}_s$  over closed points  $s \in \mathcal{A}$  is an abelian variety over  $\kappa(s)$ . Consider the Néron-Tate heights  $h_s$  on  $\mathcal{A}_s$  induced by the line bundle

$$\mathcal{L}_s := \mathcal{L}|_{\mathcal{A}_s}.$$

Note that every  $\mathcal{L}_s$  is ample. Moreover, each of them is an even line bundle. Indeed, the symmetry of  $\mathcal{L}_\eta$  implies that

$$([-1]^* \mathcal{L} \otimes \mathcal{L}^{\otimes(-1)})|_{\mathcal{A}_\eta} = [-1]^* \mathcal{L}_\eta \otimes (\mathcal{L}_\eta)^{\otimes(-1)} \cong \mathcal{O}_{\mathcal{A}_\eta}$$

is the trivial line bundle on  $\mathcal{A}_\eta$ . Then, by [Gro67, Corollaire 21.4.13], we can find some line bundle  $\mathcal{M}$  on  $S$  such that  $\pi^* \mathcal{M} \cong [-1]^* \mathcal{L} \otimes \mathcal{L}^{\otimes(-1)}$ . On considering the pullback square of the fiber  $\mathcal{A}_s$ , we easily see that

$$[-1]^* \mathcal{L}|_{\mathcal{A}_s} \cong \mathcal{L}|_{\mathcal{A}_s} = \mathcal{L}_s.$$

Thus we can define the fiberwise Néron-Tate height in a uniform way:

$$\hat{h}_{\mathcal{A}}(P) = \lim_{N \rightarrow \infty} \frac{h_s([N]P)}{N^2} \quad \text{for all closed points } P \in \mathcal{A}_s.$$

Note that in any case the height  $h_{\mathcal{A}}$  is non-negative by ampleness.

**Theorem 2.1.2.** *Keeping the notations so far. There exists a constant  $c > 0$  such that for all  $P \in \mathcal{A}(\bar{k})$  we have*

$$\left| \hat{h}_{\mathcal{A}}(P) - h(P) \right| \leq c \cdot \max\{1, h_{\bar{S}}(\pi(P))\}.$$

*Proof.* (Compare with the proof of the existence of the Néron-Tate height: [BG06, Lemma 9.2.4]) Pick any  $P \in \mathcal{A}(\bar{k})$ . Let  $\ell \geq m \geq 0$  be integers. Using telescope-sum and triangle inequality, we find

$$\left| \frac{h([2^\ell]P)}{4^\ell} - \frac{h([2^m]P)}{4^m} \right| \leq \sum_{q=m}^{\ell-1} \left| \frac{h([2^{q+1}]P) - 4h([2^q]P)}{4^{q+1}} \right|.$$

Applying Theorem 2.1.1 to the summands, we find a constant  $c_1 > 0$  such that the right hand side is bounded by

$$c_1 \lambda \sum_{q=m}^{\ell-1} \frac{1}{4^{q+1}} \leq c_1 \lambda \frac{1}{4^m}, \quad \lambda := \max\{1, h_{\bar{S}}(\pi(P))\}.$$

This means that  $(\frac{h([2^\ell]P)}{4^\ell})_{\ell \geq 1}$  is a Cauchy sequence with limit  $\hat{h}_{\mathcal{A}}(P)$ , simply by definition of the Néron-Tate height. By taking  $m = 0$  and considering  $\ell \rightarrow \infty$  we obtain the Height Inequality 2.1.2.  $\square$

## 2.2 Extension of [2]

In the previous section we have considered the duplication morphism. However for using the height machines, we need that they are defined over the compactification  $\overline{\mathcal{A}}$  constructed in sub-section 1.2. It is not always possible to extend it directly but we bypass this problem by going to the products via the graph morphism and then identifying the image sub-scheme with  $\mathcal{A}$  using the separatedness of  $\mathcal{A}$ . We consider the graph morphism corresponding to [2], namely:

$$\mathcal{A} \xrightarrow{\Gamma_{[2]}} \mathcal{A} \times_S \mathcal{A} \hookrightarrow \overline{\mathcal{A}} \times_{\bar{S}} \overline{\mathcal{A}}.$$

Note that as  $\mathcal{A}$  is separated, the first morphism is a closed immersion while the second one is an open immersion. We denote by  $\overline{\mathcal{A}}'$  the Zariski closure of  $\mathcal{A}$  under this immersion. Then note that due to  $\Gamma_{[2]}$  being a closed immersion, we can identify  $\mathcal{A}$  as an open dense subscheme  $\overline{\mathcal{A}}'$ . Moreover we have the two canonical projections  $p_1$  and  $p_2$  on the product  $\overline{\mathcal{A}} \times_{\bar{S}} \overline{\mathcal{A}}$  and note that by construction

$$p_1|_{\mathcal{A}} = \text{id}_{\mathcal{A}} \text{ and } p_2|_{\mathcal{A}} = [2].$$



To summarise, we have the following commutative diagram:

$$\begin{array}{ccccc} \overline{\mathcal{A}} & \xleftarrow{p_1} & \overline{\mathcal{A}}' & \xrightarrow{p_2} & \overline{\mathcal{A}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{A} & \xleftarrow{\text{id}} & \mathcal{A} & \xrightarrow{[2]} & \mathcal{A} \end{array}$$

At this point we introduce the line bundle on  $\overline{\mathcal{A}}'$  which we are going to work with. We define

$$\mathcal{F} = p_2^* \mathcal{L} \otimes (p_1^* \mathcal{L})^{\otimes -4}$$

which is an element of  $\text{Pic}(\overline{\mathcal{A}}')$ . Then note that we assume that the generic fiber  $\mathcal{L}_\eta$  is even and hence by the theorem of cubes we have that the restriction  $\mathcal{F}|_{\mathcal{A}_\eta}$  is trivial. Hence by [Gro67] Corollaire 21.4.13, we get that there is a line bundle  $\mathcal{M}$  on  $S$  such that

$$\mathcal{F}|_{\mathcal{A}} = \pi^* \mathcal{M}.$$

We wish to extend this line bundle  $\mathcal{M}$  to  $\overline{S}$ . We work towards this goal in the next section.

## 2.3 Extension of line bundle on the base

In the previous section, we came upon the existence of a line bundle  $\mathcal{M}$  on  $S$ . We wish that  $\mathcal{M}$  comes from a line bundle  $\mathcal{M}'$  from above on  $\overline{S}$ . We can not wish to have this a-priori because we do not know that  $\overline{S}$  is regular. However our idea here is to blow up  $\overline{S}$  at certain closed sub-schemes lying outside  $S$  so that we do not disturb  $S$  and pass to a new  $\overline{S}$  where this extension is possible.

We begin with a Lemma from [Har77].

**Lemma 2.3.1.** *Let us have a coherent sheaf  $\mathcal{F}$  on an open sub-scheme  $U$  of a noetherian scheme  $X$ . Also suppose  $\mathcal{G}$  is a quasi-coherent sheaf on  $X$  such that  $\mathcal{F} \subseteq \mathcal{G}|_U$ . Then there exists a coherent sheaf  $\mathcal{F}' \subseteq \mathcal{G}$  on  $X$  such that  $\mathcal{F}'|_U = \mathcal{F}$ .*

*Proof.* We refer to Exercise II.5.15 in [Har77]. □

Let us go back to the setting of our problem. With that we have the following:

**Lemma 2.3.2.** *If we have that  $\mathcal{M}$  is a coherent ideal sheaf on  $S$ , then we can find a coherent ideal sheaf  $\mathcal{M}'$  on  $\overline{S}$  such that  $\mathcal{M}'|_S = \mathcal{M}$ .*

*Proof.* The proof is a direct application of Lemma 2.3.1 with  $\mathcal{M} = \mathcal{F}$ ,  $X = \overline{S}$ ,  $U = S$  and  $\mathcal{G} = \mathcal{O}_{\overline{S}}$  (the structure sheaf of  $\overline{S}$ ). Note that the condition that  $\mathcal{M}$  is an ideal sheaf is necessary so that we can choose  $\mathcal{G}$  to be the structure sheaf itself. □

Next we prove a lemma which helps us split our Cartier divisor into effective parts using a result from Liu's book.

**Lemma 2.3.3.** *Let  $S$  be a quasi-projective variety over  $k$  and  $\mathcal{M}$  be an invertible sheaf on  $S$ . Then we can find a Cartier divisor  $D$  and effective Cartier divisors  $C$  and  $E$  in  $S$  such that*

$$D = C - E \text{ and } \mathcal{O}(D) \cong \mathcal{M}$$

*Proof.* We begin by noting that when  $S$  is regular, the lemma is very easy to deduce. When  $S$  is regular, Cartier divisors are equivalent to Weil divisors and every Weil divisor can be written as the difference of two effective Weil divisors almost by definition.

Now for the general case, when  $S$  is a quasi-projective variety we choose a Cartier divisor  $D$  on  $S$  such

that  $O(D) \cong \mathcal{M}$ . Then we can apply Lemma 7.1.31 and Prop 7.1.32 of [Liu06] to deduce that there exist effective Cartier divisors  $C$  and  $E$  such that  $D$  is linearly equivalent to  $C - E$ , or in other words

$$\mathcal{M} \cong O(D) \cong O(C - E)$$

Hence by changing  $D$  to  $C - E$  we get our claim.  $\square$

Now let us consider a Cartier divisor  $D$  on  $S$  such that  $O(D) \cong \mathcal{M}$ . Then by Lemma 2.3.3, we can choose to write  $D = C - E$  where  $C$  and  $E$  are effective divisors on  $S$ . Then we see from the construction of  $O(\cdot)$  that both  $O(-C)$  and  $O(-E)$  are invertible ideal sheaves on  $S$ . Our idea is to extend these ideal sheaves with the help of Lemma 2.3.2 and blow up in their supports one by one. More formally, we have the following lemma and its proof explains the above procedure:

**Lemma 2.3.4.** *There is a projective irreducible variety  $S'$  and a birational morphism*

$$\gamma: S' \rightarrow \bar{S}$$

*such that  $\gamma|_{\gamma^{-1}(S)}$  is an isomorphism. Moreover if we identify  $S$  as an open sub-scheme of  $S'$  via  $\gamma|_{\gamma^{-1}(S)}$  then we have a line bundle  $\mathcal{M}'$  on  $S'$  such that  $\mathcal{M}'|_S = \mathcal{M}$ .*

*Proof.* We consider the notations from the paragraph above this lemma. As explained in Lemma 2.3.3 above, using Lemma 2.3.2 we find a coherent ideal sheaf  $\mathcal{C}$  on  $\bar{S}$  such that  $\mathcal{C}|_S = O(-C)$ . Now we consider the blow-up of  $\bar{S}$

$$b_1: S_1 \rightarrow \bar{S}$$

corresponding to the closed sub-scheme of the ideal sheaf  $\mathcal{C}$ . Note  $b_1$  is an isomorphism when restricted to  $S$  because the restriction of the ideal sheaf  $\mathcal{C}$  to  $S$  is  $O(-C)$  which is an invertible sheaf and blow-ups along invertible ideal sheaves are isomorphisms. Hence we can consider  $S$  as an open sub-scheme of  $S_1$  by identification via  $b_1$ . But now we have that the inverse image ideal sheaf  $\bar{\mathcal{C}} = b_1^{-1}\mathcal{C}$  is an effective Cartier divisor by property of blow-ups (it is just the exceptional divisor corresponding to this blow up  $b_1$ ). Note also that we have

$$O(\bar{\mathcal{C}})|_S = O(-C) \quad (1)$$

Now via identification we view  $S$  as an open sub-scheme of  $S_1$  and  $E$  is an effective Cartier divisor on  $S$ . Thus we once again have that  $O(-E)$  is an invertible ideal sheaf and we can once again apply Lemma 2.3.2 to obtain a coherent ideal sheaf extension  $\mathcal{E}$  on  $S_1$  such that  $\mathcal{E}|_S = O(-E)$ . Using a similar argument as before we obtain that we can safely blow-up in the associated closed sub-scheme without disturbing  $S$ . More formally we consider the blow-up

$$b_2: S' \rightarrow S_1$$

along the closed sub-scheme corresponding to the coherent ideal sheaf  $\mathcal{E}$ . Then  $b_2$  is an isomorphism over  $S$  because the restriction of  $\mathcal{E}$  to  $S$  is the invertible ideal sheaf  $O(-E)$ . At first note that as pullbacks of Cartier divisors are again Cartier divisors under dominant maps of integral schemes (that is pullback is well defined), we get that  $\tilde{\mathcal{C}} = b_2^*\bar{\mathcal{C}}$  is an invertible sheaf on  $S'$  with  $\tilde{\mathcal{C}}|_S = O(-C)$ . Moreover, let us consider the effective Cartier divisor  $\tilde{\mathcal{E}}$  which is the exceptional divisor corresponding to the blow up  $b_2$ . Then note that arguing like above and using identifications of  $S$  via  $b_2$ , we get that

$$O(\tilde{\mathcal{E}})|_S = O(-E) \quad (2)$$

Now we are done because we take  $\mathcal{M}' = O(\tilde{\mathcal{E}}) - O(\tilde{\mathcal{C}}) \in \text{Pic}(S')$ . Then from (1) and (2), we have

$$\mathcal{M}'|_S = O(\tilde{\mathcal{E}})|_S - O(\tilde{\mathcal{C}})|_S = O(-E) - O(-C) = O(C - E) = O(D) = \mathcal{M}$$

This completes the proof by taking  $\gamma = b_1 \circ b_2$ .  $\square$

*Remark.* As we mentioned before, our goal is to pass to blow-ups where  $\mathcal{M}$  can be extended from  $S$  to the compactification of  $S$ . Using Lemma 2.3.4 we can find such blow-ups (for example  $S'$ ) and without loss of generality we will replace the old  $\bar{S}$  by  $S'$ . Now we can assume that  $\mathcal{M}$  has an extension to  $\bar{S}$ . Note that this change of base space does not effect the height of the base-space precisely because the blow-ups do not touch  $S$  (see Corollary 2.5.3 in Section 5).

The above procedure could be easily done by application of Hironaka on  $\bar{S}$  and passing to a regular compactification. However Hironaka is unknown for positive characteristic but blow-ups work just fine. This is one of the modifications that we make from the proof of Theorem 2.1.1 in Appendix A of [DGH20].

## 2.4 Height Inequality Using Blow Ups

In this section we begin by talking about the ideal sheaf of denominators of a Cartier divisor. Let us have a Cartier divisor given by the local equations  $D = (U_i, f_i)$  where  $U_i = \text{Spec}(A_i)$  are affine and  $f_i \in \text{Frac}(A_i)^\times$  are the local equations of the Cartier divisor. Then we define the ideal sheaf of denominators  $\mathcal{I}(D)$  to be

$$\mathcal{I}(U_i) = \{b \in A_i \mid b \cdot f_i \in A_i\}$$

Note that clearly each  $\mathcal{I}(U_i)$  is an ideal of  $A_i$  and they glue together because  $f_i$  and  $f_j$  differ by an unit in the intersection  $U_i \cap U_j$ . Hence gluing these piece-wise sections we get a coherent ideal sheaf  $\mathcal{I}$  which we call the ideal sheaf of denominators.

Now let us digress and prove a lemma which will be necessary for us later:

**Lemma 2.4.1.** *Let  $X$  be an integral scheme,  $U$  is an open sub-scheme of  $X$  and  $\mathcal{L} \in \text{Pic}(X)$  such that  $\mathcal{L}|_U$  is trivial. Then we can take a Cartier divisor  $D$  on  $X$  with  $O(D) \cong \mathcal{L}$  such that  $D|_U$  is the trivial cartier divisor on  $U$  and hence  $\mathcal{I}(D)|_U$  is trivial on  $U$ .*

*Proof.* From the construction of  $\mathcal{I}(D)$  it is clear to see that  $\mathcal{I}(D)|_U = \mathcal{I}(D|_U)$ . Hence to show the claim it is enough to choose a cartier divisor corresponding to  $\mathcal{L}$  such that its restriction to  $U$  is trivial. We begin by choosing any Cartier representative  $D$  of  $\mathcal{L}$ . Then as we have  $\mathcal{L}|_U$  is trivial, we get that  $D|_U$  is principal *i.e.* there is an  $f \in \kappa(X)$  such that  $D|_U = (U, f)$ . Then by replacing  $D$  with  $D' = D - (X, f)$  we get that  $D'|_U = (U, 1)$  which is the trivial Cartier divisor. However as we only change  $D$  by a principal Cartier divisor, we stay in the same isomorphism class of  $\mathcal{L}$ . Hence we get that  $\mathcal{L} \cong O(D')$  and  $\mathcal{I}(D')|_U$  along with  $D'|_U$  is trivial as wanted.  $\square$

We get a canonical structure map  $\bar{\mathcal{A}}' \rightarrow \bar{S}$  which we also denote by  $\pi$  for the sake of clarity. Now as explained in the introduction (but we repeat it for clarity and to re-iterate the importance of section 3), we know that  $\mathcal{F}|_{\mathcal{A}_\eta}$  is trivial and hence according to [Gro67], Corollaire 21.4.13, we get that

$$\mathcal{F}|_{\mathcal{A}} \cong \pi^* \mathcal{M}$$

for some line bundle  $\mathcal{M}$  on  $S$ . This is where we will use the results of section 3. We can find a line-bundle  $\mathcal{M}'$  on  $\bar{S}$  such that  $\mathcal{M}'|_S = \mathcal{M}$ . In that case let us denote by

$$\mathcal{F}' = \mathcal{F} \otimes (\pi^* \mathcal{M}')^{-1}.$$

In that case we can see that  $\mathcal{F}'|_{\mathcal{A}}$  is trivial. Hence by Lemma 2.4.1 above we can choose a Cartier divisor  $D$  on  $X = \bar{\mathcal{A}}'$  such that  $\mathcal{I}(D)|_{\mathcal{A}}$  is trivial and  $O(D) \cong \mathcal{F}'$ . Note also that by Lemma 2.4.1, we can have  $D|_{\mathcal{A}} = (\mathcal{A}, 1)$ , the trivial cartier divisor. Let us denote by  $\mathcal{S}$  the closed sub-scheme defined by the ideal sheaf  $\mathcal{I}(D)$ . Then as  $D$  restricted to  $\mathcal{A}$  is trivial, we get that  $\mathcal{S}$  lies outside  $\mathcal{A}$ . Let us blow up  $\bar{\mathcal{A}}'$  along  $\mathcal{S}$ . Note that as  $\mathcal{S}$  lies outside  $\mathcal{A}$ , this blow up

$$\gamma_1: \tilde{\mathcal{A}} \rightarrow \bar{\mathcal{A}}'$$

is an isomorphism over  $\mathcal{A}$  and hence we aren't changing our main object of interest (which was main point of the lemma above). Now as explained in [Ful98], Chapter 2, Case 3 in the proof of the Commutativity Theorem 2.4 that

$$\gamma_1^* D = C - E$$

for some effective divisor  $C$  in  $\tilde{\mathcal{A}}$  and the exceptional divisor  $E$  and we also have  $C \cup E \subseteq \gamma_1^{-1}(\text{Supp}(D))$ . For the sake of clarity, we re-iterate that we have the following diagram:

$$\begin{array}{ccc} \tilde{\mathcal{A}} & \xrightarrow{\gamma_1} & \overline{\mathcal{A}}' \\ & & \downarrow \pi \\ & & \overline{S} \end{array}$$

Note now that as  $D|_{\mathcal{A}}$  is trivial, we have that  $\text{Supp}(D) \subseteq \overline{\mathcal{A}}' - \mathcal{A}$ . Hence we get that the closed subset  $C \cup E$  lies outside  $\mathcal{A}$  if we identify  $\mathcal{A}$  with its inverse image via  $\gamma_1$ . Consider the scheme-theoretic union  $C \cup' E$  in  $\tilde{\mathcal{A}}$  which gives a scheme structure on the closed subset  $C \cup E$  (not necessarily reduced). Next consider the scheme-theoretic image of  $C \cup' E$  under the morphism  $\pi \circ \gamma_1$  and call it  $\mathcal{Y}$ . As the underlying set  $C \cup E$  of the closed sub-scheme  $C \cup' E$  is contained in  $\tilde{\mathcal{A}} - \mathcal{A}$ , we get that its scheme theoretic image  $\mathcal{Y}$  is a closed sub-scheme whose underlying set is contained in  $\overline{S} - S$ .

Next we consider the scheme-theoretic inverse image  $(\pi \circ \gamma_1)^{-1}(\mathcal{Y})$  in  $\tilde{\mathcal{A}}$  and call it  $\tilde{\mathcal{Y}}$  and finally consider the blowup of  $\tilde{\mathcal{A}}$  along  $\tilde{\mathcal{Y}}$

$$\gamma_2: \mathcal{A}' \rightarrow \tilde{\mathcal{A}}$$

Note here that as  $\mathcal{Y}$  lies outside  $S$ ,  $\tilde{\mathcal{Y}}$  lies outside  $\mathcal{A}$ . Then note that the scheme theoretic inverse image of  $\mathcal{Y}$  under  $\tilde{\pi} = \pi \circ (\gamma_1 \circ \gamma_2)$  is a Cartier closed sub-scheme and let us call it  $\mathcal{Y}_*$ . Now we consider the change of the base space *i.e.* we consider the blow up

$$\gamma_*: S' \rightarrow \overline{S}$$

along the closed sub-scheme  $\mathcal{Y}$  of  $\overline{S}$ . As  $\mathcal{Y} \subseteq \overline{S} - S$  we deduce that  $\gamma_*$  is an isomorphism over  $S$ . Then note that as the scheme theoretic inverse image  $\mathcal{Y}_*$  of  $\mathcal{Y}$  under the morphism  $\tilde{\pi}$  is a Cartier closed sub-scheme, we get the existence of a dotted row making the following commutative diagram by the universal property of blow-ups:

$$\begin{array}{ccccc} \mathcal{A}' & \xrightarrow{\gamma_2} & \tilde{\mathcal{A}} & \xrightarrow{\gamma_1} & \overline{\mathcal{A}}' \\ & \searrow \pi_* & & & \downarrow \pi \\ & & S' & \xrightarrow{\gamma_*} & \overline{S} \end{array}$$

Now we make a comment here that in the above diagram, each of the horizontal morphisms in the top row are isomorphisms over  $\mathcal{A}$ . For  $\gamma_1$  it is because the sub-scheme of denominators lies outside  $\mathcal{A}$  by Lemma 2.4.1 and for  $\gamma_2$  it is because  $\tilde{\mathcal{Y}}$  is contained in  $\tilde{\mathcal{A}} - \mathcal{A}$  (set theoretic difference), being the inverse image of a subset of  $\overline{S} - S$ . Hence we can view  $\mathcal{A}$  as an open dense subset of each of them. The morphism  $\gamma_*$  is also an isomorphism over  $S$  as  $\mathcal{Y} \subseteq \overline{S} - S$ .

We use the same letter for a Cartier closed sub-scheme and its associated effective divisor. By the property of blow-ups, we get that

$$\mathcal{Y}' = \gamma_*^{-1}(\mathcal{Y})$$

is effective cartier divisor. It is the exceptional divisor corresponding to the blow-up  $\gamma_*$ . Let us call

$$\Omega = \mathcal{Y}'$$

which is an effective divisor in  $S'$  for convenience. We claim that

$$\pi_*^*(\Omega) \pm \gamma_2^*(C - E)$$

is effective. Note it is enough to show that both

$$\gamma_2^* C \leq \pi_*^*(\mathcal{Y}') \text{ and } \gamma_2^* E \leq \pi_*^*(\mathcal{Y}')$$

The two conditions can be reformulated in the following way by [GW10] Cor.11.49):

$$\gamma_2^{-1}(C) \hookrightarrow \pi_*^{-1}(\mathcal{Y}') \text{ and } \gamma_2^{-1}(E) \hookrightarrow \pi_*^{-1}(\mathcal{Y}').$$

where the inverse-images are scheme-theoretic and the hooked-arrows mean that there exist such closed immersions. But these following conditions easily follow from diagram chasing and the construction of  $\mathcal{Y}$ . More precisely, first remember that a subscheme is always a closed subscheme of the scheme-theoretic inverse image of its scheme-theoretic image. In particular there exists a closed immersion

$$C \cup' E \hookrightarrow (\pi \circ \gamma_1)^{-1}(\mathcal{Y})$$

where the R.H.S is the scheme theoretic inverse image. Hence we obtain that there exist closed immersions

$$C \hookrightarrow (\pi \circ \gamma_1)^{-1}(\mathcal{Y}) \text{ and } E \hookrightarrow (\pi \circ \gamma_1)^{-1}(\mathcal{Y})$$

as  $C$  and  $E$  have canonical closed-immersions into their scheme-theoretic union. As closed immersions are stable under base change we get that there are closed immersions

$$\gamma_2^{-1}(C) \hookrightarrow \tilde{\pi}^{-1}(\mathcal{Y}) \text{ and } \gamma_2^{-1}(E) \hookrightarrow \tilde{\pi}^{-1}(\mathcal{Y})$$

where  $\tilde{\pi} = \pi \circ \gamma_1 \circ \gamma_2$ . But now from the commutativity of the diagram above, we easily get that  $\tilde{\pi}^{-1}(\mathcal{Y}) = \pi_*^{-1}(\mathcal{Y}')$  and the claim easily follows.

The effectiveness of the divisors  $\Lambda_{\pm} = \pi_*^*(\Omega) \pm \gamma_2^*(C - E)$  proves that outside their supports, the height is bounded by a constant. Rigorously speaking, this means that there is a constant  $B > 0$  that

$$h_{\Omega}(\pi_*(P)) \pm h_{\mathcal{F}'}(\gamma_1(\gamma_2(P))) \geq B$$

for all  $P \in \mathcal{A}'(k)$  outside the support. Upon further expansion and use of functoriality, we obtain

$$h_{\Omega}(\pi_*(P)) \pm h_{\mathcal{F}}(\gamma_1(\gamma_2(P))) \mp h_{\mathcal{M}'}(\tilde{\pi}(P)) \geq B$$

for all  $P$  outside the support. This in particular means that

$$|h_{\mathcal{F}}(\gamma_1(\gamma_2(P)))| \leq |h_{\Omega}(\pi_*(P))| + |h_{\mathcal{M}'}(\tilde{\pi}(P))| + B. \quad (2.1)$$

Next we want to use inequality (1) on all points of  $\mathcal{A}(\bar{k})$ . Hence we claim that none of the points in  $\mathcal{A}(\bar{k})$  lies in the support of  $\Lambda_{\pm}$ . Note that the support of  $\Lambda_{\pm}$  is given by  $\pi_*^{-1}(\mathcal{Y}') \cup \gamma_2^{-1}(C \cup E) = \pi_*^{-1}(\gamma_*^{-1}(\mathcal{Y})) = \tilde{\pi}^{-1}(\mathcal{Y}) \cup \gamma_2^{-1}(C \cup E) = \tilde{\pi}^{-1}(\mathcal{Y})$ . Hence if  $P \in \mathcal{A}(\bar{k})$  belongs to the supports, this implies that  $\pi_*(P) = \pi(P) \in \mathcal{Y}$ . But clearly  $\mathcal{Y}$  lies outside  $S$  due to our choice of choosing blow ups so as to be isomorphic over  $\mathcal{A}$  in the category of  $S$ -schemes and  $\pi$  lands on  $S$  when restricted to  $\mathcal{A}$ . It is even easier to see  $\mathcal{A}(\bar{k})$  has no intersection with  $\gamma_2^{-1}(C \cup E)$ . Thus we get that none of the points of  $\mathcal{A}(\bar{k})$  lie in the support.

Hence we can use the inequality (1) and identifying points using that  $\gamma_1$  and  $\gamma_2$  are isomorphisms over  $\mathcal{A}$ , we can write simply

$$|h_{([2]^* \mathcal{L} - 4 \cdot \mathcal{L})}(P)| \leq |h_{\Omega}(\pi(P))| + |h_{\mathcal{M}'}(\pi(P))| + B \text{ for all } P \in \mathcal{A}(\bar{k}).$$

From the above inequality using functoriality of Weil Height machines, we get the following inequality:

$$|h_{\mathcal{L}}([2]P) - 4 \cdot h_{\mathcal{L}}(P)| \leq |h_{\Omega}(\pi(P))| + |h_{\mathcal{M}'}(\pi(P))| + B \text{ for all } P \in \mathcal{A}(\bar{k}) \quad (2.2)$$

Note that inequality (2) is almost ready to be the inequality in Theorem 2.1.1 except that in the right hand side we have the terms  $|h_{\mathcal{M}'}|$  and  $|h_{\Omega}|$  coming from line bundles  $\mathcal{M}'$  on the projective varieties  $S'$  and  $\bar{S}$ . Each of the two heights  $h_{\mathcal{M}'}$  and  $h_{\Omega}$  can be bounded by some constant times the canonical Weil heights  $h_{S'}$  and  $h_{\bar{S}}$ . The reason is that the standard Weil heights are given by very ample line bundles into their respective projective embeddings. This will also be made more clear in the next section. Hence to get the final height inequality we have to show that the height  $h_{S'}$  is bounded by a constant times the height  $h_{\bar{S}}$  when we restrict it to  $\mathcal{A}(\bar{k})$ . We achieve this in the next section with the help of a lemma of Lang in [Lan83].

## 2.5 Last Inequality

This section serves as a bridging explanation for some assumptions we have made throughout the article. We have a recurring theme throughout the article where we pass via blow-up to a different compactification but the blow up does not touch certain open sub-schemes (that is an isomorphism over these open sub-schemes) which are typically  $\mathcal{A}$  or  $S$  for us. With help of Lemma 2.5.1 we show that the height functions only differ up to constant multiple factors when restricted to open sub-schemes over which these blow-ups are isomorphisms. On the light of the inequality in Theorem 2.1.1, this then justifies our change of base by application of blow-ups in section 3 and also shows us how to pass from inequality (2) in the previous section to the final inequality in Theorem 2.1.1.

So let

$$\pi : \mathbb{P}^{m'} \supset S' \rightarrow \bar{S} \subset \mathbb{P}^m$$

be a blow up morphism, where  $\bar{S}$  is a projective irreducible variety so that  $S'$  is also projective and irreducible variety. Moreover, we have an open subset  $S \subset \bar{S}$  which is not touched by the blow up, i.e.  $\pi^{-1}(S) \cong S$ . Consider two line bundles  $L_1$  and  $L_2$  on  $S'$ ; let  $h_1, h_2$  be the Weil heights attached to them respectively, while  $h_{\bar{S}}$  (resp.  $h_{S'}$ ) denotes the Weil height on  $\bar{S}$  (resp.  $S'$ ) induced by the standard height on  $\mathbb{P}^m$  (resp.  $\mathbb{P}^{m'}$ ) and the given closed immersion  $\bar{S} \subset \mathbb{P}^m$  (resp.  $S' \subset \mathbb{P}^{m'}$ ). We want to bound  $h_1$  and  $h_2$  in terms of  $h_{\bar{S}}$  (at least on a sub-domain).

For the next two assertions we adapt the convention that  $h_n$  denotes the standard height on  $\mathbb{P}^n$ ,  $n \in \mathbb{N}$ .

**Lemma 2.5.1.** *Let  $V$  be a locally closed irreducible sub-variety of  $\mathbb{P}^n$  and suppose there is a morphism  $f : V \rightarrow \mathbb{P}^\ell$  defined over  $k$  for some  $\ell \in \mathbb{N}$ . Then there exist constants  $c_1, c_2 > 0$  such that*

$$\forall P \in V(\bar{k}) : h_\ell(f(P)) \leq c_1 h_n(P) + c_2.$$

*Proof.* See [Lan83], Prop. 1.7 of Ch.IV. □

**Corollary 2.5.2.** *Let  $V$  be an irreducible variety over  $k$ . Let  $c$  be an ample line bundle and  $L$  any line bundle on  $V$ . Pick a height  $h_c$  associated to  $c$  such that  $h_c \geq 0$  (possible because  $c$  is ample). Then there exist constants  $\gamma_1, \gamma_2 > 0$  such that*

$$\forall P \in V(\bar{k}) : |h_L(P)| \leq \gamma_1 h_c(P) + \gamma_2.$$

*Proof.* (Cf. [Lan83], Prop. 5.4 of Ch.IV) By a theorem from Serre, we can decompose  $L = L_+ \otimes L_-^{\otimes(-1)}$  where  $L_+$  and  $L_-$  are very ample line bundles on  $V$  inducing closed immersions  $V \hookrightarrow \mathbb{P}^{m_\pm}$  for some  $m_\pm \in \mathbb{N}$ . By the Height Machine we have

$$h_L = h_{L_+} - h_{L_-} = h_{m_+} - h_{m_-}$$

as real-valued functions on  $V(\bar{k})$  up to bounded functions.

On the other hand, since we choose  $h_c$  to be non-negative, by the Height Machine we may assume without loss of generality that  $c$  is very ample, inducing a closed immersion  $V \hookrightarrow \mathbb{P}^n$  for some  $n \in \mathbb{N}$ . Note that in this case  $h_c = h_n$  as real-valued functions on  $V(\bar{k})$  up to bounded functions.

Applying Lemma 2.5.1 to the immersions  $V \hookrightarrow \mathbb{P}^{m_\pm}$ , using the additivity of the Height Machine and rearranging the terms resulting from Lemma 2.5.1 (the right hand sides are both bounded by  $\gamma h_c + O(1)$  for some  $\gamma > 0$ ), we are done. □

Let us come back to our beginning situation. We will take  $L_1 = O(\Omega)$  and  $L_2 = \gamma_*^* \mathcal{M}'$  which are line bundles on  $S'$ . (following our notations from previous sections) Applying Corollary 2.5.2 with  $V = S'$ ,  $c = \mathcal{O}_{\mathbb{P}^{m'}}(1)$ , we are able to bound  $h_1 = h_\Omega$  and  $h_2 = h_{\gamma_*^* \mathcal{M}'}$  with respect to  $h_{S'}$ . More precisely, using Corollary 2.5.2 we get positive constants  $r_1, r_2, s_1, s_2$  such that

$$|h_\Omega(P)| \leq r_1 \cdot h_{S'}(P) + s_1 \text{ and } |h_{\mathcal{M}'}(P)| \leq r_2 \cdot h_{S'}(P) + s_2 \text{ for all } P \in S(\bar{k}) \quad (2.3)$$

where we have used the identification of  $\gamma_*^{-1}(S)$  and  $S$  via  $\gamma_*$  and the functoriality of height machine. Now note that  $h_{S'}$  is non-negative. Since the closed points  $P \in \mathcal{A}(\bar{k})$  of our interest are all mapped into  $S(\bar{k}) \subset \bar{S}(\bar{k})$  and  $\gamma_*$  is an isomorphism over  $S$ , the restrictions of  $h_{S'}$  and  $h_{\bar{S}}$  on  $S(\bar{k})$  will differ only up-to constant multiples by Corollary 2.5.3 below. Hence by inequality (3) above we can absorb all the terms in inequality (2) of last section and arrive at Theorem 2.1.1.

**Corollary 2.5.3.** *Let us have two projective varieties  $S'$  and  $\bar{S}$  embedded in  $\mathbb{P}_k^m$  and  $\mathbb{P}_k^n$  respectively and let  $h_m$  and  $h_n$  be the standard Weil heights of these projective spaces respectively. Furthermore, suppose we have an open sub-scheme  $S \subseteq \bar{S}$  and a morphism*

$$\pi: S' \rightarrow \bar{S}$$

*such that  $\pi|_{\pi^{-1}(S)}$  is an isomorphism onto  $S$ . Then we have positive constants  $c_1, c_2, d_1, d_2$  such that*

$$c_1 \cdot h_n(P) - d_1 \leq h_m(P) \leq c_2 \cdot h_n(P) + d_2$$

*for all  $P \in S(\bar{k})$  (where we identify  $\pi^{-1}(S)$  and  $S$  via  $\pi$ )*

*Proof.* The proof consists of an application of Lemma 2.5.1 once with  $V = \pi^{-1}(S)$  and

$$f = \pi|_{\pi^{-1}(S)}: \pi^{-1}(S) \rightarrow \mathbb{P}_k^n$$

and once in the reverse direction with  $V = S$  and

$$f = (\pi|_{\pi^{-1}(S)})^{-1}: S \rightarrow \mathbb{P}_k^m$$

Note that for the application of the second case, it is vital that  $\pi$  is an isomorphism over  $S$ . Once we write down the two inequalities obtained this way and after some algebraic manipulations, we clearly get the inequalities of the Lemma.  $\square$

*Proof of Theorem 2.1.1.* Note that this lemma essentially shows that if we pass to a blow up of  $\bar{S}$  which does not touch  $S$ , the height functions (coming from the projective embeddings of  $\bar{S}$  and  $S'$ ) are equivalent to each other up to some universal constant multiples. In the view of the role of the height on the base space in the equality of Theorem 2.1.1, this clearly justifies the change of base by blow-up that we did in Section 3 because in the notation of the previous sections, we have  $h_m = h_{S'}$  and  $h_n = h_{\bar{S}}$ .

Now we come to obtaining final inequality in Theorem 2.1.1 from inequality (2.2). Using inequality (2.3) in inequality (2.2), we get (change the constant  $B$  from the last section if necessary)

$$|h_{\mathcal{L}}([2]P) - 4 \cdot h_{\mathcal{L}}(P)| \leq (r_1 + r_2) \cdot h_{S'}(\pi(P)) + (s_1 + s_2) + B \text{ for all } P \in \mathcal{A}(\bar{k})$$

Next we apply the inequality of Corollary 2.5.3 noting that  $h_m = h_{S'}$  and  $h_n = h_{\bar{S}}$  to get that

$$|h_{\mathcal{L}}([2]P) - 4 \cdot h_{\mathcal{L}}(P)| \leq c_2(r_1 + r_2)h_{\bar{S}}(\pi(P)) + d_2(r_1 + r_2) + (s_1 + s_2) \text{ for all } P \in \mathcal{A}(\bar{k})$$

Now finally from the above we have

$$|h_{\mathcal{L}}([2]P) - 4 \cdot h_{\mathcal{L}}(P)| \leq c \cdot \max\{1, h_{\bar{S}}(\pi(P))\} \text{ for all } P \in \mathcal{A}(\bar{k})$$

where we take  $c = 2 \cdot \max\{c_2(r_1 + r_2), d_2(r_1 + r_2) + (s_1 + s_2)\}$  which finally proves Theorem 2.1.1 as  $c$  does not depend on  $P$  clearly.  $\square$

## Chapter 3

# The Gao-Habegger Height Inequality

In this chapter, we treat what can be called as the Gao-Habegger inequality due to the authors Ziyang Gao and Phillip Habegger. This inequality was first formulated by the above mentioned authors in [GH19] with a goal to resolve the geometric Bogomolov conjecture. The nature of the inequality is to bound the fiberwise Neron-Tate height from below linearly in term of the height of the parameter space  $S$ . It is easy to see that to obtain such a lower bound in full generality is hopeless. Indeed, if we take torsion points in then it has Neron-Tate height 0 while the height below is something positive. Fortunately, this problem also indicates us about what "bad parts" we should get rid of to at least hope for such an inequality *i.e.* in particular we must get rid of the torsion points. However it turns out we must also get rid of parts coming from the constant part (the trace) and not only torsion points but also torsion cosets. Once we get rid of these parts, then we have the desired bound which we later use to obtain our main results.

The proof that we present in this thesis is not the same to either of the proofs of the inequality found in [DGH20] or [GH19] but rather a mixture of these two proofs. Broadly the proof can be divided into two parts: proving an upper bound of intersection numbers and a lower bound. For the upper bound we closely follow the proof in [GH19] which makes use of a version of Bezout's theorem due to Phillipon in [Phi86]. For the lower bound on the other hand, we follow the proof in [DGH20] using the integration of Betti forms. We begin the chapter by setting up the various immersions and line-bundles which will be used in the later parts. Next we construct the Betti forms from Betti maps and then we proceed to prove our main height inequality 3.3.1.

### 3.1 Setup for the Height Inequality

In this section, we introduce the set-up in which we will prove our main height inequality. Let  $\pi: \mathcal{A} \rightarrow S$  be an abelian scheme where  $S$  is a regular, quasi-projective and irreducible variety over  $k = \overline{\mathbb{Q}}$ . We begin by choosing immersions of  $\mathcal{A}$  into projective spaces.

#### 3.1.1 Immersions and choice of line-bundles

Suppose  $A$  is the generic fiber of  $\mathcal{A}$  which is an abelian variety over  $k(S) = K$ , the function field of  $S$ . Then we can choose a symmetric ample line bundle  $L_0$  on  $A$ . Let us denote  $L = L_0^{\otimes l}$  for some natural number  $l \geq 4$  such that  $L$  is very ample. Then by Theorem 9 of [Mum11] we get that the global sections of  $L$  determines a projectively normal closed immersion  $A \hookrightarrow \mathbb{P}_K^n$  for some  $n$ .

By Theoreme XI 1.3a in [Ray06], we get an  $S$ -ample line bundle  $\mathcal{L}$  on  $\mathcal{A}$  such that the generic fiber of  $\mathcal{L}$  *i.e.* the restriction of  $\mathcal{L}$  to the generic fiber  $A$  is  $L$ . By taking suitably high powers and changing  $L$



and  $\mathcal{L}$  by their powers, we can assume that  $L$  is very ample and  $\mathcal{L}$  is very ample relative to  $S$ . As  $S$  itself is quasi-projective, we deduce that there exists a very ample line bundle  $\mathcal{E}$  on  $S$  arising from an immersion of  $S$  into  $\mathbb{P}_k^m$  for some  $m$ . Hence by Summary 13.71 of [GW10], we get a closed immersion

$$\mathcal{A} \hookrightarrow \mathbb{P}_k^n \times S \rightarrow S$$

arising from the line bundle  $\mathcal{L} \otimes \pi^* \mathcal{E}$  as  $\mathcal{A}$  is proper over  $S$ . We denote by  $\overline{S}$  the Zariski closure of  $S$  in  $\mathbb{P}_k^m$  and by  $\overline{\mathcal{A}}$  the Zariski closure of  $\mathcal{A}$  in  $\mathbb{P}_k^n \times \overline{S} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^m$ . Both  $\overline{\mathcal{A}}$  and  $\overline{S}$  are irreducible, projective varieties.

### 3.1.2 Height Functions

In this sub-section, we are going to introduce the various height functions that we are going to deal with. Previously we have seen that  $\mathcal{A}$  is presented with a closed immersion into  $\mathbb{P}_k^n \times S$  over  $S$ . As  $S$  itself is quasi-projective, we have an immersion  $S \hookrightarrow \mathbb{P}_k^m$  for some  $m$  and let us call by  $\overline{S}$  the Zariski closure of  $S$  in  $\mathbb{P}_k^m$ . Then overall we get an immersion

$$\mathcal{A} \hookrightarrow \mathbb{P}_k^n \times \mathbb{P}_k^m$$

Hence for every  $P \in \mathcal{A}(k)$ , we can write  $P = (P', \pi(P))$  for  $P' \in \mathbb{P}_k^n(k)$  and  $\pi(P) \in \mathbb{P}_k^m(k)$ . We have the canonical Weil heights on the projective spaces  $\mathbb{P}_k^n$  and  $\mathbb{P}_k^m$  both of which we denote by  $h$  and then we define the naive height on  $\mathcal{A}(k)$  as

$$h_{na}(P) = h(P') + h(\pi(P))$$

If we take the line bundle  $\mathcal{E}$  in the previous sub-section such that it gives the immersion of  $S$  in  $\mathbb{P}_k^m$ , then using the functorial properties of the Height machine, we easily get that the naive height  $h_{na}$  belongs to the height function class of the line bundle  $\mathcal{L} \otimes \pi^* \mathcal{E}$  which in turn is the restriction to  $\mathcal{A}$  of the standard twist  $\mathcal{O}(1, 1)$  on  $\mathbb{P}_k^n \times \mathbb{P}_k^m$ . Note then that the restriction of  $h_{na}$  to each special fiber  $\mathcal{A}_s$  is a height function in the class of the line bundle

$$(\mathcal{L} \otimes \pi^* \mathcal{E})|_{\mathcal{A}_s} = \mathcal{L}|_{\mathcal{A}_s}$$

Suppose we denote by  $[-1]$  the negation morphism of the abelian scheme  $\mathcal{A} \rightarrow S$ . Then we see that the generic fiber of the line bundle  $([-1]^* \mathcal{L} \otimes \mathcal{L})$  is  $[-1]^* L \otimes L$  which is trivial as  $L$  was originally chosen to be symmetric. Hence by Corollaire 21.4.13 of [Gro67], we get a line bundle  $\mathcal{K}$  on  $S$  such that

$$[-1]^* \mathcal{L} \otimes \mathcal{L} \cong \pi^* \mathcal{K}$$

Hence once again we get that

$$[-1]^*(\mathcal{L}|_{\mathcal{A}_s}) \otimes \mathcal{L}|_{\mathcal{A}_s} = \pi^* \mathcal{K}|_{\mathcal{A}_s}$$

is trivial for all  $s \in S(k)$ . Hence we get that the line bundles  $\mathcal{L}|_{\mathcal{A}_s} = \mathcal{L}_s$  are all symmetric very-ample line-bundles on the abelian varieties  $\mathcal{A}_s$ . Hence we have that canonical Neron-Tate heights  $\hat{h}_{\mathcal{A}_s, \mathcal{L}_s}$  corresponding to the bundles  $\mathcal{L}_s$  and we define

$$\hat{h}_{\mathcal{A}}(P) = \hat{h}_{\mathcal{A}_s, \mathcal{L}_s}(P) \text{ for all } P \in \mathcal{A}(k)$$

where  $s = \pi(P)$ . In the next sub-section we work on extending the map  $[N]$  to the compactification  $\overline{\mathcal{A}}$  via the graph morphism.

### 3.1.3 Extending $[N]$ to compactifications

As  $\mathcal{A} \rightarrow S$  is an abelian scheme, we have the "multiplication by N" morphism

$$[N]: \mathcal{A} \rightarrow \mathcal{A}$$

over  $S$ . However  $[N]$  gives only a rational morphism on the compactification  $\overline{\mathcal{A}}$ . To by-pass this problem we use an elementary but elegant method using graph morphisms which has been repeatedly used throughout the work.

Let us denote by

$$\gamma_N: \mathcal{A} \rightarrow \mathcal{A} \times_S \mathcal{A}$$

the graph morphism of  $[N]$ . As  $\pi$  is separated, we immediately get that  $\gamma_N$  is a closed immersion. Let  $X$  be an irreducible closed subvariety of  $\mathcal{A}$  of dimension  $d$ . We denote by  $\Gamma_N$  and  $X_N$  the scheme theoretic image under the map  $\gamma_N$  of  $\mathcal{A}$  and  $X$  respectively. It is easy to see that  $\gamma_N|_{\mathcal{A}}$  and  $\rho_1|_{\gamma_N}$  gives isomorphisms between the schemes  $\mathcal{A}$  and  $\Gamma_N$  by definition of the graph morphism where  $\rho_1$  is the projection on the first and the third factors from  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ . Note that  $\mathcal{A} \times_S \mathcal{A}$  is immersed into  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m = \mathbb{P}_{\overline{S}}^n \times_{\overline{S}} \mathbb{P}_{\overline{S}}^n$  and the projections  $\rho_1$  and  $\rho_2$  to the first and third factors and the last two factors respectively are the projections of  $\mathcal{A} \times_S \mathcal{A}$  when restricted. Let us denote by  $\overline{\Gamma}_N$  the Zariski closure of  $\Gamma_N$  in  $\overline{\mathcal{A}} \times_{\overline{S}} \overline{\mathcal{A}} \subseteq \mathbb{P}_{\overline{S}}^n \times_{\overline{S}} \mathbb{P}_{\overline{S}}^n = \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ . Note that then  $\Gamma_N \cong \mathcal{A}$  can be thought of as an open dense sub-scheme of  $\overline{\Gamma}_N$ . Hence we can think of  $\overline{\Gamma}_N$  as a compactification of  $\mathcal{A}$ . More over we have the morphism  $\rho_2|_{\overline{\Gamma}_N}$  which is, in some sense, an extension of  $[N]$  because we have

$$\rho_2|_{\overline{\Gamma}_N} \circ (\rho_1|_{\Gamma_N})^{-1} = [N]$$

by definition of the graph morphism. Let  $\overline{X}_N$  be the Zariski closure of  $X_N$  in  $\mathbb{P}_k^{m,n,n} = \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ . It is an irreducible closed sub-variety of  $\mathbb{P}_k^{m,n,n}$  with dimension  $\dim(\overline{X}_N) = \dim(X_N) = \dim(X) = d$ . We introduce certain line bundles on  $\overline{X}_N$  on which we will be applying the Theorem of Siu later on. We have seen above that  $\overline{X}_N$  is an irreducible closed sub-variety of the multi-projective space  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ . Then we have the two projections  $\rho_1$  and  $\rho_2$  from  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ . We define

$$\mathcal{F} = \rho_2^* \mathcal{O}(1, 1)|_{\overline{X}_N}$$

where  $\mathcal{O}(1, 1)$  is the standard multi-twisted sheaf on the multi-projective space  $\mathbb{P}_k^n \times \mathbb{P}_k^m$ . Similarly we define

$$\mathcal{M} = \mathcal{O}(0, 0, 1)|_{\overline{X}_N}$$

Lastly we are going to talk about the heights functions of the line bundles  $\mathcal{F}$  and  $\mathcal{M}$  before finishing this section.

Let  $P \in \mathcal{A}(k) \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^m(k)$  be written as  $P = (P', \pi(P))$  where  $P' \in \mathbb{P}_k^n(k)$  and  $\pi(P) \in \mathbb{P}_k^m(k)$ . Then the point  $[N]P$  has the form  $[N]P = (P'_N, \pi(P))$  as  $[N]$  is a morphism over  $S$ . Then the graph morphism  $\Gamma_N$  is given by

$$\Gamma_N(P) = (P', P'_N, \pi(P)) \in \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m(k)$$

We had already seen that the naive height  $h_{na}$  on  $\mathcal{A}(k)$  belonged to the height class of the line bundle  $\mathcal{L} \otimes \pi^* \mathcal{K}$  where  $\mathcal{K}$  is the line bundle on  $S$  giving the immersion of  $S$  into  $\mathbb{P}_k^m$ . Hence we can say that  $h_{na}$  is the restriction of a height function on  $\overline{\mathcal{A}}$  belonging to the height class of the line-bundle  $\mathcal{O}(1, 1)|_{\overline{\mathcal{A}}}$ . Hence by functorial properties of the height machine we get that  $h_{na} \circ \rho_2$  is a height function in the class of  $\mathcal{F}$ . Looking at the description of  $\Gamma_N$  as above, we get that the function  $h_{na} \circ \rho_2$  restricted to  $X_N$  is the function

$$P \mapsto h_{na}(P'_N, \pi(P)) = h_{na}([N]P)$$

and arguing similarly we get that the function

$$P \mapsto h(\pi(P))$$

is a height function in the class of the line bundle  $\mathcal{M}$  restricted to  $X_N$ . Hence by estimating intersection numbers involving  $\mathcal{F}$  and  $\mathcal{M}$  we can assert bigness of them (Theorem of Siu) which in turn gives inequality corresponding the two heights described above.

## 3.2 Betti Maps and Forms

One of the principal strategies of proving height inequalities is to consider global sections of big line-bundles and to obtain an inequality (lower bound) outside the support of this section. Hence in the course of this we need to prove that certain line bundles are big. This question can be translated to a question of inequalities of intersection numbers with the help of Theorem 2.2.13 in [Laz17]. Using the fact that integration of forms and top-self intersection coincide in the compact (here projective) case, we can show the desired inequalities by integrating forms and estimating. One of the principal tools needed for that is the so called *Betti form* which arise from foliations with respect to *Betti map*.

### 3.2.1 Construction of the Betti maps

We start by recalling the setup on which we will be working. We have an abelian scheme  $\pi: \mathcal{A} \rightarrow S$  over a regular, irreducible, quasi-projective variety  $S$  defined over  $\mathbb{C}$ . We denote by  $\mathcal{A}^{an}$  (resp  $S^{an}$ ) the analytification of the respective algebraic varieties. When there is no ambiguity we will omit the subscript. We fix a point  $s_0 \in S(\mathbb{C})$ . Roughly speaking we will use the fact that  $\mathcal{A}$  is locally a trivial bundle with fiber  $\mathcal{A}_{s_0}$  (due to Ehresmann) to obtain a real analytic map from a neighbourhood of  $\mathcal{A}_{s_0}$  in  $\mathcal{A}$  to  $\mathcal{A}_{s_0}$  which will also have complex analytic fibers because  $\pi$  is holomorphic. We start our discussion with a theorem from complex geometry:

**Theorem 3.2.1** (Ehresmann). *Suppose  $\pi: \mathcal{A} \rightarrow S$  is a proper submersion of differentiable manifolds and let  $S$  be contractible with a fixed base point  $s_0$ . Then there is a diffeomorphism*

$$\phi: \mathcal{A} \rightarrow \mathcal{A}_{s_0} \times S$$

over  $S$  where  $\mathcal{A}_{s_0}$  denotes the fiber of  $\pi$  over  $s_0$ .

*Proof.* See Theorem 9.3 in [Voi02]. □

In fact, when the family is complex *i.e.* when  $\pi$  is a proper holomorphic submersion, we can say more.

**Lemma 3.2.2.** *Suppose  $\pi: \mathcal{A} \rightarrow S$  is a proper holomorphic submersion of complex differentiable manifolds and let  $S$  be contractible with a fixed base point  $s_0$ . Then up to replacing  $S$  by a neighbourhood of  $s_0$  there is a diffeomorphism*

$$\phi: \mathcal{A} \rightarrow \mathcal{A}_{s_0} \times S$$

over  $S$ . Moreover, if  $p_2$  is the projection of the R.H.S to  $\mathcal{A}_{s_0}$ , then for all  $p \in \mathcal{A}_{s_0}$ , the fiber  $(p_2 \circ \phi)^{-1}(p)$  is a complex analytic subset of  $\mathcal{A}$

*Proof.* See Prop 9.5 in [Voi02]. □

We come back to our setting where  $\pi: \mathcal{A} \rightarrow S$  is an abelian scheme and we fixed a base point  $s_0 \in S(\mathbb{C})$ . As  $S^{an}$  is a complex manifold, we can choose a simply connected (in the complex topology) open neighbourhood  $U$  of  $s_0$  in  $S$ . Note that  $\pi: \pi^{-1}(U) \rightarrow U$  is a smooth morphism and hence induces a surjective linear map between tangent spaces. Hence the analytification of  $\pi$  is a submersion in the sense of differential geometry. Moreover by the definition of abelian schemes, we get that  $\pi$  (more precisely the analytification of  $\pi$ ) is proper and hence a proper submersion between the complex manifolds  $\mathcal{A}$  and  $S$ . As  $\pi$  comes from a morphism of algebraic varieties, we also get that  $\pi$  is holomorphic. Hence by Lemma 3.2.2 we get a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times \mathcal{A}_{s_0} \\ & \searrow \pi & \downarrow p_1 \\ & & U \end{array}$$

where  $\phi$  is a real analytic(not necessarily holomorphic) diffeomorphism and  $p_1$  is the canonical projection.

We are going to use these local charts  $U$  around the base point  $s_0$  to construct real analytic maps  $\beta_U^{s_0}$  from  $\pi^{-1}(U)$  to the  $g$ -dimensional complex torus which will have desirable properties. More precisely

**Theorem 3.2.3.** *Suppose  $\mathcal{A}$ ,  $S$ ,  $s_0$ ,  $U$  be as in the above discussion. Then there exists real analytic maps*

$$\beta_U^{s_0} : \pi^{-1}(U) \rightarrow \mathcal{A}_{s_0}$$

*satisfying the following:*

1. *the real analytic map*

$$\pi \times \beta_U^{s_0} : \pi^{-1}(U) \rightarrow U \times \mathcal{A}_{s_0}$$

*is a diffeomorphism.*

2. *for all  $p \in \mathcal{A}_{s_0}$ , the fiber  $\beta_U^{s_0^{-1}}(p)$  is complex analytic subset of  $\pi^{-1}(U)$ .*

3. *the map*

$$\beta_U^{s_0} |_{\mathcal{A}_s} : \mathcal{A}_s \rightarrow \mathcal{A}_{s_0}$$

*is a real analytic isomorphism of real Lie groups for all  $s \in S(\mathbb{C})$*

*Proof.* Let us denote by  $p_2$  the second projection from  $U \times \mathcal{A}_{s_0}$  to  $\mathcal{A}_{s_0}$ . Then we define the Betti maps as

$$\beta_U^{s_0} = p_2 \circ \phi$$

where  $\phi$  is trivialisation arising from Theorem 3.2.1. As both  $\phi$  and  $p_2$  are real analytic, we easily obtain that  $\beta_U^{s_0}$  is real analytic.

By observing that  $\phi = \pi \times \beta_U^{s_0}$  from construction of the Betti map we immediately deduce 1.

For 2 note that  $\pi$  is holomorphic, we get 2 from Lemma 3.2.2 as  $\beta_U^{s_0} = p_2 \circ \phi$ .

Lastly the fact that the restriction in point 3 is a diffeomorphism comes from the fact that  $\phi$  is a diffeomorphism over  $U$ . However it does not immediately give that it is also an isomorphism of real Lie groups. We sketch the argument of that as follows:

We denote by  $f_s$  the restriction of  $\phi$  to the fiber  $\mathcal{A}_s$ . Then as  $f_s$  is a diffeomorphism, we get an isomorphism of first homologies

$$(f_s)_* : H_1(\mathcal{A}_s, \mathbb{Z}) \rightarrow H_1(\mathcal{A}_{s_0}, \mathbb{Z})$$

As each of the fibers is an complex abelian variety, we get that  $H_1(\mathcal{A}_s, \mathbb{Z})$  is a real basis of  $\mathcal{A}_s$  for each  $s \in S(\mathbb{C})$ . Hence each point  $p \in \mathcal{A}_s$  has real co-ordinates with respect to the real basis  $H_1(\mathcal{A}_s, \mathbb{Z})$ . However as  $(f_s)_*$  is an isomorphism, we get that  $(f_s)_*^{-1}(H_1(\mathcal{A}_{s_0}, \mathbb{Z}))$  is also a real basis of  $\mathcal{A}_s$  for each  $s$ . Then the Betti map in our construction is nothing but the map which sends a point  $p \in \mathcal{A}_s$  to the real co-ordinates in  $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$  of  $p$  w.r.t to the basis  $(f_s)_*^{-1}(H_1(\mathcal{A}_{s_0}, \mathbb{Z}))$ . Note that with this description, the restriction of the Betti map to a fiber is nothing but a matrix multiplication and hence clearly a morphism of the underlying additive groups which completes our claim. This later description is detailed more in [GH19].

□

### 3.2.2 Transverse Foliations from Betti maps

The property (ii) in Theorem 3.2.3 shows that the fibers of the Betti map are complex analytic. By construction, the Betti map looks like a projection upto a diffeomorphism and hence they give rise to a foliation which are transverse to the fibers  $\mathcal{A}_s$ . We use this fact to pull back suitable forms from the abelian variety  $\mathcal{A}_{s_0}$  to a form on  $\pi^{-1}(U)$  which will be our Betti forms.

More precisely, we define a foliation  $\mathcal{F}$  by specifying the local leaves as  $\mathcal{F}_{U,p} = \beta_U^{\pi(p)^{-1}}(p)$  for  $p \in \mathcal{A}$ . It is easy to see that  $\mathcal{F}$  is actually a transverse foliation as  $\beta_U^s$  looks like projections upto a diffeomorphism. Hence we get that for all  $p \in \mathcal{A}$ , we have

$$\mathcal{T}_p(\mathcal{A}) = \mathcal{T}_p(\mathcal{A}_{\pi(p)}) \oplus T_p(\mathcal{F})$$

where  $\mathcal{T}$  denotes the tangent space. If we denote by  $P_p$  the projection onto the first factor in the above, then it is easy to see that  $P_p = (D\beta_U^{\pi(p)})_p$  where  $\mathcal{D}$  denotes the induced map between tangent spaces

Apriori  $P_p$  is just a map of real vector spaces. However we first note that the restriction of  $\beta_U^{\pi(p)}$  to the fiber  $\mathcal{A}_{\pi(p)}$  is identity by construction. Hence we get that  $p \in \beta_U^{\pi(p)^{-1}}(p) = \mathcal{F}_{U,p}$ . As the foliation  $\mathcal{F}$  has  $\mathcal{F}_{U,p}$  as the local leaves, we get that  $\mathcal{T}_p(\mathcal{F}) = \mathcal{T}_p(\beta_U^{\pi(p)^{-1}}(p))$  which is a complex sub vector space of  $\mathcal{T}_p(\mathcal{A})$  because  $\mathcal{F}_{U,p}$  is complex analytic from part (ii) of Theorem 3.2.3. We already know that  $\mathcal{A}_{\pi(p)}$  is complex analytic and hence  $\mathcal{T}_p(\mathcal{A}_{\pi(p)})$  is also a complex sub-vector space of  $\mathcal{T}_p(\mathcal{A})$ . The upshot of all this is that the induced map of tangent spaces  $P_p$  from Betti maps is complex linear even if the Betti maps aren't holomorphic in general.

### 3.2.3 Translation invariant forms on complex abelian varieties

The famous Hodge decomposition of the cohomology groups of a compact Kahler manifold gives us the notions of  $(p, q)$ -forms. However in the case of an abelian variety *i.e.* a complex torus we get more and we can associate a translation invariant form to every cohomology class of such  $(p, q)$ -forms. We use this property and pull back these special forms to get our Betti form. The translation invariance gives that on torsion sections the Betti forms vanish, a property which is crucial in the proof Theorem 3.3.1 as these Betti forms detect the "bad" torsion points where the inequality doesn't hold.

Suppose we have a complex abelian variety  $A$  of dimension  $g$ . Then it is well known that we can write  $A = \mathbb{C}^g / \Lambda$  for a lattice  $\Lambda$  in  $\mathbb{C}^g$ . For simplification we call  $V = \mathbb{C}^g$ . Call the co-ordinates of  $V$  as  $v_1, v_2 \dots v_g$ . Then  $dv_1, dv_2 \dots dv_g$  are translation invariant 1-forms on  $V$ . As they are translation invariant, in particular they are invariant under translation by elements of  $\Lambda$ . Hence we get that  $dv_i$ 's arise as pullbacks of certain forms on  $\mathbb{C}^g / \Lambda = A$  which we also call  $dv_i$ . Passing to real co-ordinates and taking complex conjugates, we also get the corresponding conjugate forms  $\overline{dv_i}$ . It is easy to check that  $\{dv_1 \dots dv_g, \overline{dv_1} \dots \overline{dv_g}\}$  are linearly independent over  $\mathbb{C}$ .

From Corollary 1.3.4 in [BL13] we get that the first cohomology  $H^1(A, \mathbb{C})$  has dimension  $2g$  over  $\mathbb{C}$ . Hence the linear independency shows that the family  $\{dv_1, \dots dv_g, \overline{dv_1} \dots \overline{dv_g}\}$  is a basis for  $H^1(A, \mathbb{C})$ . Corollary 1.3.4 above also tells us that

$$H^n(A, \mathbb{C}) \cong \bigwedge^n H^1(A, \mathbb{C})$$

Under the de-Rahm isomorphism between ordinary cohomology and de-Rahm cohomology the cup-product of ordinary cohomology corresponds to the wedge product of differential forms. Hence we get that the forms  $dv_I \wedge \overline{dv_J}$  form a basis of  $H^n(A, \mathbb{C})$  where  $I = \{i_1, \dots i_p\}$ ,  $J = \{j_1, \dots j_q\}$  are subsets of  $\{1, \dots g\}$  and

$$dv_I = dv_{i_1} \wedge dv_{i_2} \dots \wedge dv_{i_p} \text{ and } \overline{dv_J} = \overline{dv_{j_1}} \wedge \dots \wedge \overline{dv_{j_q}}$$

Let us denote by  $IF^n(A)$  the vector space of translation invariant  $n$ -forms on  $A$ . Then proposition 1.3.5 in [BL13] states that

$$H^n(A, \mathbb{C}) = IF^n(X)$$

Hence we get that for any  $\phi \in IF^n(X)$  we can write

$$\phi = \sum_{I,J} a_{IJ} d_I \wedge \overline{dv_J}$$

where  $a_{IJ} \in \mathbb{C}$ . When  $|I| = p$  and  $|J| = q$  for all  $I, J$  in the summand above, we call that  $\phi$  is an *invariant form of type  $(p, q)$* . We denote by  $IF^{p,q}(X)$  the vector space of invariant  $(p, q)$ -forms on  $X$ . The normal Hodge decomposition for compact Kahler manifolds gives us the decomposition

$$H^n(A, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X)$$

where  $H^{p,q}(X)$  is the space of  $(p, q)$ -forms on  $A$ . The proof of Proposition 1.4.3 in [BL13] shows that

$$IF^{p,q}(X) \cong H^{p,q}(X)$$

Thus we can choose an unique canonical invariant  $(p, q)$ -form to every cohomology class of  $(p, q)$ -forms very much similar to the uni-graded case. In particular we have the following lemma which will pave the way to the construction of Betti forms:

**Lemma 3.2.4.** *Suppose  $A$  is a complex abelian variety and  $L$  is a line bundle on  $A$ . Then we can find an unique translation invariant closed  $(1, 1)$ -form  $\phi$  in the Chern class  $c_1(L)$ . In terms of co-ordinates, we have*

$$\phi = \sum_{i,j} a_{ij} dv_i \wedge \overline{dv_j}$$

where  $v_i$  are the local co-ordinates and  $a_{ij} \in \mathbb{C}$  are complex numbers and this form satisfies

$$[N]^* \phi = N^2 \cdot \phi$$

Moreover, if  $L$  is ample,  $\phi$  is a positive form.

*Proof.* The Chern class  $c_1(L)$  is a cohomology class of  $(1, 1)$ -form on  $A$  i.e.  $c_1(L) \in H^{p,q}(X)$ . Now as  $A$  is a complex abelian variety, the statement follows easily from our discussion above. The form is closed as the co-efficients  $a_{ij}$  are constant complex numbers. Lastly as the multiplication by  $N$  morphism in complex abelian varieties is just the ordinary multiplication co-ordinate wise, we get that

$$[N]^* \omega = N^{p+q} \cdot \omega$$

for any invariant  $(p, q)$ -form  $\omega$ . □

### 3.2.4 Betti Forms

Suppose we have an abelian scheme  $\mathcal{A} \rightarrow S$  and an  $S$ -ample line bundle  $\mathcal{L}'$  on  $\mathcal{A}$ . The restriction of  $\mathcal{L}'$  to the fibers  $\mathcal{A}_s$  together with Lemma 3.2.4 gives rise to translation invariant positive  $(1, 1)$ -forms  $\omega_s$  on each fiber  $\mathcal{A}_s$ . We pull back these forms locally by the Betti maps to obtain a global form  $\omega$  on  $\mathcal{A}$  which we call the Betti form.

Recall from sub-section 2 that we described the complex linear map  $P_p$  as the projection

$$\mathcal{T}_p(\mathcal{A}) = \mathcal{T}_p(\mathcal{A}_{\pi(p)}) \oplus \mathcal{T}_p(\mathcal{F}) \rightarrow \mathcal{T}_P(\mathcal{A}_{\pi(p)})$$

From Lemma 3.2.4 and fixing an  $S$ -ample line bundle  $\mathcal{L}'$  on  $\mathcal{A}$ , we get canonical translation invariant closed  $(1, 1)$ -forms  $\omega_s$  corresponding to the Chern class of  $\mathcal{L}'|_{\mathcal{A}_s}$  on each fiber  $\mathcal{A}_s$ . Note that as  $\mathcal{L}'$  is  $S$ -ample, each  $\mathcal{L}'|_{\mathcal{A}_s}$  is ample and hence each  $\omega_s$  is a positive form. Next we define a global form  $\omega$  on  $\mathcal{A}$  as follows:

at any point  $p \in \mathcal{A}$  the associated bi-linear form to  $\omega$  is given by

$$\omega(v_1, v_2) = \omega_{\pi(p)}(P_p(v_1), P_p(v_2)) \text{ for } v_1, v_2 \in \mathcal{T}_p(\mathcal{A})$$

We argued in sub-section 2 that  $P_p = (d\beta_U^{\pi(p)})_p$  is the induced map between tangent spaces at  $p$  corresponding to some local Betti map around  $\pi(p)$ . Thus given any  $p \in \mathcal{A}$ , if  $U$  is a simply connected base as in Theorem 3.2.1 and  $\beta_U^{\pi(p)}$  is the corresponding Betti map, from the definition of  $\omega$  we deduce that

$$\omega|_{\pi^{-1}(U)} = \beta_U^{\pi(p)*} \omega_{\pi(p)}$$

Next we gather some important properties of the Betti form needed later for estimates.

**Lemma 3.2.5.** *Let  $\mathcal{A}$ ,  $S$  and  $\omega$  be as described above. Then  $\omega$  is a closed, semi-positive  $(1,1)$ -form such that*

$$[N]^* \omega = N^2 \cdot \omega$$

where  $[N]$  is the multiplication by  $N$  morphism on  $\mathcal{A}$ .

*Proof.* Note that as  $P_p$  is complex linear, we easily get that the pull-back of a type  $(1,1)$ -form by  $P_p$  is again  $(1,1)$ . To note that  $\omega$  is semi-positive, take any  $p \in \mathcal{A}$  and  $v \in \mathcal{T}_p^{1,0}(\mathcal{A})$ . We have

$$-i \cdot \omega(v, \bar{v}) = -i \cdot \omega_{\pi(p)}(P_p(v), P_p(\bar{v})) = -i \cdot \omega_{\pi(p)}(P_p(v), \overline{P_p(v)}) \geq 0$$

as each  $\omega_s$  is a positive form and  $P_p$  is complex linear ( $i = \sqrt{-1}$ ). Note that as  $P_p$  is not in general injective, we can only say that  $\omega$  is semi-positive. Next we need to show that  $\omega$  is a closed form. This is where we use the local description of  $\omega$  as the pull-backs by Betti maps. We saw earlier that

$$\omega|_{\pi^{-1}(U)} = \beta_U^{\pi(p)*} \omega_{\pi(p)}$$

We already know that the form  $\omega_{\pi(p)}$  is closed and  $\beta_U^{\pi(p)}$  is a smooth maps. As pull-backs of closed forms under smooth maps is again closed, we deduce that  $\omega|_{\pi^{-1}(U)}$  is closed. As the open sets  $\pi^{-1}(U)$  cover  $\mathcal{A}$  as we vary  $p$ , we get that  $\omega$  is locally closed and hence globally a closed form.. Hence we only need to show that  $[N]^* \omega = N^2 \cdot \omega$ .

Note that from point 3 of Theorem 3.2.3, we get that the restriction of the Betti maps fiber-wise respect the group structure. Hence we get the following diagram:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times \mathcal{A}_{s_0} \\ [N] \downarrow & & \downarrow \text{id} \times [N]_{s_0} \\ \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times \mathcal{A}_{s_0} \end{array}$$

where  $[N]_{s_0}$  denotes the multiplication-by- $N$  morphism restricted on  $\mathcal{A}_{s_0}$ . Hence we get from the local description of the Betti form that

$$[N]^*(\omega|_U) = [N]^* \beta_U^{s_0*} \omega_{s_0} = \phi_U^*(\text{id} \times [N]_{s_0})^* p_2^* \omega_{s_0}$$

where we used the commutativity of the above diagram in the second equality. But as  $p_2 \circ (\text{id} \times [N]_{s_0}) = [N]_{s_0}$ , we get that

$$[N]^*(\omega|_U) = \phi_U^*([N]_{s_0}^* \omega_{s_0}) = \phi_U^*(N^2 \cdot \omega_{s_0}) = N^2 \cdot \phi_U^*(\omega_{s_0}) = N^2 \cdot \omega|_U$$

where in the second equality we have used the homogeneity of pull-back by  $[N]$  as indicated in Theorem 3.2.3. Thus we showed the claim locally and hence globally we have  $[N]^* \omega = N^2 \cdot \omega$ .  $\square$

### 3.3 The Height Inequality

We recall the set-up described in section 1. We have a closed immersion of  $\mathcal{A}$  into  $\mathbb{P}_S^n$  over  $S$  such that the generic fiber satisfies the properties described in section 1. We also have that  $S$  is a quasi-projective irreducible regular variety immersed in  $\mathbb{P}_k^m$  with the Zariski closure  $\overline{S}$  an irreducible projective variety. We assume  $k = \overline{\mathbb{Q}}$ . Furthermore let  $X$  be an irreducible subvariety of  $\mathcal{A}$  of dimension  $d$  such that  $\pi|_X: X \rightarrow S$  is dominant. We recall the Betti form  $\omega$  on  $\mathcal{A}$  constructed in section 2. With the following set-up, we can state the main height inequality that we are going to prove:

**Theorem 3.3.1.** *Suppose  $X^{an}$  contains a smooth point  $P$  such that  $(\omega|_{X^{sm,an}})^{\wedge d} > 0$  at  $P$  and  $N$  is positive power of 2. Then there is a Zariski open and dense subset  $U_N$  (dependent on  $N$ ) of  $X$  defined over  $k$  and a constant  $c_2(N)$  such that*

$$h_{na}([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N)$$

where  $c_1 > 0$  is a constant independent of  $N$ .

In this section we prove this height inequality. The general strategy is to use the Theorem of Siu to conclude that certain line bundles are big and then prove the inequality outside the zero locus of some global section. To use the Theorem of Siu we make certain estimates on the intersection numbers of  $\mathcal{F}$  and  $\mathcal{M}$  from section 1 using integration against the Betti form on one hand and positivity results on the other.

To begin with suppose  $N$  is a power of 2. Then following the notation of section 1, we have the closure of the image of the graph morphism  $\overline{X_N}$  which is an irreducible projective subvariety of  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ . We had introduced the line-bundles  $\mathcal{F}$  and  $\mathcal{M}$ .

Suppose we call  $P_0$  the point  $P$  mentioned in Theorem 3.3.1 where the top-wedge of the Betti form is positive and call  $s_0 = \pi(P_0)$  a point in  $S_0$ . Furthermore suppose  $U$  is a contractible, relatively compact open neighbourhood of  $s_0$  in  $S^{an}$ . We construct a smooth bump function

$$\vartheta: \mathbb{P}_k^{m,an} \rightarrow [0, 1]$$

such that  $\vartheta(s_0) = 1$  and  $\vartheta$  has compact support  $K \subseteq U$  by realising  $S$  as a quasi-projective complex variety inside  $\mathbb{P}_k^m$ . Finally we set  $\theta = \vartheta \circ \rho_3$  where  $\rho_3$  is the projection on the third factor from  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$  to  $\mathbb{P}_k^m$ . It is easy to see by construction that  $\rho_2$  restricted to  $\mathcal{A}$  is  $\pi$ . (We will also denote by  $\vartheta$  and  $\theta$  their respective restrictions to  $S^{an}$  and  $\mathcal{A}^{an}$ ).

#### 3.3.1 The lower bound on self-intersection of $\mathcal{F}$

We begin with a lemma on an analogue of Serre's theorem for differential forms over relatively compact bases.

**Lemma 3.3.2.** *Suppose  $\alpha$  is a strictly positive  $(1,1)$ -form on  $\mathcal{A}^{an}$  and let  $\omega$  be the Betti form. Suppose  $U$  is a relatively compact open subset of  $S$  and  $\mathcal{A}_U = \pi^{-1}(U)$ . Then there exists a constant  $C > 0$  such that*

$$(C\alpha - \omega)|_{\mathcal{A}_U} \geq 0$$

is a semi-positive  $(1,1)$ -form.

*Proof.* As  $\alpha$  is strictly positive on  $\mathcal{A}^{an}$ , we can cover  $\mathcal{A}^{an}$  with an open complex chart  $\{U_j\}$  such that on each  $U_j$  we have the diagonal form

$$i \cdot \alpha|_{U_j} = \sum_i \alpha_i dz_i \wedge \overline{dz_i}$$

where  $z_i$  are the local co-ordinates,  $\alpha_i$  are strictly positive smooth real functions on  $U_j$  and  $i = \sqrt{-1}$ . We can shrink the charts  $U_i$  using that  $\mathcal{A}^{an}$  is locally compact and assume that  $\alpha_i$  are strictly positive



on the closures  $\overline{U_j}$ . Shrinking the charts and using that  $\omega$  is semi-positive on  $\mathcal{A}^{an}$  (Lemma 3.2.5) we can also assume that

$$i \cdot \omega|_{U_j} = \sum \beta_i dz_i \wedge \overline{dz_i}$$

with  $\beta_i$  non-negative smooth real functions on  $U_i$ . Now note that as  $\pi$  is proper and  $U$  is relatively compact, we get that  $\mathcal{A}_U$  is also relatively compact in  $\mathcal{A}$ . Let us denote by  $\overline{\mathcal{A}_U}$  the compact closure of  $\mathcal{A}_U$  in  $\mathcal{A}$ . Then we get that finite sub-family  $\{U_i\}_{i=1}^k$  covers  $\overline{\mathcal{A}_U}$  and hence  $\mathcal{A}_U$  in particular. Let us denote by  $V_j = U_j \cap \mathcal{A}_U$  for  $j = 1 \dots k$ . Then  $\{V_j\}_j$  is an open cover of  $\mathcal{A}_U$  such that on each of them both  $i\alpha$  and  $i\omega$  have the diagonal forms. Now as  $\alpha_i$  is strictly positive on the closures  $\overline{U_j}$  we get that  $\alpha_i$  are strictly positive on the closures  $\overline{V_j}$ . But as  $V_j \subseteq \mathcal{A}_U$  and  $\mathcal{A}_U$  is relatively compact, we get that each  $\overline{V_j}$  is compact. Using that  $\alpha_i$  are strictly positive on  $\overline{V_j}$  and that  $\{V_j\}$  is a finite family, we can find an  $\epsilon > 0$  such that  $\alpha_i \geq \epsilon$  for all  $i$ . Using similar reasoning, we get that there is a  $D > 0$  such that  $\beta_i \leq D$  for all  $i$ . Hence by choosing  $C > D/\epsilon$  we get that

$$C\alpha_i - \beta_i \geq 0 \text{ for all } i$$

This implies that on each  $V_j$  we have the local diagonal form

$$i(C\alpha - \omega) = i \cdot C\alpha - i \cdot \omega = \sum_i (C\alpha_i - \beta_i) dz_i \wedge \overline{dz_i}$$

which is an equivalent condition for an  $(1,1)$ -form to be positive. Hence we get that  $C\alpha - \omega$  is locally semi-positive for each  $V_j$  and as  $\{V_j\}$  cover  $\mathcal{A}_U$ , we get that

$$(C\alpha - \omega)|_{\mathcal{A}_U} \geq 0$$

□

Next we come to the lower bound on the self intersection of  $\mathcal{F}$  that is the main goal of this subsection. For this part we go to the base change of  $\overline{\mathbb{Q}}$  to  $\mathbb{C}$  and hence we assume  $k = \mathbb{C}$ .

**Lemma 3.3.3.** *Suppose  $X^{an}$  contains a smooth point  $P_0$  such that  $(\omega|_{X^{an}})^{\wedge d} > 0$  in  $P_0$ . Then there exists a constant  $\kappa > 0$  independent of  $N$  such that*

$$\mathcal{F}^d \geq \kappa \cdot N^{2d}$$

for all  $N$

*Proof.* Let  $P_0, s_0, \vartheta, \theta, U$  and  $K$  be as described in the beginning of section 3. We extend  $\vartheta$  to a smooth function on whole  $(\mathbb{P}_k^m)^{an}$  and then by post-composition with the second projection we extend  $\theta$  to whole  $(\mathbb{P}_k^n \times \mathbb{P}_k^m)^{an}$ .

We have the canonical Fubini study form on every projective space which is in the Chern class of the Serre one-twist. We have the analytification of the Segre embedding

$$i: \mathbb{P}_k^n \times \mathbb{P}_k^m \rightarrow \mathbb{P}_k^{m,n} \rightarrow \mathbb{P}_k^{(m+1)(n+1)-1}$$

Let us pull-back the Fubini study form on the R.H.S above under this Segre embedding and then restrict the resulting form on  $\overline{\mathcal{A}}$  the closure of  $\mathcal{A}$  in  $\mathbb{P}_k^{m,n}$ . We call this form by  $\alpha$  which will serve the role of  $\alpha$  in the previous lemma. Note that  $\alpha$  is in the Chern class of the very ample line bundle  $\mathcal{O}(1,1)$  restricted to  $\overline{\mathcal{A}}$  and hence is strictly positive on  $\mathcal{A}^{an}$ .

Now as  $U$  is relatively compact, we get by using Lemma 3.3.2 that there is a constant  $C > 0$  such that

$$(C\alpha - \omega)|_{\mathcal{A}_U} \geq 0$$

We consider the the form

$$C\theta\alpha - \theta\omega$$

which is a-priori a  $(1, 1)$ -form on  $\mathcal{A}^{an}$ . As  $\theta$  is non-negative and  $C\alpha - \omega$  is semi-positive on  $\mathcal{A}_U$ , we get that  $C\theta\alpha - \theta\omega$  is also semi-positive on  $\mathcal{A}_U$ . However  $\theta$  has compact support  $\pi^{-1}(K) \subseteq \pi^{-1}(U) = \mathcal{A}_U$  AS  $\pi$  is proper and hence we get that  $C\theta\alpha - \theta\omega$  is 0 outside  $\mathcal{A}_U$ . Overall we get that

$$C\theta\alpha - \theta\omega \geq 0$$

is a semi-positive  $(1, 1)$ -form on  $\mathcal{A}^{an}$ . The multiplication by  $N$  map  $[N]$  is a holomorphic map as it arises as the analytification of an algebraic morphism of varieties and hence pull-back under  $[N]$  of a semi-positive  $(1, 1)$ -form is again a semi-positive  $(1, 1)$  form. Hence we get that

$$\beta = [N]^*(C\theta\alpha - \theta\omega) = C[N]^*(\theta\alpha) - [N]^*(\theta\omega) = C[N]^*(\theta\alpha) - N^2(\theta\omega) \geq 0$$

is a semi-positive  $(1, 1)$ -form where we have used that pull-back by  $[N]$  commutes with  $\theta$  as  $\theta \circ [N] = \theta$  and the homogeneity property of the Betti form mentioned in Lemma 3.2.5. We denote

$$\gamma = C[N]^*(\theta\alpha) \text{ and } \delta = N^2(\theta\omega)$$

so that  $\beta = \gamma - \delta$ . Note that both  $\gamma$  and  $\delta$  have compact support inside  $\mathcal{A}_U$  as the same is true for  $\theta$ . Hence we can integrate the forms  $\gamma^{\wedge d}$  and  $\delta^{\wedge d}$  on any  $d$ -dimensional smooth sub-variety of  $\mathcal{A}^{an}$  and hence in particular on  $X^{sm, an}$ . we claim that

$$\int_{X^{sm, an}} \gamma^{\wedge d} \geq \int_{X^{sm, an}} \delta^{\wedge d} \quad (3.1)$$

We note that

$$\int_{X^{sm, an}} (\gamma^{\wedge d} - \delta^{\wedge d}) = \int_{X^{sm, an}} (\beta + \delta)^{\wedge d} - \delta^{\wedge d} = \int_{X^{sm, an}} \left( \sum_{i=0}^d \binom{d}{i} \beta^{\wedge i} \wedge \delta^{\wedge d-i} \right) - \delta^{\wedge d} = \sum_{i=1}^d \binom{d}{i} \int_{X^{sm, an}} \beta^{\wedge i} \wedge \delta^{\wedge d-i} \geq 0$$

where in the second equality above we have used the fact that both  $\beta$  and  $\delta$  are  $(1, 1)$ -forms and hence of even degree and the wedge product is commutative between forms of even degree. The last inequality follows as both  $\beta$  and  $\delta$  are semi-positive forms on  $\mathcal{A}^{an}$  and hence  $\beta^{\wedge i} \wedge \delta^{\wedge d-i}$  is a semi-positive top form on  $X^{sm, an}$  which has non-negative integral for each  $i$ . Hence we get the claimed inequality.

At first let us estimate the R.H.S of the above inequality (3.1). We easily get that

$$\int_{X^{sm, an}} \delta^{\wedge d} = N^{2d} \int_{X^{sm, an}} (\theta\omega)^{\wedge d} = \kappa' N^{2d} \quad (3.2)$$

where  $\kappa' = \int_{X^{sm, an}} (\theta\omega)^{\wedge d} > 0$  as  $\theta(P_0) = 1$  and  $(\omega|_{X^{sm, an}})^{\wedge d} > 0$  at  $P_0$ . Hence we already have one side of the inequality in Lemma 3.3.3. Next we relate the L.H.S of inequality (3.1) with the intersection number  $\mathcal{F}^d$ .

First note that as pull-back by smooth maps commutes with the exterior wedge product of forms, we have that  $\gamma^{\wedge d} = C^d([N]^*(\theta\alpha))^{\wedge d} = C^d[N]^*(\theta\alpha)^{\wedge d}$ . Hence we deduce

$$\left( \int_{X^{sm, an}} \gamma^{\wedge d} \right) / C^d = \int_{X^{sm, an}} [N]^*(\theta\alpha)^{\wedge d} = \int_{X^{sm, an}} (\rho_2|_{\Gamma_N} \circ \rho_1|_{\Gamma_N}^{-1})^*(\theta\alpha)^{\wedge d} = \int_{X^{sm, an}} (\rho_1|_{\Gamma_N}^{-1})^* \rho_2|_{\Gamma_N}^* (\theta\alpha)^{\wedge d} \quad (3.3)$$

where  $\rho_1$  and  $\rho_2$  are the projections in the notion of section 1. In section 1, we have seen that  $\rho_1|_{X_N}: X_N \rightarrow X$  is an isomorphism. Hence by passing to the analytification we get that

$$\rho_1|_{\Gamma_N^{an}}: X_N^{an} \rightarrow X^{an}$$

is a bi-holomorphic map. Hence by the change of variables formula we get that the last term of the equation (3.3) is equal to

$$\int_{X_N^{sm, an}} (\rho_2|_{\Gamma_N})^*(\theta\alpha)^{\wedge d}$$

Next we consider the analytification of the closure of  $X_N$  which we call  $\overline{X_N}^{an}$  which is a complex analytic subset of  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ . We note that  $X_N^{sm,an} \subseteq \overline{X_N}^{an}$  and the difference is a proper closed subset of  $\overline{X_N}^{an}$  and hence of dimension strictly smaller than  $d$ . As the integrand in the last display is a  $d$ -form, we get that the integral of it over the boundary is zero and hence

$$\int_{X_N^{sm,an}} (\rho_2|_{\Gamma_N})^*(\theta\alpha)^{\wedge d} = \int_{\overline{X_N}^{an}} (\rho_2|_{\Gamma_N})^*(\theta\alpha)^{\wedge d} \quad (3.4)$$

Now note that  $\alpha$  is the restriction to  $\overline{A}^{an}$  of the Fubini form in the Chern class of  $\mathcal{O}(1,1)$  in  $\mathbb{P}_k^{m,n}$  which we also call  $\alpha$  now and  $\rho_2|_{\Gamma_N}$  also extends to a global morphism from  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$  to  $\mathbb{P}_k^{m,n}$ . Hence the last term of equation (3.4) is equal to

$$\int_{\overline{X_N}^{an}} \rho_2^*(\theta\alpha)^{\wedge d} \leq \int_{\overline{X_N}^{an}} (\rho_2^*\alpha)^{\wedge d} \quad (3.5)$$

where we have the inequality as  $\theta$  takes value in  $[0,1]$  and hence  $\theta \leq 1$ . Now note that  $\rho_2^*\alpha$  is in the Chern class of the line bundle  $\rho_2^*\mathcal{O}(1,1)$  and has compact support in  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$  as the latter is compact. Hence by Theorem 11.21 of [Voi02] we get that in the compact case the integration of the top-wedge and the top-intersection number is the same. Hence we get that the last term of inequality (3.5) is equal to

$$\rho_2^*\mathcal{O}(1,1)^d \cdot [\overline{X_N}] \quad (3.6)$$

Let us denote by  $i_N$  the inclusion of  $\overline{X_N}$  in  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ . Then by-projection formula we have

$$i_N^*(\rho_2^*\mathcal{O}(1,1))^d = \rho_2^*\mathcal{O}(1,1)^d[(i_N)_*[\overline{X_N}]] = \rho_2^*\mathcal{O}(1,1)^d[\overline{X_N}] \quad (3.7)$$

Note that the first term of equation (3.7) is  $\mathcal{F}^d$  as  $\mathcal{F} = \rho_2^*\mathcal{O}(1,1)|_{\overline{X_N}} = i_N^*\rho_2^*\mathcal{O}(1,1)$ . Hence combining equalities and inequalities ((3.7)), (3.6), (3.5), (3.4) and inequality (3.3), we get that  $(\int_{X_N^{sm,an}} \gamma^{\wedge d})/C^d \leq \mathcal{F}^d$ . But then by Inequality (3.1) and equation (3.2), we get that

$$\mathcal{F}^d \geq \frac{\kappa'}{C^d} N^{2d}$$

which finishes the proof by taking  $\kappa = \frac{\kappa'}{C^d}$  and noting that  $\kappa' > 0$  hence  $\kappa > 0$ .  $\square$

*Remark.* Note that it was crucial that the top form  $\omega|_{X_N}^{\wedge d}$  did not vanish identically to have the constant  $\kappa$  to be strictly positive. This fact exhibits the importance of the property of the Betti form that it vanishes on torsion sections and indicates what "bad parts" we should get rid of so that we can hope for the inequality in Theorem 3.3.1.

### 3.3.2 The upper bound on $\mathcal{F}^{d-1} \cdot \mathcal{M}$

We keep the notations from the previous sub-sections. In particular we have the irreducible closed sub-variety  $X$  in  $\mathcal{A}$  of dimension  $d$  and for each  $N$  we have the associated scheme theoretic image of the graph morphism  $X_N$  and its closure  $\overline{X_N}$  which is an irreducible closed projective sub-variety of the tri-projective space  $\mathbb{P}_k^{n,n,m}$ . Finally we had defined  $\mathcal{F} = \mathcal{O}(0,1,1)|_{\overline{X_N}}$  and  $\mathcal{M} = \mathcal{O}(0,0,1)|_{\overline{X_N}}$ . We begin by defining the Hilbert polynomials in multi-projective spaces.

**Definition 3.3.1.** Suppose  $Z$  is an irreducible closed sub-variety in the tri-projective space  $\mathbb{P}_k^{n,n,m}$ , We define the *Hilbert polynomials*  $H_Z$  as

$$H_Z(U, V, W) = \sum_{i+j+k=\dim Z} \binom{\dim Z}{i, j, k} \mathcal{O}(1,0,0)^i \cdot \mathcal{O}(0,1,0)^j \cdot \mathcal{O}(0,0,1)^k [Z] U^i V^j W^k \in \mathbb{Z}[U, V, W]$$

With this definition, we have our main result which is an analogue of Bezout's theorem for multi-projective spaces due to Phillipon.

**Theorem 3.3.4.** *Suppose  $T$  is an irreducible component of the intersection of  $Z$  with tri-homogenous polynomials  $F_0, \dots, F_k$  of tri-degree at most  $(D_1, D_2, D_3)$ . Then*

$$H_T(D_1, D_2, D_3) \leq H_Z(D_1, D_2, D_3)$$

*Proof.* See Proposition 3.3 of [Phi86]. □

Now we come to the main inequality that we prove in this sub-section.

**Lemma 3.3.5.** *Assume  $d \geq 1$ . Then there exists a constant  $c > 0$  depending only on  $X$  and its embeddings into various projective spaces such that*

$$\mathcal{F}^{d-1} \cdot \mathcal{M} \leq c \cdot N^{2(d-1)}$$

when  $N = 2^l$  is a power of two.

*Proof.* We begin by describing a polynomial model for the duplication morphism [2]. Note that the line-bundle  $L$  on the generic fiber  $A$  considered in sub-section 1.1 is even, ample and hence satisfies  $[2]^*L = L^{\otimes 4}$ . Also recall that  $L$  was chosen so that the embedding of  $A$  into  $\mathbb{P}_{k(S)}^n$  is projectively normal. Hence by result of Serre, Corollaire 2 Appendix II of [WBS87] we get homogenous polynomials  $f_0, \dots, f_n$  of degree 4 with no common zeroes in the generic fiber  $A$  such that [2] is represented by these polynomials *i.e.*

$$[2](P) = [f_0(P) : \dots : f_n(P)] \in A \hookrightarrow \mathbb{P}_{k(S)}^n.$$

Then note that we can write

$$f_i = \frac{F_i}{H}$$

where  $F_i$  are bi-homogenous polynomials of bi-degree  $(4, D')$  and  $H$  is homogenous of degree  $D'$ . This comes from the fact that the co-efficients of  $f_i$  are elements of  $k(S) = k(\overline{S})$  and the latter are just quotients of homogenous polynomials of equal degree in  $k[S_0, \dots, S_m]$  as  $S$  is embedded in  $\mathbb{P}_k^m$ . So overall we have that  $F_i(\mathbf{X}, \mathbf{S})$  are bi-homogenous polynomials of bi-degree  $(4, D')$  where  $\mathbf{X}$  are the projective co-ordinates of  $\mathbb{P}_k^n$  and  $\mathbf{S}$  are the projective co-ordinates of  $\mathbb{P}_k^m$  such that  $f_i = \frac{F_i}{H}$  represent [2] on  $A$  for some homogenous polynomial  $H(\mathbf{S})$  of degree  $D'$ .

We consider the intersection of the zero-set of  $H$  in  $\mathbb{P}_k^m$  with  $S$  and call it  $Z'$ . First note that as  $f_i$  are well defined on the generic fiber  $A$ ,  $H$  cannot vanish on all of  $S$ . Hence  $Z'$  is a proper closed subset of  $S$ . We call  $Z = \pi^{-1}(Z')$  which is a proper closed subset of  $\mathcal{A}$ . Furthermore as  $H$  does not vanish on  $\pi(Z)$  and [2] is well-defined on  $A$  via  $f_i$ , by scaling up the co-ordinates  $f_i$ , we can assume that

1. the polynomials  $\{F_i\}$  have no common zero on  $\mathcal{A}/Z(k)$
2. if  $(P, s) \in \mathcal{A}/Z(k) \subset \mathbb{P}_k^n \times \mathbb{P}_k^m(k)$  then

$$[2](P.s) = ([F_0(P, s) : \dots : F_n(P, s)], s)$$

Note furthermore that [2] maps  $\mathcal{A}/Z$  into itself as it is a morphism over  $S$  and  $\pi^{-1}(\pi(Z)) = Z$  by construction of  $Z$ .

Now that we have a polynomial model of [2] on (almost)  $\mathcal{A}$  we want to re-iterate them to give a model of  $[2^l]$ . Let us denote by  $F_0^l(\mathbf{X}, \mathbf{S}) \dots F_n^l(\mathbf{X}, \mathbf{S})$  the polynomials for the description of  $[2^l]$ . Then we see from definition that inductively

$$F_i^{l+1}(\mathbf{X}, \mathbf{S}) = F_i(F_0^l(\mathbf{X}, \mathbf{S}), \dots, F_n^l(\mathbf{X}, \mathbf{S}); \mathbf{S})$$

Now suppose that  $F_i^l$  is bi-homogenous of bi-degree  $(D_l, D'_l)$ . Then from the above inductive definition and the fact that  $F_i = F_i^1$  are bi-homogenous of bi-degree  $(4, D')$  we get by easy algebra that  $F_i^{l+1}$  are bi-homogenous of bi-degree  $(D_{l+1}, D'_{l+1})$  with recursion formula

$$D_{l+1} = 4 \cdot D_l \text{ and } D'_{l+1} = 4 \cdot D'_l + D'$$

From these we deduce that

$$D_l = 4^l \text{ and } D'_l = \frac{4^l - 1}{3} \cdot D' \leq 4^l \cdot D'$$

So both the degrees grow as  $4^l = N^2$  upto some constant factor, a fact that will be of assistance to us. Next we want to view  $\overline{X_N}$  as an irreducible component of the intersection of  $\overline{X} \times \mathbb{P}_k^n$  and some carefully chosen polynomials inside the tri-projective space  $\mathbb{P}_k^{n,n,m}$ . More precisely, we consider  $X \subset \overline{X} \hookrightarrow \mathbb{P}_k^n \times \mathbb{P}_k^m$ . Note that by hypothesis, we have that  $\pi|_X$  is dominant on  $S$ . As  $\pi(Z)$  is a proper closed subset of  $S$  and  $\{F_i\}$  have no common zeroes on  $\mathcal{A}/Z$ , we get that there is some  $i$  such that  $F_i$  doesn't vanish identically on  $X$  and we might assume that  $i = 0$ . Next consider  $\overline{X} \times \mathbb{P}_k^n$  as an irreducible closed sub-variety of  $\mathbb{P}_k^{n,n,m}$  by permuting co-ordinates. We consider the family of  $n$  tri-homogenous polynomials

$$g_j = F_j^l(\mathbf{X}, \mathbf{S})Y_0 - Y_j F_0^l(\mathbf{X}, \mathbf{S})$$

in the tri-projective space  $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$  where  $Y_i$  are the projective co-ordinates of the middle factor and  $j$  varies from 1 to  $n$ . Note that there is an open dense subset  $U$  in  $X$  where  $F_0^l$  does not vanish and suppose  $U_N$  is the image of  $U$  under the graph morphism for  $N = 2^l$ . Then if  $(P, s) \in U(k) \subset \mathbb{P}_k^n \times \mathbb{P}_k^m(k)$  and  $\text{Gr}$  is the graph morphism then

$$\text{Gr}(\mathbf{X}, \mathbf{S}) = ([X_0 : \dots X_n], [F_0^l(\mathbf{X}, \mathbf{S}) : \dots F_n^l(\mathbf{X}, \mathbf{S})], \mathbf{S})$$

and thus we see that  $\text{Gr}(U) = U_N$  satisfies

$$\frac{Y_j}{Y_0} = \frac{F_j(\mathbf{X}, \mathbf{S})}{F_0(\mathbf{X}, \mathbf{S})}$$

and hence we get that  $U_N$  is contained in the intersection of  $\overline{X} \times \mathbb{P}_k^n$  and the tri-homogenous polynomials  $g_j$ . As  $U_N$  is dense in  $\overline{X_N}$  we get that  $\overline{X_N}$  is also contained and hence is an irreducible component of the above intersection.

Now we are in a place to apply Theorem 3.3.4 with  $T = \overline{X_N}$  and  $Z = \overline{X} \times \mathbb{P}_k^n$ . We start with the quantity that we need to bound from above *i.e.*

$$\mathcal{F}^{d-1} \mathcal{M} = \mathcal{O}(0, 1, 1)^{d-1} \mathcal{O}(0, 0, 1) [\overline{X_N}] \quad (3.8)$$

where the equality follows from projection formula. Using linearity of intersection numbers we get that the term above is equal to

$$\sum_{i+j=d-1} \binom{d-1}{i} \mathcal{O}(0, 1, 0)^i \mathcal{O}(0, 0, 1)^{j+1} [\overline{X_N}] \leq \sum_{i+j=d, j>0} \binom{d}{i} \mathcal{O}(0, 1, 0)^i \mathcal{O}(0, 0, 1)^j [\overline{X_N}] \quad (3.9)$$

Note that the index  $j \geq 1$  in the sum of R.H.S above. Hence we can estimate the R.H.S as

$$D'_l \cdot \left( \sum_{i+j=d, j>0} \binom{d}{i} \mathcal{O}(0, 1, 0)^i \mathcal{O}(0, 0, 1)^j [\overline{X_N}] \right) \leq \sum_{i+j=d} \binom{d}{i} \mathcal{O}(0, 1, 0)^i \mathcal{O}(0, 0, 1)^j [\overline{X_N}] (D'_l)^j \quad (3.10)$$

Note that the R.H.S of equality (3.10) consists of summands in the Hilbert polynomial  $H_{\overline{X_N}}(D_l, 1, D'_l)$  (maybe not all of them) as defined in Definition 3.3.1. As all the terms in an Hilbert polynomial is non-negative, we get that the R.H.S of Inequality (3.10) can be bounded from above by

$$H_{\overline{X_N}}(D_l, 1, D'_l) \leq H_{\overline{X} \times \mathbb{P}_k^n}(D_l, 1, D'_l) \quad (3.11)$$

where the above inequality follows from Theorem 3.3.4 by taking  $T = \overline{X_N}$  and  $Z = \overline{X} \times \mathbb{P}_k^n$ . Hence we aim to bound the R.H.S from above.

By definition we have

$$H_{\overline{X} \times \mathbb{P}_k^n}(D_l, 1, D'_l) = \sum_{i+j+k=m+d} \binom{n+d}{i, j, k} \mathcal{O}(1, 0, 0)^i \cdot \mathcal{O}(0, 1, 0)^j \cdot \mathcal{O}(0, 0, 1)^j [\overline{X} \times \mathbb{P}_k^n] \cdot D_l^i \cdot (D'_l)^k \quad (3.12)$$

Now we write the cycle class

$$[\overline{X} \times \mathbb{P}_k^n] = \sum_{i'+k'=n+m-d} a_{i'k'} \mathcal{O}(1, 0, 0)^{i'} \mathcal{O}(0, 0, 1)^{k'}$$

for  $a_{i'k'} \geq 0$  where equality above means rational equivalence of cycle classes. This follows from the fact that the ring of cycle classes (under intersection product) of  $\mathbb{P}_k^n \times \mathbb{P}_k^m$  is generated by  $\mathcal{O}(0, 1)$  and  $\mathcal{O}(1, 0)$  as an algebra and the  $a_{ij}$ 's are positive due to positivity of the cycle class  $[\overline{X} \times \mathbb{P}_k^n]$ . Hence combining the above with Equation (3.12) we get

$$H_{\overline{X} \times \mathbb{P}_k^n}(D_l, 1, D'_l) = \sum_{i'+k'=n+m-d} a_{i'k'} \sum_{i+j+k=n+d} \binom{n+d}{i, j, k} \mathcal{O}(1, 0, 0)^{i+i'} \mathcal{O}(0, 1, 0)^j \mathcal{O}(0, 0, 1)^{k+k'} \cdot D_l^i \cdot (D'_l)^k \quad (3.13)$$

Now note that if  $i + i' \leq n$  then  $\mathcal{O}(1, 0, 0)^{i+i'}$  vanishes, if  $k + k' \leq m$  then  $\mathcal{O}(0, 0, 1)^{k+k'}$  vanishes and if  $j \leq n$ , then  $\mathcal{O}(0, 1, 0)^j$  vanishes. However for each summand above, we have

$$(i + i') + j + (k + k') = 2n + m = n + n + m$$

and hence we get that the summand of Equation (3.13) is non-zero only when  $i + i' = n$ ,  $k + k' = m$  and  $j = n$ . Hence we get that R.H.S of Equation (3.13) is equal to

$$\sum_{i'+k'=n+m-d} a_{i',k'} \cdot \binom{n+d}{n-i', n, m-k'} \mathcal{O}(1, 0, 0)^n \mathcal{O}(0, 1, 0)^n \mathcal{O}(0, 0, 1)^m \cdot D_l^{n-i'} \cdot (D'_l)^{m-k'} \quad (3.14)$$

Note that as  $\mathcal{O}(1, 0, 0)^n \mathcal{O}(0, 1, 0)^n \mathcal{O}(0, 0, 1)^m [\mathbb{P}_k^{n,n,m}] = 1$ , we get that R.H.S of Equality (3.14) is equal to

$$\sum_{i'+k'=n+m-d} a_{i',k'} \cdot \binom{n+d}{n-i', n, m-k'} \cdot D_l^{n-i'} \cdot (D'_l)^{m-k'} \leq \sum_{i'+k'=n+m-d} a_{i',k'} \cdot \binom{n+d}{n-i', n, m-k'} \cdot (4^l)^{n-i'} \cdot (4^l \cdot D')^{m-k'} \quad (3.15)$$

where the inequality above holds as  $D_l \leq 4^l$  and  $D'_l \leq 4^l \cdot D'$  by our recursive formulas above. Now using the fact that  $i' + k' = n + m - d$ , we get that the R.H.S of Inequality (3.16) can be bounded as

$$\sum_{i'+k'=n+m-d} a_{i',k'} \cdot \binom{n+d}{n-i', n, m-k'} \cdot (4^l)^{n-i'} \cdot (4^l \cdot D')^{m-k'} \leq C(n, d) \cdot \left( \sum_{i'+k'=n+m-d} a_{i'k'} \right) \cdot (4^l)^d \cdot D'^m \quad (3.16)$$

where  $C(n, d)$  is an upper bound for the binomial co-efficients  $\binom{n+d}{i, j, k}$  independent of  $N = 2^l$  (for example we can take  $C(n, d) = (n+d)^{n+d}$ ). Note that the sum  $(\sum_{i'+k'=n+m-d} a_{i'k'})$  is also independent of  $N$  as the  $a_{i'k'}$  are the co-efficients of the cycle class  $[\overline{X} \times \mathbb{P}_k^m]$  in a canonical form. Hence we get a

constant  $c' > 0$  independent of  $N$  such that together with Inequalities and equalities (3.10), (3.11), (3.12), (3.13), (3.14) and (3.16), we get that

$$D'_l \cdot \left( \sum_{i+j=d, j>0} \binom{d}{i} \mathcal{O}(0, 1, 0)^i \mathcal{O}(0, 0, 1)^j [\overline{X_N}] \right) \leq c' \cdot (4^l)^d \quad (3.17)$$

where  $c' = C(n, d) \cdot (\sum_{i'+k'=n+m-d} a_{i'k'}) \cdot D'^m$ . Note that  $D'_l = \frac{4^l-1}{3} D' \geq \frac{4^l \cdot D'}{4}$  and hence we get from Equation (3.17) that

$$\sum_{i+j=d, j>0} \binom{d}{i} \mathcal{O}(0, 1, 0)^i \mathcal{O}(0, 0, 1)^j [\overline{X_N}] \leq c \cdot (4^l)^{d-1} = c \cdot N^{2(d-1)} \quad (3.18)$$

where  $c = \frac{4c'}{D'} > 0$  is also independent of  $N$ . Hence combining Inequality (3.18) with Inequality (3.9) and Equality (3.8), we get that

$$\mathcal{F}^{d-1} \cdot \mathcal{M} \leq c \cdot N^{2(d-1)}$$

which finishes the proof.  $\square$

### 3.4 Proof of the Gao-Habegger Height Inequality

In this sub-section, we finally prove the central height inequality in [DGH20] due to Gao and Habegger. We compare the two bounds of opposite nature obtained in two previous sub-sections and apply a theorem of Siu to conclude that a certain difference of line-bundles is big and hence (after possibly multiplying by an integer) has non-zero global sections. Finally we get a lower bound outside the zero locus of one such global section obtaining Theorem 3.3.1.

Let  $X$  be an irreducible closed sub-variety of dimension  $d$ . If  $d = 0$ , then  $X$  is finite being a variety and hence the assertion of Theorem 3.3.1 is obvious. Thus we may assume  $d \geq 1$ . For some  $l \in \mathbb{N}$ , let us set  $N = 2^l$  and  $X_N, \overline{X_N}$  be as in previous subsections. In particular  $\overline{X_N}$  is an irreducible closed sub-variety in  $\mathbb{P}_k^{n,n,m}$  with  $\dim(\overline{X_N}) = d$ .

Let  $\kappa > 0$  and  $c > 0$  be the absolute constants in the inequalities of Lemma 3.3.3 and Lemma 3.3.5 respectively. Let us choose an integer  $q \in \mathbb{N}$  such that

$$0 < \frac{cd}{q} < \kappa \quad (3.19)$$

Then we have

$$d(\mathcal{M}^{\otimes N^2} \cdot (\mathcal{F}^{\otimes q})^{d-1}) = dq^{d-1} N^2 (\mathcal{M} \cdot \mathcal{F}^{d-1}) \leq dq^{d-1} N^2 (cN^{2(d-1)}) = \left(\frac{cd}{q}\right) \cdot q^d N^{2d} < q^d (\kappa N^{2d}) \leq q^d \mathcal{F}^d = (\mathcal{F}^{\otimes q})^d \quad (3.20)$$

Hence comparing the first and the last terms in the chain of inequalities in (3.20) we can apply the Theorem of Siu as in Theorem 2.2.13 of [Laz17] to conclude that the line bundle

$$\mathcal{G} = \mathcal{F}^{\otimes q} - N^2 \mathcal{M}$$

is a big line-bundle. Once again by using Lemma 2.2.3 of [Laz17] we can find an integer  $c'(N)$  (dependent on  $N$ ) such that  $\mathcal{G}^{\otimes c'(N)}$  has a non-zero global section. Let us call by  $U_N$  the complement of the zero-locus of this global section in  $\overline{X_N}$ . Then by elementary Height theory, we get that

$$q \cdot (h_{\mathcal{F}}(P) - \frac{N^2}{q} h_{\mathcal{M}}(P)) = h_{\mathcal{F}^{\otimes q}}(P) - N^2 h_{\mathcal{M}}(P) = h_{\mathcal{F}^{\otimes q} - N^2 \mathcal{M}}(P) = h_{\mathcal{G}}(P) \text{ for all } P \in U_N(\overline{k}) \quad (3.21)$$

and

$$c'(N) h_{\mathcal{G}}(P) = h_{\mathcal{G}^{\otimes c'(N)}}(P) \geq s(N) \text{ for all } P \in U_N(k) \quad (3.22)$$

for some positive constant  $s(N)$  dependent  $N$  where (3.22) follows as  $U_N(k)$  lies outside the support of a certain global section. Combining Equalities in (3.21) and the inequality (3.22) we get that

$$q \cdot (h_{\mathcal{F}}(P) - \frac{N^2}{q} h_{\mathcal{M}}(P)) \geq \frac{s(N)}{c'(N)} \quad (3.23)$$

Hence rearranging the terms, we get that

$$h_{\mathcal{F}}(P) \geq c_1 N^2 h_{\mathcal{M}}(P) + c_2(N) \quad (3.24)$$

where  $c_1 = \frac{1}{q}$  and  $c_2(N) = \frac{s(N)}{qc'(N)}$ . Note that  $q$  and hence  $c_1$  is independent of  $N$  as  $\kappa$ ,  $d$  and  $c$  are independent of  $N$ . On the other hand  $c_2(N)$  is dependent on  $N$ . Now recalling the descriptions of  $h_{\mathcal{F}}$  and  $h_{\mathcal{M}}$  as in the end of sub-section 3.1.3 of section 3.1, we get that

$$h_{na}([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N) \text{ for all } P \in U_N(\bar{k}) \quad (3.25)$$

by replacing  $c_2(N)$  with  $-c_2(N)$  and hence proving Theorem 3.3.1.



## Chapter 4

# Uniformity of Cardinality in a pencil of curves

In this chapter, we show an uniform upper bound on the number of  $k$ -rational points in a family of curves  $\mathcal{C}$  parameterized by a smooth curve  $S$ . We will closely follow the treatment in [DGH19], one of the earlier works of the same authors pre-dating [DGH20]. The main improvement in the second paper from the first one is that the authors were able to generalise to parameter spaces of arbitrary dimension and not only curves. As a result they could show this uniformity over any curve (above a certain absolute modular height) by simply taking the parameter space to be the fine Moduli space of principally polarized abelian varieties with level  $l$  structure.

One of the crucial steps in proving the Gao Habegger inequality( Theorem 3.3.1 is described in section 5 of [GH19]. This step involves showing that when the base space  $S$  is one-dimensional, an irreducible sub-variety  $X$  of  $\mathcal{A}$  is "degenerate" if and only if it is "generically special" as in the sense defined in [GH19]. Unfortunately more work is required in the case of higher dimensions. To avoid going into moduli spaces, we treat the case of one dimensional parameter space in this thesis and we refer to [DGH20] for the case of parameter spaces of arbitrary dimension.

### 4.1 The main theorem

In this section, we state the main theorem that we are going to prove regarding the uniform bound on a pencil of curves in this chapter. Suppose we have an irreducible, quasi-projective smooth projective variety  $\mathcal{C}$  defined over  $k$  along with a smooth morphism  $\pi: \mathcal{C} \rightarrow S$  such that the geometric fibers  $\pi^{-1}(s) = \mathcal{C}_s$  are all smooth projective curves of genus  $g \geq 2$  and  $S$  is smooth, irreducible curve defined over  $k$ . Then  $S$  is embedded in some projective space and corresponding to this embedding, we get a canonical Weil height which we denote by  $h$ . With the following set up we have the following theorem

**Theorem 4.1.1.** *There is a constant  $c > 0$  depending on  $\mathcal{C}$ ,  $\pi$  and the embedding of  $S$  into a projective space with the following property:*

*Suppose  $s \in S(k)$  with  $h(s) \geq c$  and  $\mathcal{C}_s$  is embedded into its Jacobian  $\text{Jac}(\mathcal{C}_s)$  via a  $k$ -point  $P_0$ . Furthermore, suppose  $\Gamma$  is a finitely generated subgroup of  $\text{Jac}(\mathcal{C}_s)$  of rank  $r$ . Then*

$$\#((\mathcal{C}_s(k) - P_0) \cap \Gamma) \leq c^{1+r}$$

The above theorem gives an upper bound on a family of curves in terms of a constant which is independent of  $s$ . In works( [DGH20]) the authors generalise this to higher dimensional base spaces and generalise the uniform bound to all curves instead of pencils. In this chapter we will focus on proving a different version of this theorem which we state next.

**Theorem 4.1.2.** *Suppose  $\pi: \mathcal{A} \rightarrow S$  be an abelian scheme and  $\mathcal{C}$  be an irreducible closed sub-variety of  $\mathcal{A}$  which dominates  $S$  and has a geometrically irreducible smooth curve as generic fibre. Furthermore we assume that  $S$  is a smooth, irreducible curve over  $k$ . Then there exists a positive constant  $c = c(\mathcal{C}, S, \pi)$  such that the following holds:*

*If  $s \in S(k)$  with  $h(s) \geq c$ , then  $\mathcal{C}_s$  is integral. Moreover if  $\mathcal{C}_s$  is not a coset in  $\mathcal{A}_s$  and  $\Gamma$  is finitely generated subgroup of  $\mathcal{A}_s$  of rank  $r$ , then*

$$\#(\mathcal{C}_s(k) \cap \Gamma) \leq c^{1+r}$$

Theorem 4.1.1 looks very similar to Theorem 4.1.2 and follows from the later rather un-complicatedly but still some arguments are needed which we add at the end of the chapter. For now we focus on proving Theorem 4.1.2 simply because the set-up is the same as we have worked in till now in this article.

We make a third and final reduction before moving on to the next sub-section. Suppose we denote by  $\mathcal{C}_\eta$  to be the generic fiber of  $\mathcal{C}$ . This is an irreducible closed sub-variety of the generic fiber  $A = \mathcal{A} \times_S k(S)$ . If the base change  $(\mathcal{C}_\eta)_{\overline{k}}$  is contained in a translate of  $A^{k(S)/k} \times_k \overline{k(S)}$  (now viewed as an abelian sub-variety of  $A$ ) then Theorem 4.1.2 can be tackled as indicated in paragraph 5.2 on page 13 of [DGH19]. Hence from this point onwards, we will also assume that

$$(\mathcal{C}_\eta)_{\overline{k}} \text{ is not contained in any translate of the trace } A^{k(S)/k} \otimes_k \overline{k(S)} \quad (4.1)$$

## 4.2 Lower bound on fiberwise Neron-Tate height

We begin by combining the results of Chapter 1 and 2 to obtain a lower bound on the fiber-wise Neron-Tate height in terms of the height of the base space. Let us have an abelian scheme  $\mathcal{A} \rightarrow S$  as before where  $S$  is a quasi-projective variety defined over  $k = \overline{\mathbb{Q}}$ . We do not assume  $\dim(S) = 1$  yet for this section. Hence we have both Theorem 2.1.1 and Theorem 3.3.1 at our disposal and combining them we have

**Lemma 4.2.1.** *Suppose  $\mathcal{A} \rightarrow S$  be as above and  $\mathcal{X}$  is an irreducible closed sub-variety of  $\mathcal{A}$  dominating  $S$  with*

$$(\omega|_{\mathcal{X}^{sm,an}})^{\wedge d} > 0$$

*at some smooth point  $P \in X^{sm,an}(k)$ ,  $\omega$  is the Betti form and  $d = \dim(X)$ . Then we have a constant  $C > 0$  and a Zariski open dense subset  $U$  of  $\mathcal{X}$  such that*

$$h(\pi(P)) \leq C(1 + \hat{h}_{\mathcal{A}}(P)) \text{ for all } P \in U(k)$$

*Proof.* We can apply Theorem 2.1.1 and let  $c_0$  be the constant  $c$  mentioned in that theorem. Then we have

$$|\hat{h}_{\mathcal{A}}(P) - h_{na}(P)| \leq c_0(1 + h(\pi(P))) \Rightarrow \hat{h}_{\mathcal{A}}(P) \geq h_{na}(P) - c_0(1 + h(\pi(P))) \text{ for all } P \in \mathcal{X}(k) \quad (4.2)$$

Hence applying inequality (4.2) with  $[N]P$  instead of  $P$  where  $N = 2^l$  is a power of 2, we get that there is a Zariski open dense subset  $U_N$  of  $X$  such that

$$\hat{h}_{\mathcal{A}}([N]P) \geq h_{na}([N]P) - c_0(1 + h(\pi(P))) \Rightarrow \hat{h}_{\mathcal{A}}([N]P) \geq c_1 N^2 h(\pi(P)) - c_0(1 + h(\pi(P))) - c_2(N) \quad (4.3)$$

for all  $P \in U_N(k)$  where in the implication, we used Theorem 3.3.1 and the constants  $c_1$  and  $c_2(N)$  are from the theorem (note that all our hypotheses are satisfied). Hence re-writing Equality 4.3, we get

$$\hat{h}_{\mathcal{A}}([N]P) = N^2 \hat{h}_{\mathcal{A}}(P) \geq (c_1 N^2 - c_0) h(\pi(P)) - (c_0 + c_2(N)) \quad (4.4)$$

$$\Rightarrow \hat{h}_{\mathcal{A}}(P) \geq (c_1 - \frac{c_0}{N^2}) h(\pi(P)) - \frac{c_0 + c_2(N)}{N^2} \quad (4.5)$$

where in the first equality we used the homogeneity of the Neron-Tate height as  $\mathcal{L}$  has even ample fibers on closed points. Now we choose  $N$  to be a power of 2 with  $N^2 \geq \frac{2c_0}{c_1} \Rightarrow \frac{c_0}{N^2} \leq \frac{c_1}{2}$  and we fix this  $N$  and the open subset  $U_N$  of  $X$ . Applying this to Inequality (4.4), we get

$$\hat{h}_{\mathcal{A}}(P) \geq \frac{c_1}{2} h(\pi(P)) - \frac{c_0 + c_2(N)}{N^2} \text{ for all } P \in U_N(k) \quad (4.6)$$

Then we are finished from inequality (4.6) by taking  $C = \max\{\frac{c_1}{2}, |\frac{c_0 + c_2(N)}{N^2}|\}$  and  $U = U_N$ .  $\square$

### 4.3 Degenerate and Generically Special Sub-varieties

In this section, we give two definitions and an equivalence between these two assuming that the base space is a smooth curve. We will refer to [GH19] for proofs and definitions mostly.

Let  $\pi: \mathcal{A} \rightarrow S$  be an abelian scheme as before and  $S$  is a smooth affine curve over  $k = \overline{\mathbb{Q}}$ . We have the following definition

**Definition 4.3.1.** Let  $X$  be an irreducible closed sub-variety of  $\mathcal{A}$  of dimension  $d$ .  $X$  is said to be *non-degenerate* if

$$(\omega|_{X^{sm,an}})^{\wedge d} \neq 0$$

where  $\omega$  is the Betti form.  $X$  is called *degenerate* if it is not non-degenerate.

*Remark.* Note that  $(\omega|_{X^{sm,an}})^{\wedge d} \neq 0$  implies that  $(\omega|_{X^{sm,an}})^{\wedge d} > 0$  at some smooth point  $P$  and hence the hypothesis of Theorem 3.3.1 is satisfied.

Next we define the Chow  $K/k$ -trace of an abelian variety defined over  $K$  and  $K$  is the function field of a smooth projective variety defined over  $k$ .

**Definition 4.3.2.** Suppose  $K = k(S)$  is the function field of an irreducible quasi-projective variety  $S$  defined over  $k$  and let  $A$  be an abelian variety defined over  $\overline{K}$ , the algebraic closure of  $K$ . Then we define the Chow  $\overline{K}/k$ -trace of  $A$  as the final object in the category of pairs  $(H, i)$  where  $H$  is an abelian variety defined over  $k$  and  $i$  is a homomorphism of abelian varieties from  $H \times_k \overline{K}$  to  $A$ . We denote the trace by  $A^{\overline{K}/k}$ .

*Remark.* The trace can be defined by exactly the same way for any *primary* extension of fields  $K/k$  but for our purposes, it is enough to consider the case of a function which is primary automatically over  $k$ . Moreover when  $\text{char}(k) = 0$ , we have that the canonical morphism  $i$  from  $A^{\overline{K}/k} \times_k \overline{K}$  to  $A$  is actually a closed immersion and hence we can treat the trace as a closed sub-variety of  $A$  via  $i$ . For these details and more, refer to Brian Conrad's notes [Con06].

Next we give a second definition for the sub-variety  $\mathcal{X}$  which will eventually be equivalent to degeneracy for one dimensional base.

**Definition 4.3.3.** Let  $\mathcal{A} \rightarrow S$  and  $\mathcal{X}$  be as above. Then  $\mathcal{X}$  is said to be *generically special* if the geometric generic fibre  $X = \mathcal{X} \times_S \overline{k(S)}$  can be written as

$$X = \cup_{i=1}^t Z_i \times_k \overline{k(S)} + B_i + P_i$$

where  $A = \mathcal{A} \times_S \overline{k(S)}$  is the geometric generic fibre of  $\mathcal{A}$ ,  $Z_i$  is an irreducible closed sub-variety of  $A^{\overline{k(S)}/k}$ ,  $B_i$  is an abelian sub-variety of  $A_{\overline{k(S)}}$  and  $P_i \in A(\overline{k(S)})$  is a torsion point. For any arbitrary closed sub-variety  $\mathcal{X}$ , we define

$$\mathcal{X}^* = \mathcal{X} / \cup_{Z \in \text{GS}(\mathcal{X})} Z$$

where  $\text{GS}(\mathcal{X})$  is the set of generically special sub-varieties of  $\mathcal{A}$  contained in  $\mathcal{X}$ .

*Remark.* An important aspect of the construction is that  $\mathcal{X}^*$  is Zariski open. However it is far from obvious from the immediate definitions. This follows as a consequence of a uniform version of the Manin-Mumford conjecture. For more details, we refer to [GH19].

Having defined the above two notions, we next state that when  $\dim(S) = 1$ , then the above notions are equivalent. More precisely

**Theorem 4.3.1.** *Suppose  $\mathcal{A}$ ,  $S$  and  $\mathcal{X}$  be as above such that  $\mathcal{X}$  dominates  $S$ . Assume furthermore that  $S$  is a curve. Then  $\mathcal{X}$  is degenerate if and only if it is generically special.*

*Proof.* For the proof, we refer to Theorem 5.1 of [GH19].  $\square$

*Remark.* The equivalence of Theorem 4.3.1 gives us a way to translate the geometric notion of *degeneracy* completely into algebraic terms using generical speciality. Degeneracy allows us to use the Gao-Habegger inequality on an open subset of the image of the Faltings-Zhang map whereas being generically special helps us to show that the image of the Faltings Zhang map itself is not generically special and hence that the open set on which we apply Gao-Habegger inequality is actually non-empty. Thus with the help of this equivalence we can use Theorem 3.3.1 non-trivially to obtain a contradiction

## 4.4 Two Reduction steps

In this section, we take a quick detour into two reduction steps which allows us to simplify things for us. We note that we can always assume that  $S$  is an affine curve by excluding finitely many points from the curve  $S$ . The first goal of this section is to show that we can make a finite base change from  $S$  to a smooth, affine curve  $S'$  over  $S$  and it is enough to prove Theorem 4.1.2 for the base change. Suppose  $S' \rightarrow S$  be a finite morphism and  $S'$  is a smooth, irreducible, affine curve. Then note that  $K' = k(S')$  is a finite extension of  $K = k(S)$ . More over if  $s' \in S'(k)$  is a closed point lying over  $s \in S(k)$ , then we can easily see that the fibres  $\mathcal{A}_s$  and  $(\mathcal{A} \times_S S')_{s'}$  are isomorphic and hence can be identified. Moreover as the morphism  $\mathcal{A}' = \mathcal{A} \times_S S' \rightarrow S'$  is proper and smooth (being base changes of the abelian scheme  $\mathcal{A} \rightarrow S$ ) we get that  $\mathcal{A}' \rightarrow S'$  is also an abelian scheme as each fiber is an abelian variety. Similarly we can identify the fibers  $\mathcal{C}_s$  and  $(\mathcal{C} \times_S S')_{s'}$  for  $s'$  lying over  $s$ .

Suppose  $A$  is the generic fiber of  $\mathcal{A}$  which is an abelian variety over  $K = k(S)$  and  $A_{\overline{K}} = A \times_K \overline{K}$  is the base change of  $A$  to  $\overline{K}$ . Now by Theorem 2.4 of [Sil92], we can find a finite extension  $K'/K$  such that all elements of  $\text{Hom}(A_{\overline{K}}^{\overline{K}/k} \otimes_k \overline{K}, A_{\overline{K}})$  is defined over  $K'$  (we obtain  $K'$  by adjoining the co-ordinates of all 3-torsions of  $A_{\overline{K}}$ ). Now we can assume  $S$  is a smooth-affine curve and hence we get another smooth affine curve  $S'$  finite over  $S$  which corresponds to the extension  $K'/K$  which we obtain by just taking the normalisation of  $S$  in  $K'$ . Then we have the following lemma

**Lemma 4.4.1.** *We can find a finite affine, irreducible non-singular curve  $S'$  over  $S$  such that the following holds;*

*Suppose we have shown Theorem 4.1.2 for  $\mathcal{C}' = \mathcal{C} \times_S S'$  as irreducible sub-variety of the abelian scheme  $\mathcal{A}' \rightarrow S'$ . Then the theorem is also true for  $\mathcal{C}$ ,  $\mathcal{A}$  and  $S$  and*

$$\text{All elements of } \text{Hom}(A_{\overline{K}}^{\overline{K}/k} \otimes_k \overline{K}, A_{\overline{K}}) \text{ are defined over } K' = k(S') \quad (4.7)$$

*Proof.* First we note that  $S'$  and  $S$  are embedded in some projective spaces  $\mathbb{P}_k^m$  and  $\mathbb{P}_k^n$ . Then by using Lemma 2.5.1, we can find a constant  $C > 0$  such that

$$h_S(s) \leq C h_{S'}(s') \quad (4.8)$$

for all  $s \in S(k)$  and  $s' \in S'(k)$  lying over  $s$ . Now suppose we have proved Theorem 4.1.2 for the base change with constant  $c$ . Then consider the constant  $c_S = c \cdot C$ . If  $h_S(s) \geq c_S$ , then by inequality (4.8),

we get that  $h_{S'}(s') \geq c$ . Note that from Lemma 37.25.1 of [Stab] we deduce that as all the fibers  $\mathcal{C}'_{s'} = \mathcal{C}_s$  are irreducible, we get that the generic fiber  $\mathcal{C}'_\eta$  is also irreducible. This in turn implies that  $\mathcal{C} \times_S S'$  is irreducible as  $\mathcal{C} \times_S S' \rightarrow S'$  is dominant and hence the generic fiber  $(\mathcal{C}_\eta)_{K'}$  is dense in  $\mathcal{C} \times_S S'$  and the closure of an irreducible space is irreducible. Moreover we can assume  $(\mathcal{C}_\eta)_{K'}$  is geometrically irreducible by point 7 of Lemma 37.22.8 of [Staa] (by possibly excluding finitely many points and then passing to a finite base change again of  $S'$ ). Hence all the hypotheses are satisfied. Now the rest of the theorem clearly follows by identification of the fibers  $\mathcal{C}_s$  and  $\mathcal{C}'_{s'}$  and by changing the constant  $c$  to  $c_S$  thus we obtain the first part about the reduction. The part about the homomorphisms clearly follows from the discussion before the Lemma.  $\square$

As an upshot of this lemma, we will make a reduction. Note that the assumption 4.7 in particular implies that the canonical morphism

$$A^{K/k} \otimes_k \overline{K} \rightarrow A_{\overline{K}}^{\overline{K}/k} \otimes_k \overline{K}$$

is an isomorphism, Indeed by uniqueness of universal properties, we just need to produce an unique map in the opposite direction using the universal property. Indeed by assumption (4.7) the canonical morphism

$$A_{\overline{K}}^{\overline{K}/k} \otimes_k \overline{K} \rightarrow A_{\overline{K}}$$

comes as a base change of a homomorphism

$$A_{\overline{K}}^{\overline{K}/k} \otimes_k K \rightarrow A$$

and the above homomorphism in turn yields a map

$$A_{\overline{K}}^{\overline{K}/k} \otimes_k K \rightarrow A_{\overline{K}}^{\overline{K}/k} \otimes_k K$$

by universal property of traces. The base change of the above homomorphism to  $\overline{K}$  gives us the required inverse homomorphism. As by Lemma 4.4.1 we can change to  $S'$  from  $S$ , we can assume in particular that the trace of  $A$  does not increase if we pass to a finite extension of  $K$ .

Now we come to the second reduction. For reasons to be clear later, we want that the special fibers  $\mathcal{C}_s$  on which we do estimates should not be contained in proper cosets of the abelian variety  $\mathcal{A}_s$ . It turns out that this is a "generic property" *i.e.* it is enough to show that the generic fiber  $\mathcal{C}_\eta$  is not contained in a proper coset. This is the reduction that we make in this section.

So on the contrary, suppose that  $\mathcal{C}_\eta$  is contained in a proper coset of  $A$ . Hence there exists a finite extension  $K'/K$  such that

$$(\mathcal{C}_\eta)_{K'} \subseteq P + A'$$

where  $A'$  is an abelian sub-variety of  $A_{K'}$  and  $P \in A'(K')$ . There is a smooth affine curve  $S'$  finite over  $S$  which corresponds to the extension  $K'/K$  (the normalisation of  $S$  in  $K'$ ) We consider the Neron model of  $A'$  over  $S'$  and call it  $\mathcal{A}'$ . Then note that as  $A'$  is an abelian sub-variety of  $A_{K'}$  which is the generic fiber of the abelian scheme  $\mathcal{A} \times_S S'$ , by the Neron mapping property we get that there is a closed immersion (after possibly excluding finitely many points from  $S$ )

$$\mathcal{A}' \hookrightarrow \mathcal{A} \times_S S'$$

because they induce a closed immersion on the generic fibers and hence they induce a closed immersion on the fibers of some open set of  $S'$  by Theorem 8.10.5 in [Gro66]. Note that  $(\mathcal{C}_\eta)_{K'} - P$  is contained in  $A'$  and hence we can consider the Zariski closure of it in  $\mathcal{A}'$  and call it  $\mathcal{C}'$ . We claim that  $\mathcal{A}'$  is an abelian scheme over  $S'$  except for finitely many points in  $S'$ .

**Lemma 4.4.2.** *Suppose  $\mathcal{A}'$  and  $S'$  be as above. Then there is an open subset  $U'$  of  $S'$  such that*

$$\mathcal{A}'_{U'} \rightarrow U'$$

*is an abelian scheme.*

*Proof.* Note that  $\mathcal{A}'$  is the Neron model of the abelian variety  $A'$  over  $k(S')$  and hence it is a smooth commutative group scheme over  $S'$  with generic fiber  $A'$ . Moreover the generic fiber  $A'$  being an abelian variety, it is geometrically irreducible. Hence by Lemma 37.25.5 of [Stab], we get that there is an open set  $U'$  of  $S'$  such that  $\mathcal{A}'$  has geometrically irreducible (and hence geometrically connected and hence an abelian variety) special fibers for all  $s$  in  $U$ . Hence we just need that  $\mathcal{A}'_{U'} \rightarrow U'$  is proper. Clearly it is enough to show that  $\mathcal{A}' \rightarrow S'$  is proper. Thus we need to show that  $A'$  has good reduction at all points  $v$  of  $S'$ . But note that  $A'$  is an abelian sub-variety of  $A_{K'}$ .  $A_{K'}$  is the generic fiber of  $\mathcal{A} \times_S S'$  which is an abelian scheme over  $S'$  being the base change of the abelian scheme  $\mathcal{A} \rightarrow S$ . Hence  $A_{K'}$  has good reduction at all places  $v$  of  $S'$ . Thus we also get that  $A'$  also has good reduction at all places  $v$  of  $S'$  by Corollary 3 of [Joh] just by considering the exact sequence

$$0 \rightarrow A' \rightarrow A_{K'} \rightarrow A_{K'}/A' \rightarrow 0$$

which completes the proof and shows that  $\mathcal{A}'_{U'} \rightarrow U'$  is an abelian scheme.  $\square$

Note that as the coset was considered proper, we have  $\dim(A') < \dim(A)$  and thus by induction on dimension, we can assume that we can show the theorem for the surface  $\mathcal{C}'_{U'}$  contained in the abelian scheme  $\mathcal{A}'_{U'}$  over  $U'$  which is also a smooth affine curve (the zero dimensional case is obvious). Next as  $\mathcal{A}'$  is the Neron model of  $A'$ , we get that

$$A'(K') \cong \mathcal{A}'(S')$$

and hence we can find a section  $\sigma_P$  corresponding to  $P$  of  $\mathcal{A}'$  over  $S$ . Then note that we have a natural identification

$$\mathcal{C}'_{s'} = (\mathcal{C}_{S'})_{s'} - \sigma_{s'} \quad (4.9)$$

where  $\sigma_{s'}$  is the point induced by  $\sigma_P$  at  $s' \in U'(k)$ . Suppose  $c$  is the constant of Theorem 4.1.2 applied to  $\mathcal{A}'_{U'} \rightarrow U'$  and let  $\Gamma$  be a subgroup of rank  $r$  in  $\mathcal{A}_s(k)$  for some  $s \in S$ . Consider the subgroup  $\Gamma_w = \Gamma + \sigma_{s'}$  which is finitely generated of rank at-most  $(1+r)$ . Then  $\Gamma_1 = \Gamma_w \cap \mathcal{A}'_{s'}$  is a subgroup of  $\mathcal{A}'_{s'}$  of rank at-most  $1+r$  (note that  $\mathcal{A}'_{s'}$  is a subgroup) for some  $s' \in S'(k)$  lying over  $s$ . Now suppose that  $s' \in U'(k)$ , then using the identification (4.9) we get

$$\#((\mathcal{C}_{S'})_{s'} - \sigma_{s'}) \cap \Gamma_1 = \#(\mathcal{C}'_{s'} \cap \Gamma_1) \leq c^{2+r} \quad (4.10)$$

as  $\text{Rk}(\Gamma_1) \leq 1+r$ . But note that as  $\sigma_{s'} \in \Gamma_w$ , we get that

$$((\mathcal{C}_{S'})_{s'} - \sigma_{s'}) \cap \Gamma_w = (\mathcal{C}_{S'})_{s'} \cap \Gamma_w \Rightarrow \#((\mathcal{C}_{S'})_{s'} - \sigma_{s'}) \cap \Gamma_w = \#(\mathcal{C}_{S'})_{s'} \cap \Gamma_w \leq c^{2+r}$$

Now note that

$$P \in \mathcal{C}_s \cap \Gamma \Rightarrow P \in \mathcal{C}_s \cap \Gamma_w \Rightarrow P \in (\mathcal{C}_s - \sigma_{s'}) \cap \Gamma_w \Rightarrow P \in \mathcal{C}'_{s'} \cap \Gamma_1$$

where in the second implication we used  $\sigma_{s'} \in \Gamma_w$  and in the last implication we used the identification we used (4.9) noting  $(\mathcal{C}_{S'})_{s'} = \mathcal{C}_s$  for  $s'$  lying over  $s$ . Hence we get using (4.10) that

$$\#(\mathcal{C}_s \cap \Gamma) \leq \#(\mathcal{C}'_{s'} \cap \Gamma_1) \leq c^{2+r} \quad (4.11)$$

whenever  $s$  is in the image of  $U'(k) \subseteq S'(k)$ . We can obtain a  $C_1$  large enough so that  $h(s) \geq C_1$  will imply that  $s$  is in the image of  $U'(k)$  because the complement of  $U'$  in  $S'$  is finite. Hence by changing  $c$  to  $\max\{c^2, C_1\}$  we get Theorem 4.1.2 for  $\mathcal{C}$ . Hence finally we can assume

$$C_\eta \text{ is not contained in any proper coset of } A \quad (4.12)$$

## 4.5 Bound on Large points

In this section, we collect some results due to Mumford and Vojta which then combine to give a bound on the cardinality of "large points" *i.e.* points whose Neron Tate heights are larger than a given cut-off. It turns out that the bound is uniform. The height cut-off in the definition of largeness can be shown bounded above linearly by the Weil height of the base space when we look at a pencil of curves. This crucial fact will later be used to bound the total number of points (both small and large).

Suppose  $A$  is an abelian variety defined over a number field  $k$  equipped with a symmetric and ample line bundle. We can assume hence that  $A$  is immersed in some projective space  $\mathbb{P}_k^n$  as a projectively normal sub-variety. There is the canonical Weil height on  $\mathbb{P}_k^n$  which we call  $h$  and its restriction gives a height function on  $A$ . By Tate's limiting process, we get that the canonical Neron Tate height  $\hat{h}$  and by the theory of heights, we get a constant  $c_{NT} \geq 0$  such that

$$|\hat{h}(P) - h(P)| \leq c_{NT} \text{ for al } P \in A(k)$$

Furthermore, the addition morphism from  $A \times A$  to  $A$  is defined by bi-homogenous polynomials and let  $h_1$  be the logarithmic height of these polynomials as defined in [BG06]. Next we define the bi-linear form associated to  $\hat{h}$  as

$$\langle P, Q \rangle = \frac{1}{2}(\hat{h}(P + Q) - \hat{h}(P) - \hat{h}(Q))$$

It can be shown (as in [BG06]) that this bi-linear form induces an inner product  $|P| = \hat{h}(P)^{\frac{1}{2}}$ . Suppose  $C$  is an irreducible curve contained in  $A$  and let  $A$  be defined over a number field of degree  $n$ . Then , we have

**Theorem 4.5.1** (Vojta, Remond). *There exists a constant  $c$  dependent only on  $n$  such that the following holds:*

*Let  $c_1 = cdeg(C)^2$  and  $c_2 = cdeg(C)^6$ . Suppose  $C$  is not a coset in  $A$ . Suppose  $P, Q \in C(k)$*

$$\langle P, Q \rangle \geq (1 - \frac{1}{c_1})|P||Q| \text{ and } |Q| \geq c_2|P|$$

*Then the point  $P$  must satisfy*

$$|P|^2 \leq cdeg(C)^{20} \max\{1, h(C), c_{NT}, h_1\}$$

*where  $h(C)$  is the height of the sub-variety  $C \subseteq \mathbb{P}_k^n$  as defined in [BGS94] and  $deg(C)$  is the degree as a sub-variety of  $\mathbb{P}_k^n$ .*

We need another result due to Mumford which, in a way, acts as a counter of the above and these two bounds of opposite nature give us the bound on the cardinality. Before stating the theorem we quickly define the concept of a stabilizer which will also come handy in later sections.

**Definition 4.5.1.** Suppose  $X$  is a sub-variety of an abelian variety  $A$  defined over  $k$ . Then we define the stabliser  $\text{Stab}(C)$  as

$$\text{Stab}(C) = \cap_{P \in C(k)} (C - P) = \{P \in A(k) \mid C + P \subseteq C\}$$

Note that from the first definition it is clear that  $\text{Stab}(C)$  is a sub-variety of  $A$  and from the second description, we get that it is an algebraic sub-group of  $A$ . With this we can state the result due to Mumford.

**Theorem 4.5.2.** *Assume the same hypothesis as Theorem 4.5.1 and let  $c_1 = cdeg(C)^2$ . Then there exists a constant  $c$  dependent only on  $n$  such that that following holds:*

*If  $P, Q \in C(k)$  such that  $P - Q \notin \text{Stab}(C)(k)$  and*

$$\langle P, Q \rangle \geq (1 - \frac{1}{c_1})|P||Q| \text{ and } ||P| - |Q|| \leq \frac{1}{cdeg(C)}|P|$$



Then

$$|P|^2 \leq c \deg(C)^3 \max\{1, h_1, c_{NT}, h(C)\}$$

These two tools can be used together with some elementary arguments to obtain the following corollary

**Corollary 4.5.3.** *There exists a constant  $c$  dependent only on  $n$  and  $\deg(C)$  such that the following holds:*

*Suppose  $C$  is not a coset in  $A$  and  $\Gamma$  is a finitely generated subgroup of  $A(k)$  of rank  $r$ . Then*

$$\#\{P \in C(k) \cap \Gamma \mid |P|^2 \geq c \max\{1, h_1, c_{NT}, h(C)\}\} \leq c^r$$

*Proof.* For a proof using Mumford and Vojta's results, we refer to Corollary 2.3 in [DGH19].  $\square$

Finally we generalise the above corollary to a family of curves inside a family of abelian varieties, Suppose we have an abelian scheme  $\pi: \mathcal{A} \rightarrow S$  with a closed immersion of  $\mathcal{A}$  inside  $\mathbb{P}_k^n \times S$  as described before. We have the Neron-Tate heights fiberwise  $\hat{h}$  and we have the Weil height on the base space curve  $S$  coming from an embedding of  $S$  into some  $\mathbb{P}_k^m$ . Then we have

**Corollary 4.5.4.** *Suppose  $\mathcal{C}$  is an irreducible closed sub-variety of  $\mathcal{A}$  which dominates  $S$  and has a geometrically irreducible generic fibre. Then there exists a constant  $c$  dependent on  $\pi, \mathcal{C}$ , the embedding of  $S$  and  $n$  such that the following holds:*

*Suppose  $s \in S(k)$  and  $\Gamma$  is a finitely generated sub-group of  $\mathcal{A}_s(k)$  of rank  $r$ . Then*

$$\#\{P \in \mathcal{C}_s(k) \cap \Gamma \mid \hat{h}(P) > c \max\{1, h(s)\}\} \leq c^r$$

*Proof.* This follows from Corollary 4.5.3 by noting that all the terms  $h_1, h(C)$  and  $c_{NT}$  are bounded above by  $h(s)$  linearly. This assertion in turn follows from the Arithmetic Bezout theorem. We refer to [DGH19].  $\square$

## 4.6 A Preliminary Lemma

In this section, we prove certain lemmas considering abelian varieties and closed irreducible curves embedded in them. Suppose  $A$  is an abelian variety defined over  $k = \overline{\mathbb{Q}}$  and  $C$  is a closed irreducible curve contained in  $A$  (although the arguments here work for all algebraically closed fields  $k$ ). We define the Faltings-Zhang map as

$$F: A \times A \times A \rightarrow A \times A$$

$$F(P_0, P_1, P_2) = (P_1 - P_0, P_2 - P_0)$$

Later we are going to use these lemmas on the fibres  $\mathcal{X}_s = F(\mathcal{C}_s \times \mathcal{C}_s \times \mathcal{C}_s)$  inside  $\mathcal{A}_s \times \mathcal{A}_s$  to show that they are not generically special and hence non-degenerate. Let us also consider an abelian subvariety  $H$  of  $A$  which will later arise as the trace  $A^{K/k}$  of the generic fiber of  $\mathcal{A}$ .

**Lemma 4.6.1.** *Suppose  $C$  is not a coset in  $A$ . Let  $B$  be an abelian sub-variety of  $A^2$  and  $Z$  an irreducible closed sub-variety of  $H^2$  such that*

$$Z + B + (P_1, P_2) = (C - P)^2$$

*for some  $P, P_1, P_2 \in C(k)$ . Then  $C$  is contained in a translate of  $H$*

*Proof.* Let  $q$  be the projection from  $A^2$  to the first co-ordinate. Then we get that

$$C - P = q(Z) + q(B) + P_1$$



Then note that

$$q(B) + (C - P) = q(B) + (q(Z) + q(B) + P_1) = q(Z) + (q(B) + q(B)) + P_1 = q(Z) + q(B) + P_1 = C - P$$

as  $B$  is an abelian sub-variety. Hence by definition of  $\text{Stab}(C)$  we get that

$$q(B) \subseteq \text{Stab}(C)$$

But as we have  $\dim(C) = 1$ , from the definition of the stabiliser, we get that  $\dim(\text{Stab}(C)) \leq 1$ . Now note that  $q(B)$  is irreducible as  $B$  is an abelian sub-variety. Moreover, as it is contained inside the stabiliser, we get that  $\dim(q(B)) \leq 1$ . If  $q(B)$  has dimension 1, then we must have  $\dim(\text{Stab}(C)) = 1$  and this immediately gives

$$\text{Stab}(C) = C - P$$

for all  $P \in C(k)$ . Hence  $C = \text{Stab}(C) + P$  and this clearly shows  $C$  is a coset. Hence we must have  $\dim(q(B)) = 0 \Rightarrow q(B) = 0$  as  $q(B)$  is an abelian sub-variety. Hence we get that

$$C - P = q(Z) + P_1 \subseteq q(H^2) + P_1 = H + P_1$$

and this finishes the proof.  $\square$

**Lemma 4.6.2.** *Suppose  $C$  is not a coset in  $A$ . Let  $B$  be an abelian sub-variety of  $A^2$  and  $(Q_1, Q_2) \in A^2(k)$  such that  $B + (Q_1, Q_2) \subseteq F(C^3)$ . Let  $P_1, P_2 \in A(k)$  be such that*

$$F(C \times \{P_1\} \times \{P_2\}) \subseteq B + (Q_1, Q_2)$$

*Then we must have*

$$Q_1 - P_1 = Q_2 - P_2$$

*Proof.* Let  $\Delta: A \rightarrow A^2$  be the diagonal map. Then the condition gives us

$$(P_1 - P, P_2 - P) \in B + (Q_1, Q_2) \text{ for all } P \in C(k)$$

and hence we get that

$$\Delta(C) \subseteq ((P_1 - Q_2, P_2 - Q_2) + B) \cap \Delta(A) \Rightarrow C \subseteq \Delta^{-1}(((P_1 - Q_2, P_2 - Q_2) + B) \cap \Delta(A)) = \Delta_B$$

Next we claim that  $\Delta_B$  of the above display is a finite union of cosets of dimension  $\dim(B \cap \Delta(A))$ . At first note that if  $P, Q \in C(k)$  belong in  $\Delta_B$ , then clearly

$$(P - Q, P - Q) \in B \cap \Delta(A) \Rightarrow P - Q \in \Delta^{-1}(B)$$

Hence we get that  $C$  is contained in a coset of  $\Delta^{-1}(B)$ . Note that  $\Delta^{-1}(B)$  is an algebraic group (not necessarily connected) and hence by the structure theorem, as  $k$  is algebraically closed, we get that  $\Delta^{-1}(B)$  is a finite union of cosets (of the component containing 0) each of whose dimension is  $\dim(\Delta^{-1}(B)) = \dim(B \cap \Delta(A))$  and this completes our claim.

Now as  $C \subseteq \Delta_B$  is an irreducible curve of dimension 1 and  $\Delta_B$  is an union of translates of dimension  $\dim(B \cap \Delta(A)) \leq \dim(B)$  we must have  $\dim(B) \geq \dim(B \cap \Delta(A)) \geq 2$  as otherwise  $C$  will be equal to one of the translates which is forbidden by hypothesis.

Let us first assume that  $\dim(B) = 2$ . Then by irreducibility of  $B$ , we must have that  $B \subseteq \Delta(A)$  and hence we get that

$$(P_1 - P, P_2 - P) \in \Delta(A) + (Q_1, Q_2) \Rightarrow P_1 - Q_1 - P = P_2 - Q_2 - P \Rightarrow P_1 - Q_1 = P_2 - Q_2 \text{ for all } P \in C(k)$$

and hence we are done. Hence we might assume that  $\dim(B) \geq 3$ .

Now note that  $\dim(F(C^3)) \leq 3$  being the image of a 3-dimensional irreducible variety. We also have by hypothesis that

$$B + (Q_1, Q_2) \subseteq F(C^3)$$

which together with  $\dim(B) \geq 3$  implies that

$$B + (Q_1, Q_2) = F(C^3) \quad (4.13)$$

as both sides are irreducible and hence also  $\dim(B) = 3$ . Note that this implies

$$\{P'_1 - P'_0 - Q_1\} \times (C - P'_0 - Q_2) \subseteq B \quad (4.14)$$

for any  $P'_0, P'_1 \in C(k)$ . Let  $q_1$  and  $q_2$  be the first and second projections from  $A^2$  to  $A$ . Then it is easy to see that each fiber of  $q_1|_B$  is a coset in  $A^2$ , namely a translate of  $\{0\} \times q_2(B)$ . Hence all of them are equi-dimensional. But (4.14) shows that  $C - P'_0 - Q_2$ , an one dimensional irreducible sub-variety is contained in a fiber of  $q_1|_B$ . As  $C$  is not a coset, this forces that the dimension of at least one fiber and hence each fiber of  $q_1|_B$  is at least 2. Hence by the Fiber dimension theorem, we get that

$$\dim(q_1(B)) = \dim(B) - \dim(q_1|_B^{-1}(b)) \leq 3 - 2 = 1 \text{ for all } b \in q_1(B)$$

On the other hand from (4.13) we get that

$$q_1(B) + Q_1 = q_1(F(C^3)) = C - C \quad (4.15)$$

This implies that

$$\dim(C - C) = \dim(q_1(B)) \leq 1 \quad (4.16)$$

But note that

$$C - P \subseteq C - C \text{ for all } P \in C(k) \Rightarrow 1 = \dim(C - P) \leq \dim(C - C)$$

This together with (4.16) readily shows that  $\dim(C - C) = 1$  and  $C - P = C - C$  for all  $P \in C(k)$  (note both  $C - C$  and  $C - P$  are irreducible). Hence by definition we get that  $\text{Stab}(C) = C - P$  for all  $P \in C(k)$  which contradicts the hypothesis that  $C$  is not a coset and hence finishes the proof.  $\square$

We finish this section with one last lemma which sort of combines the previous two and is the main result of this section.

**Lemma 4.6.3.** *Suppose  $C$  is not contained in any proper coset of  $A$  and  $C \neq A$ . Let  $\psi: A \rightarrow H$  be a homomorphism of abelian varieties such that  $\psi|_H: H \rightarrow H$  is an isogeny. Let  $\psi \times \psi: A^2 \rightarrow H^2$  be the square of  $\psi$  and let  $B$  be an abelian sub-variety of  $A^2$  contained in the kernel of  $\psi \times \psi$  and let  $Z$  be an irreducible closed sub-variety of  $H^2$ . Further more let  $Q_1, Q_2 \in C(k)$  such that*

$$Z + B + (Q_1, Q_2) \subseteq F(C^3)$$

*If  $H \neq A$ , then  $(C - P)^2 \not\subseteq Z + B + (Q_1, Q_2)$  for all  $P \in C(k)$ .*

*Proof.* Suppose  $H \neq A$  and  $(C - P)^2 \subseteq Z + B + (Q_1, Q_2) \subseteq F(C^3)$ . Then as  $\dim((C - P)^2) = 2$  and  $\dim(F(C^3)) \leq 3$  we must have equality in the left or right. But if  $(C - P)^2 = Z + B + (Q_1, Q_2)$  then by Lemma 4.6.1  $C$  is contained in a translate of  $H$  which is a proper coset as  $H \neq A$ . Hence we must have that

$$Z + B + (Q_1, Q_2) = F(C^3) \quad (4.17)$$

Now as  $C$  is not contained in a proper coset, we can find  $P_1, P_2 \in C(k)$  such that

$$P_1 - P_2 \notin Q_1 - Q_2 + H(k)$$

Note that (4.17) implies

$$F(C \times \{P_1\} \times \{P_2\}) \subseteq Z + B + (Q_1, Q_2) \quad (4.18)$$

We claim that  $Z$  is not a coset in  $H^2$ . Suppose the contrary. Then we can write  $Z = B' + (S_1, S_2)$  for some abelian sub-variety  $B'$  of  $H^2$  and from above we get

$$F(C \times \{P_1\} \times \{P_2\}) \subseteq (B + B') + (Q_1 + S_1, Q_2 + S_2) \quad (4.19)$$

Now Lemma 4.6.2 gives us that

$$Q_1 + S_1 - P_1 = Q_2 + S_2 - P_2 \Rightarrow (P_1 - Q_1) = (Q_1 - Q_2) + (S_1 - S_2)$$

which contradicts our choice of  $P_1, P_2$  as  $(S_1, S_2) \in H^2(k)$ . Hence we get that in particular  $\dim(H) \geq 1$  and  $Z \neq H^2$ .

Suppose  $\dim(H) = 1$ . Note that  $\psi(C) \subseteq H$  is an irreducible variety of dimension at-most 1. Moreover if  $\psi(C)$  is a point, then  $C$  is contained in a coset of the kernel of  $\psi$  which is an algebraic sub-group. But this is impossible by hypothesis that  $C$  is not contained in a coset and hence we get that  $\psi(C)$  must have dimension at least 1. Hence we must have  $\psi(C) = H$ . Now if we apply  $\psi^2 = \psi \times \psi$  to both sides of (4.18),

$$(\psi^2(P_1), \psi^2(P_2)) - \psi^2(\Delta(C)) \subseteq \psi^2(Z) + (\psi(Q_1), \psi(Q_2)) \quad (4.20)$$

as  $B$  is contained in the kernel of  $\psi^2$ . Hence  $\psi^2(\Delta(C)) = \Delta(\psi(C)) = \Delta(H)$  is an irreducible variety of dimension 1 (as  $\Delta$  is a closed immersion) contained in a translate of  $-\psi^2(Z)$ . Hence we get  $\dim(\psi^2(Z)) \geq 1$ . Again as  $Z \neq H^2$ , we get that  $\dim(\psi^2(Z)) \leq \dim(Z) \leq \dim(H^2) - 1 = 1$ . Hence we get that  $\dim(\psi^2(Z)) = 1$  and it is equal to a translate of  $\Delta(H)$ , in particular a coset. But by hypothesis  $\psi^2|_{H^2}$  is an isogeny and as  $Z \subseteq H^2$ , we get that  $Z$  is also a coset which we have already discarded.

Hence we can assume  $\dim(H) \geq 2$ . This implies that  $\dim(A) \geq 3$  as  $H \neq A$  and both are irreducible. We claim that  $\text{Stab}(C - C) \neq C - C$ . Otherwise we have that for any  $P \in C(k)$ , we have  $C \subseteq (C - C) + P = \text{Stab}(C - C) + P$  which is impossible as  $\text{Stab}(C - C)$  is an algebraic subgroup and  $C$  is not contained in any proper coset of  $A$ . Hence we must have  $\dim(\text{Stab}(C - C)) \leq 1$  as  $\dim(C - C) \leq 2$  being the image of  $C \times C$  under the difference morphism.

Suppose  $\text{Stab}(C - C) = 0$ . Then from (4.17) applying the first projection  $q_1$ , we get

$$q_1(Z) + q_1(B) + Q_1 = C - C \quad (4.21)$$

Doing the same trick as in the proof of Lemma 4.6.1, we get that  $q_1(B)$  is an irreducible abelian subvariety contained in  $\text{Stab}(C - C)$  which is 0-dimensional. Hence we must have that  $q_1(B) = 0$ . Thus from (4.21) we get that

$$C - C = q_1(Z) + Q_1 \subseteq q_1(H^2) + Q_1 = H + Q_1 \Rightarrow C \subseteq H + Q_1 + P \text{ for all } P \in C(k) \quad (4.22)$$

Thus  $C$  is contained in a translate of  $H$  which is impossible by hypothesis.

Finally suppose  $\dim(\text{Stab}(C - C)) = 1$ . Consider the component of it containing 0. It is a connected commutative algebraic group of dimension 1 which is a closed sub-variety of an abelian variety and hence proper. Thus this component must be an elliptic curve  $E$ . We consider the quotient abelian variety with the quotient homomorphism  $f: A \rightarrow A/E$ . Note that  $f(C)$  has dimension 1. Otherwise  $f(C)$  would be a single point and hence  $C$  would be contained in a translate of  $E$ . We claim that  $\dim(f(C) - f(C)) = 1$ . Note that  $f(C) - f(C) = f(C - C)$  and hence its dimension is at most 2 as  $\dim(C - C) \leq 2$ . Next we claim that each fiber of  $f|_{C-C}$  contains a translate of  $E$ . Indeed as  $E \subseteq \text{Stab}(C - C)$ , we get that if  $f(P) = P'$  for some  $P \in (C - C)(k)$ , then  $f(P + P_E) = P'$  and  $P + P_E \in (C - C)(k)$  for all  $P_E \in E(k)$ . Hence by Fiber dimension theorem, we get that  $\dim(f(C - C)) \leq 1$  because equality holds in Fiber-Dimension theorem holds in an open dense subset. But as  $f(C - C)$  is irreducible, it being zero dimensional means it is a single point which in turn means that  $C - C$  is contained in a translate of  $E$  which is a proper coset as  $\dim(A) \geq 3$  and this clearly contradicts hypothesis. Hence we get  $\dim(f(C) - f(C)) = 1$ . We have then  $f(C) - f(P) \subseteq f(C) - f(C)$  for all  $P \in C(k)$  and the L.H.S is one dimensional as  $f(C)$  is one-dimensional. Hence we get that  $f(C) - f(P) = f(C) - f(C)$  for all  $P \in C(k)$ . Hence we get that the stabiliser of  $f(C)$  is  $f(C) - f(P)$  and hence  $f(C)$  is contained in a proper coset which in turn implies that  $C$  is contained in a proper coset as  $f$  is a quotient map. This gives us the contradiction and finally finishes the proof.

## 4.7 Decomposition of the generic fiber

In this section, we want to decompose the generic fiber of our abelian scheme into abelian subvarieties so that we can apply the lemmas of the previous section. Suppose  $\pi: \mathcal{A} \rightarrow S$  be an abelian scheme and let  $A$  be the generic fiber which is an abelian variety defined over  $K = k(S)$ , the function field of the smooth affine curve  $S$  defined over  $k$ . Note that from our reduction to hypothesis 4.7 we can assume that the trace of  $A$  doesn't increase when we pass to  $\bar{K}$ . Let  $H = A^{K/k}$  be the trace which is an abelian variety defined over  $k$  and we can view  $H_K = H \otimes_k K$  as an abelian sub-variety of  $A$  via the canonical morphism as  $\text{char}(k) = 0$ . Then by Poincaré's complete reducibility theorem as in Theorem 8.9.3 in [BG06] we get an abelian subvariety  $G$  of  $A$  defined over  $K$  such that  $G + H_K = A$  and  $\#(H_K \cap G) = m < \infty$ . We claim that  $G^{K/k} = 0$ . Indeed there is a canonical closed immersion

$$i_G: G^{K/k} \otimes_k K \hookrightarrow G \hookrightarrow A$$

and hence by the universal property of traces,  $i_G$  factors as

$$G^{K/k} \otimes_k K \xrightarrow{i_H} H_K \hookrightarrow A$$

Hence  $i_G \times i_H$  gives a morphism

$$i_G \times i_H: G^{K/k} \otimes_k K \rightarrow H_K \times_A G = H_K \cap G$$

Now as the target of the above map is finite and the domain is irreducible, we get that  $i_G \times i_H$  is constant and hence  $i_G$  is the constant zero map and it shows  $G^{K/k} = 0$ .

We can define an isogeny

$$s: H_K \times G \rightarrow A$$

$$s(P, Q) = P + Q$$

On the other direction we define an isogeny as follows: Let  $P_A = P_H + P_G$  be some decomposition for  $P_A \in A$ ,  $P_H \in H_K$  and  $P_G \in G$ . Then we define

$$\phi: A \rightarrow H_K \times G$$

$$\phi(P_A) = (m \cdot P_H, m \cdot P_G)$$

Note that this is well defined as if  $P'_H$  and  $P'_G$  are another decomposition, then  $P_A - P'_A = P'_G - P_G \in H_K \cap G$  and hence  $m \cdot (P_G - P'_G) = m \cdot (P_A - P'_A) = 0$ . We can also deduce it from the fact that the addition map  $s: H_K \times G \rightarrow A$  is an isogeny and then just take the dual of this isogeny in which easily see that  $m = \deg(s)$ . Then note that if  $p$  is the projection from  $H_K \times G$  to  $H_K$ , then we get a homomorphism

$$\psi: A \rightarrow H_K$$

$$\psi(P) = p(\phi(P))$$

Note that  $\phi|_{H_K} = [m]$ , the multiplication by  $m$  and hence an isogeny as  $H_K$  is an abelian variety. It is also easy to check that  $\psi(G) = 0$ .

We have the isogeny  $\phi^2 = \phi \times \phi$  from  $A^2$  to  $H_K^2 \times G^2$ . By projecting on to  $H_K^2$ , we get that isogeny

$$\psi^2 = \psi \times \psi: A^2 \rightarrow H_K^2$$

Next we claim that there is no non-zero homomorphism of abelian varieties between  $H_K^2$  and  $G^2$ . This is because any homomorphism from  $H_K^2$  to  $G^2$  factors through the  $K/k$ -trace of  $G^2$  which is 0 as the  $K/k$ -trace of  $G$  is 0. Hence we can decompose any abelian subvariety of  $H_K^2 \times G^2$  as product of abelian sub-varieties of  $H_K^2$  and  $G^2$ . Hence due to the isogenies  $s^2$  and  $\phi^2$ , we can decompose any abelian sub-variety  $B$  of  $A^2$  as  $B = B' + B''$  where  $B'$  and  $B''$  are abelian subvarieties of  $H_K^2$  and  $G^2$  respectively. Moreover as  $\psi(G) = 0$ , we can also have  $\psi^2(B'') = 0$ .

We can extend this morphism  $\psi^2$  on the generic fiber to the corresponding abelian schemes. Note that  $S \times H^2$  is the Neron model of  $H_K^2$  and hence we get a closed immersion of  $S \times H$  inside  $\mathcal{A} \times_S \mathcal{A}$  by the Neron mapping property (after possibly excluding finitely many points). On each special fiber  $\mathcal{A}_s$  it gives a homomorphism  $\psi_s$  to  $H$  which is an isogeny when restricted to  $H$ . Next we consider the Faltings-Zhang map on the abelian scheme

$$F: \mathcal{A} \times_S \mathcal{A} \times_S \mathcal{A} \rightarrow \mathcal{A} \times_S \mathcal{A}$$

which fiberwise is the Faltings Zhang map considered in the previous section. Now suppose that  $\mathcal{C}$  is an irreducible closed subvariety of  $\mathcal{A}$  dominating  $S$  such that the generic fibre  $\mathcal{C}_\eta$  is a geometrically irreducible curve. We consider  $\mathcal{X} = F(\mathcal{C} \times_S \mathcal{C} \times_S \mathcal{C})$ . We look upon the degeneracy of  $\mathcal{X}$  next. Suppose  $\mathcal{X}^*$  be as defined in Definition 1.2 of [GH19]. Then we know from Proposition 1.3 of [GH19] that  $\mathcal{X}^o = \mathcal{X}/\mathcal{X}^*$  is Zariski closed and the geometric generic fiber

$$(\mathcal{X}^o)_\eta \otimes_K \overline{K} = \cup_{i=1}^t Z_i \otimes_k \overline{K} + B_i + P_i$$

where  $B_i$ ,  $Z_i$  and  $P_i$  are as in Definition 4.3.3. But as we have assumed that the trace of  $A$  and hence  $A^2$  doesn't increase, we may pass onto a finite extension of  $K$  and using Lemma 4.4.1, we can actually assume that

$$(\mathcal{X}^o)_\eta = \cup_{i=1}^t Z_i \otimes_k K + B_i + P_i$$

where  $B_i$  are abelian subvarieties of  $A$ ,  $P_i$  torsion points in  $A(K)$  and  $Z_i \times_k K$  is geometrically irreducible. Furthermore, we can assume that  $\psi^2(B_i) = 0$  because we can decompose any  $B_i$  into  $B'_i + B''_i$  with  $B'_i \subseteq H_K^2$  and  $B''_i \subseteq G$ . Then  $Z_i \times_k K + B'_i$  is an irreducible closed sub-variety of  $H_K^2$  and  $\psi^2(B''_i) \subseteq \psi^2(G) = 0$ .

Let us call by  $\mathcal{B}_i$  the Neron model of  $B_i$  and let  $\mathcal{Z}_i$  be the Zariski closure of  $Z_i \otimes_k K$  in  $\mathcal{A} \times_S \mathcal{A}$ . Then note that as the generic fiber of  $\mathcal{Z}_i$  is geometrically irreducible, we get by Lemma 37.25.5 of [Stab] that the special fiber  $Z_{i,s}$  of  $\mathcal{Z}_i$  is also geometrically irreducible for all  $s$  in  $U(k)$  for some open  $U$  in  $S$ . As  $S$  is a curve, this means that except for finitely many  $s$ ,  $Z_{i,s}$  is geometrically irreducible. By similar logic we also get the special fibers  $B_{i,s}$  of  $\mathcal{B}_i$  are irreducible for all but finitely many  $s$ . Hence in total, we get that the the special fibers  $\mathcal{X}_s^o$  satisfies

$$\mathcal{X}_s^o = \cup_{i=1}^t Z_{i,s} + B_{i,s} + P_{i,s}$$

with  $Z_{i,s}$  an irreducible closed sub-variety of  $H^2$ ,  $B_{i,s}$  be an abelian sub-variety of  $\mathcal{A}_s^2$  with  $\psi_s^2(B_{i,s}) = 0$  and  $P_{i,s}$  a torsion point of  $\mathcal{A}_s$  for all but finitely many  $s \in S(k)$ . We end this section with a modification of Lemma 4.2.1. In particular, we claim that the inequality works all over  $\mathcal{X}^*$  and not just some open dense subset of  $\mathcal{X}$ .

**Lemma 4.7.1.** *Suppose we have the hypothesis of Lemma 4.2.1. Then we have the inequality for all  $P \in \mathcal{X}^*(k)$*

*Proof.* From Theorem 4.3.1, we know that  $\mathcal{X}^*$  is non-empty as  $\mathcal{X}$  is non-degenerate. Hence by Lemma 4.2.1, we can find a proper closed subset  $Z \subseteq \mathcal{X}$  such that the inequality holds for all  $P \in \mathcal{X}/Z(k)$ . Decompose  $Z = \cup_i Z_i$  into irreducible components  $Z_i$ . Then as  $\dim(Z_i) \leq \dim(\mathcal{X}) - 1$ , by induction we get that the inequality holds for  $\cup_i Z_i^*$ . Hence the inequality holds for

$$\mathcal{X}/Z(k) \cup_i Z_i^*(k)$$

and thus it is enough to show

$$\mathcal{X}^*(k) \subseteq \mathcal{X}/Z(k) \cup_i Z_i^*(k) \Leftrightarrow \cup_i (Z - Z_i^*) \subseteq \mathcal{X}^o$$

But note that  $\cup_i (Z - Z_i)^*$  is just the union of all generically special sub-varieties contained in  $Z$  and they are all contained in  $\mathcal{X}$  trivially and hence in  $\mathcal{X}^o$  which finishes the proof.  $\square$

We finish this section by stating a theorem which shall be the main tool used in the next section to bound the cardinality of rational points in small balls. This theorem comes from a slight modification of the inequality stated above together with the equivalence stated in Theorem 4.3.1.

**Theorem 4.7.2.** *Let us have the notations and set-up used in this section. Then there is a constant  $c_1 \geq 1$  independent of  $s \in S(k)$  such that if  $P_1, P_2, P \in \mathcal{C}_s(k)$  with  $F(P_1, P_2, P) \notin \cup_{i=1}^t (Z_{i,s} + B_{i,s} + P_{i,s})$  for some  $s \in S(k)$  then*

$$h(s) \leq c_1 \cdot \max\{1, \hat{h}(P_1 - P), \hat{h}(P_2 - P)\}$$

*Proof.* The theorem follows from Lemma 4.7.1 readily by taking  $c_1 = 2 \cdot C$  where  $C$  is the constant of Lemma 4.7.1 and noting that  $(\mathcal{X}^0)_s = (\mathcal{X}/\mathcal{X}^*)_s = \cup_{i=1}^t (Z_{i,s} + B_{i,s} + P_{i,s})$ .  $\square$

## 4.8 Bounding the cardinality in small balls

We begin by stating the main result of this section.

**Lemma 4.8.1.** *Suppose  $\pi: \mathcal{A} \rightarrow S$  is an abelian scheme where  $S$  is a smooth, irreducible, affine curve and  $\mathcal{C}$  is an irreducible closed sub-variety of  $\mathcal{A}$  such that the generic fiber  $\mathcal{C}_\eta$  is a geometrically irreducible curve not contained in any translate of  $A^{k(S)/k} \otimes_k \bar{k}(S)$ . Then there are constants  $c_2, c_3 \geq 1$  with the following property:*

*Let  $s \in S(k)$  be such that  $h(s) \geq c_3$ . Then the special fiber  $\mathcal{C}_s$  over  $s$  is integral. Furthermore if  $\mathcal{C}_s$  is not a coset in  $\mathcal{A}_s$ , then*

$$\#\{Q \in \mathcal{C}_s(k) \mid \hat{h}(Q - P) \leq \frac{h(s)}{c_3}\} \leq c_2$$

Before going to the proof of Lemma 4.8.1, we prove a lemma which we later use to show that "not being contained in a proper coset" is (in some sense) a generic property.

**Lemma 4.8.2.** *Suppose  $A$  is an abelian variety and  $C$  is an irreducible closed curve contained in  $A$ . Then  $C$  is not contained in a proper coset if and only if*

$$(C - C) + (C - C) + \dots (C - C) = A$$

where the sum on the left is taken  $\dim(A)$ -fold.

*Proof.* Suppose  $C$  is not contained in any proper coset. We write  $(C - C)^1 = C - C$  and

$$(C - C)^i = (C - C)^{i-1} + (C - C)$$

Then we claim that  $\dim(C - C)^i \geq i$  for all  $i \leq \dim(A)$ . This then clearly proves the claim by taking  $i = \dim(A)$  as each  $(C - C)^i$  is an irreducible closed sub-variety contained in  $A$ . We argue by induction. For  $i = 1$ , the claim easily follows by irreducibility of  $C - C$  and by the fact that  $C - P \subseteq C - C$  for all  $P \in C(k)$ . Now suppose we have proved the claim for  $i - 1$ . Then note that

$$(C - C)^i = (C - C)^{i-1} + (C - C) \supseteq (C - C)^{i-1}$$

Hence once again by irreducibility of  $(C - C)^i$  and induction hypothesis it is enough to show that

$$(C - C)^i \neq (C - C)^{i-1}$$

Suppose the contrary. This then clearly gives by definition of stabilisers that

$$(C - P) \subseteq (C - C) \subseteq \text{Stab}((C - C)^{i-1}) \text{ for all } P \in C(k)$$

As  $\text{Stab}(C - C)$  is an algebraic sub-group and  $C - P$  is irreducible, the structure theorem of algebraic groups gives us that  $C - P$  and hence  $C$  is contained in a coset. Moreover this coset is always proper if

$\text{Stab}((C-C)^{i-1}) \neq A$  which then contradicts hypothesis. Hence we must have that  $\text{Stab}((C-C)^{i-1}) = A$  and this clearly shows that

$$(C-C)^{i-1} = (C-C)^i = A$$

and the claim follows.

Conversely if  $C$  is contained in a proper coset of an abelian sub-variety  $B$ , then clearly  $(C-C)^i \subseteq B$  for all  $i$  and hence the dimension is never full as claimed in the lemma.  $\square$

*Proof of Lemma 4.8.1.* We recall that by hypothesis (4.12), we might assume that  $\mathcal{C}_\eta$  is not contained in any proper coset of the generic fiber of  $A$ . This in turn with Lemma 4.8.2 implies that the sum  $(C_\eta - C_\eta)^i = A$  where we use the notation of the previous lemma and  $i = \dim(A)$ . Then note that the Zariski closure of  $(C_\eta - C_\eta)^i$  in  $\mathcal{A}$  is an irreducible closed subvariety that has generic fiber equal to  $A$ . As the generic fiber of a dominant morphism is dense in the domain scheme, we must have that the Zariski closure is  $\mathcal{A}$  itself. Let us call this Zariski closure as  $\mathcal{C}^i$ . Then note that the special fiber of  $(\mathcal{C}^i)$  over  $s$  is exactly equal to the sum  $(\mathcal{C}_s - \mathcal{C}_s)^i$  and hence this sum is dense in  $\mathcal{A}_s$ . However due to the properness of the addition and difference morphisms, we also get that  $(\mathcal{C}_s - \mathcal{C}_s)^i$  is closed in  $\mathcal{A}_s$  as it is the image of a product of copies of  $\mathcal{C}_s$  under a combination of difference and addition morphisms. Hence we get that  $(\mathcal{C}_s - \mathcal{C}_s)^i = \mathcal{A}_s$ . Finally once again applying Lemma 4.8.2, we get that  $\mathcal{C}_s$  is not contained in a proper coset for all  $s \in S(k)$ .

Next suppose  $A^{K/k} = \mathcal{A}_s$  for some  $s \in S(k)$ . From the universal property of traces, it is easy to deduce that this implies that the canonical morphism from  $A^{K/k} \otimes_k K \rightarrow A$  is an isomorphism. This in turn clearly contradicts the hypothesis that we made at the beginning of this chapter as in (4.1). Hence we can assume that  $H = A^{K/k} \neq \mathcal{A}_s$  for all  $s \in S(k)$ .

Note that as  $\mathcal{C}_\eta$  is geometrically irreducible, by Lemma 37.25.5 of [Stab] we can deduce that  $\mathcal{C}_s$  is geometrically irreducible and hence integral for almost all  $s \in S(k)$  (as  $S$  is a curve). Thus we can take the constant  $c_3$  such that  $h(s) \geq c_3$  implies integrality of  $\mathcal{C}_s$ . Furthermore, we choose  $c_3 > c_1$  where  $c_1$  is the constant of Theorem 4.7.2.

Next suppose that  $\mathcal{C}_s$  is not a coset in  $\mathcal{A}_s$ . In particular, we have  $\mathcal{C}_s \neq \mathcal{A}_s$ . In the first paragraph, we argued that  $\mathcal{C}_s$  is not contained in any proper coset of  $\mathcal{A}_s$  for any  $s$ . Furthermore, in the second paragraph we argued that  $H \neq \mathcal{A}_s$  for any  $s$ . Hence by our constructions in the previous section with  $Z_{i,s}$ ,  $B_{i,s}$  and  $\psi_s$ , we can apply Lemma 4.6.3 to obtain that

$$(\mathcal{C}_s - P)^2 \not\subseteq (Z_{i,s} + B_{i,s} + P_{i,s}) \quad (4.23)$$

for all  $s \in S(k)$  and  $i$ . Note that each of the terms  $(Z_{i,s} + B_{i,s} + P_{i,s})$  is irreducible (being the image of a product of irreducible varieties under addition) and  $(\mathcal{C}_s - P)^2$  is also irreducible. Hence we get from (4.23) that

$$(\mathcal{C}_s - P)^2 \not\subseteq \cup_{i=1}^t (Z_{i,s} + B_{i,s} + P_{i,s}) \quad (4.24)$$

Note that (4.24) shows that all irreducible components of the intersection

$$(\mathcal{C}_s - P)^2 \cap (\cup_{i=1}^t (Z_{i,s} + B_{i,s} + P_{i,s})) \quad (4.25)$$

are of dimension at most 1. We are interested in bounding the number of irreducible components in this intersection uniformly on  $s$ . From an appropriate version of Bezout's theorem, we can deduce easily that it is enough to bound the degrees of  $(\mathcal{C}_s - P)^2$  and  $(Z_{i,s} + B_{i,s} + P_{i,s})$  uniformly on  $s$ . Now from Theorem 2 of the lecture notes [Bej19] we deduce that these degrees are a locally constant function on  $S$ . But note that  $S$  is just a curve and hence this implies that these degrees are constant on all but finitely many  $s \in S(k)$  and hence we get that the degrees are bounded uniformly on  $S$ . Hence Bezout gives us a constant  $k_1$  such that the number of irreducible components of the intersection in (4.25) is bounded above by  $k_1$  which is independent of  $s$ . (note that this argument also uses that  $t$  is uniform w.r.t  $s$  which is true thanks to uniform Manin-Mumford, see [GH19])

Let us call  $q$  the canonical projection onto the second factor from  $\mathcal{A}_s \times \mathcal{A}_s$  to  $\mathcal{A}_s$ . Let us call by  $Y$  an irreducible component of the intersection in (4.25) which we have argued already to be of dimension



atmost 1. Then the dimension of  $q(Y)$  is atmost 1 as well. Suppose  $\dim(q(Y)) = 1$ . Then note that as  $q$  is a base change of the structure morphism from  $\mathcal{A}$  to  $S$ , it is flat. Moreover, both  $Y$  and  $q(Y)$  is integral and hence we get that each fiber of  $q|_B$  has dimension 0 and hence they are finite. Note that each of these fibers can be seen as an intersection of  $Y$  and  $\{x\} \times \mathcal{A}_s$  for some  $x \in \mathcal{A}_s(k)$  and hence by arguments similar to before, we can find another constant  $k_2$  such that the cardinality of each of these finite fibers is bounded above by  $k_2$  which is independent of  $s$ . Finally we set  $c_2 = k_1 k_2$ .

Now suppose that the cardinality of the set in Lemma 4.8.1, which we call  $S$  for convenience, is greater than  $c_2$ . Suppose  $Y$  is an irreducible component of the intersection (4.25) such that  $\dim(q(Y)) = 0$ . Then as  $Y$  is irreducible, we have that  $q(Y)$  is also irreducible of dimension 0 and hence a single point. Note that by choice of  $k_1$  there are atmost  $k_1$  many such 0-dimensional  $q(Y)$ 's and hence as  $c_2 \geq k_1$  we can choose a point  $P_1$  in  $S$  such that  $P_1 - P \neq q(Y)$  for any irreducible component  $Y$  of the intersection in (4.25). Next we claim that

$$(P_1 - P, P_2 - P) \notin \cup_i (Z_{i,s} + B_{i,s} + P_{i,s}) \text{ for some } P_2 \in S \quad (4.26)$$

Suppose the contrary. Then clearly  $P_1 - P$  is in  $q(Y)$  for some irreducible component  $Y$  of the intersection in (4.25). By our choice of  $P_1$ , we have that  $q(Y) \neq P_1 - P$  and hence due to dimensionality reason, we must have that  $\dim(Y) = 1$ . Hence if (4.26) does not hold, then we get that the finite fiber of  $q|_Y$  over  $P_1 - P$  has at least  $c_2 = k_1 k_2 \geq k_2$  many points and  $q(Y)$  has dimension 1 which clearly contradicts our choice of  $k_2$  and hence finishes the proof of the claim in (4.26).

Now we choose  $P_1, P_2$  from  $S$  which satisfied (4.26). Then from Theorem 4.7.2, the definition of the set  $S$  and  $h(s) \geq c_3$ , we have

$$\frac{h(s)}{c_1} \leq \max\{1, \hat{h}(P_1 - P), \hat{h}(P_2 - P)\} \leq \frac{h(s)}{c_3} \quad (4.27)$$

which contradicts our choice of taking  $c_3$  so large such that  $c_3 > c_1$  and finishes the proof.  $\square$

*Remark.* The arguments show that the constant  $c_2$  can be chosen  $\max\{k_1, k_2\}$  instead of  $k_1 k_2$ .

## 4.9 Uniformity in a pencil with an embedding: The non-isotrivial case

In this section, we prove Theorem 4.1.2 in the non-isotrivial case *i.e.* we assume that the generic fiber  $\mathcal{C}_\eta$  is not contained in a translate of  $A^{K/k} \otimes_k K$  where  $K = k(S)$  together with an abelian scheme inside which  $\mathcal{C}$  is embedded. Note that we can consider only base changes to  $K$  and not  $\bar{K}$  because of our reduction in (4.7). As explained before (4.1) the iso-trivial case can be tackled using known methods. We begin by stating a packing lemma due to Remond concerning the covering of a large ball in an euclidean space using small balls. Let  $\Gamma$  be a finitely generated abelian group which for our purposes will be the  $\Gamma$  mentioned in Theorem 4.1.2. Then  $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$  is a finite-dimensional euclidean space which we also call  $\Gamma$  by abuse of notation. Then we have the following theorem

**Theorem 4.9.1** (Remond). *Let  $0 < r_1 \leq R_1$  be positive reals and  $M$  be a subset of the ball of radius  $R$  in the euclidean space  $\Gamma$ . Then there exists a finite subset  $\Sigma \subseteq M$  of cardinality atmost  $(1 + \frac{2R_1}{r_1})^r$  such that the balls of radius  $r_1$  centered at elements of  $\Sigma$  cover  $M$  where  $r$  is the rank of  $\Gamma$ .*

*Proof.* For proof of this fact, we refer to Lemme 6.1 of [R  m00]  $\square$

Our strategy to prove Theorem 4.1.2 is to bound the large points using arguments of Vojta and Mumford (as in Crollary 4.5.4) while bounding the small points inside each ball of radius  $r_1$  and then using Theorem 4.9.1 to conclude the proof by appropriately choosing  $r_1$  and  $R_1$ . We produce the bound for large points before. Let  $\mathcal{A}$ ,  $S$  and  $\mathcal{C}$  be as before with  $S$  being 1-dimensional.



**Corollary 4.9.2.** *There exists a constant  $c_1 \geq 1$  dependent on  $\mathcal{C}$  but not on  $s$  with the following property:*

*If  $\mathcal{C}_s$  is not a coset in  $\mathcal{A}_s$ , then we have*

$$\#\{P \in \mathcal{C}_s(k) \cap \Gamma \mid \hat{h}(P) > c_1 \max\{1, h(s)\}\} \leq c_1^r$$

where  $r$  is the rank of  $\Gamma$ .

This is exactly the same as the statement of Corollary 4.5.4 and we re-state it in this section for the sake of clarity and continuity. Finally we are in a position to prove Theorem 4.1.2.

*Proof of Theorem 4.1.2.* First we begin by choosing the constant  $c$  of Theorem 4.1.2 to be  $c_1$ . Now suppose  $s \in S(k)$  satisfies  $h(s) \geq c = c_1$ . Using Corollary 4.9.2 we get that

$$\#\{P \in \mathcal{C}_s(k) \cap \Gamma \mid \hat{h}(P) > c \cdot h(s)\} \leq c^r \quad (4.28)$$

as we assumed that  $h(s) \geq c \geq 1$ . Hence we need to bound the cardinality of points in  $\mathcal{C}_s(k) \cap \Gamma$  with height bounded by the cut-off  $c \cdot h(s)$ . In other words, we have to bound the number of points of  $\mathcal{C}_s$  in the ball of radius  $R_1 = \sqrt{c \cdot h(s)}$  in the euclidean space  $\Gamma$ . We apply Theorem 4.9.1 with this  $R_1$ ,  $r_1 = \sqrt{\frac{h(s)}{c_3}}$  and  $M = \{P \in \mathcal{C}_s(k) \cap \Gamma \mid \hat{h}(P) \leq c \cdot h(s)\}$  where  $c_3$  is the constant from Lemma 4.8.1. Hence we get that  $\mathcal{C}_s(k) \cap \Gamma$  can be covered by at-most  $(1 + 2\sqrt{cc_3})^r$  balls of radius  $r_1$  centered at points of  $M$ . But note that by Lemma 4.8.1 we conclude that each such ball contain atmost  $c_2$  many points of  $\mathcal{C}_s(k)$ . (we can apply Lemma 4.8.1 due to the non-isotriviality assumption) Hence we get that

$$\#\{P \in \mathcal{C}_s(k) \cap \Gamma \mid \hat{h}(P) \leq c \cdot h(s)\} \leq c_2 \cdot (1 + 2\sqrt{cc_3})^r \leq 3^r C^{1+r} \quad (4.29)$$

where  $C = \max\{c, c_2, c_3\}$ . This combined with (4.28) gives us

$$\#\{P \in \mathcal{C}_s(k) \cap \Gamma\} \leq c^r + 3^r C^{1+r} \leq c_0^{1+r} \quad (4.30)$$

where  $c_0 = 2 \cdot \max\{1, 3C\}$  and this concludes the proof of Theorem 4.1.2 with  $c = c_0$ .  $\square$

## 4.10 Uniformity in pencils without an embedding

In this section, we prove our central Theorem 4.1.1 of this chapter. The main difference of this formulation from Theorem 4.1.2 is that  $\mathcal{C}$  does not already come with an embedding into the abelian scheme  $\mathcal{A}$  and we do not require that  $\mathcal{C}_s$  is not a coset. Instead we require that all the fibers  $\mathcal{C}_s$  are smooth projective curves of genus  $g \geq 2$  which is absolutely necessary as we do not have finiteness of  $\#\{\mathcal{C}_s(k) \cap \Gamma\}$  from Falting's theorem otherwise.

We begin with an elementary lemma which shows that a curve cannot be a translate of an abelian variety unless it is an elliptic curve.

**Lemma 4.10.1.** *Any abelian variety  $A$  over  $k$  of dimension 1 is an elliptic curve.*

*Proof.* We use the definition that an elliptic curve is a smooth projective curve of genus  $g = 1$  where  $g = \dim_k(H^0(A, \Omega_A))$  where  $\Omega_A$  is the Kahler sheaf of  $A$ . Now as  $A$  is an abelian variety, it is smooth and hence from algebraic geometry, we get that  $H^0(A, \Omega_A)$  is a  $k$ -vector space of dimension 1 and hence  $g = 1$ . Obviously  $A$  is smooth and projective and hence the claim follows.  $\square$

**Corollary 4.10.2.** *Suppose  $C$  is a projective smooth curve contained in an abelian variety  $A$  defined over  $k$  such that there is an abelian sub-variety  $B$  of  $A$  and  $P \in A(k)$  with  $C = B + P$ . Then  $C$  has genus 1.*

*Proof.* Note that as  $C$  is a curve and translation is an isomorphism, we get that  $\dim(B) = 1$ . Hence by Lemma 4.10.1  $B$  is an elliptic curve and hence in, particular it has genus 1. Now the claim follows from the fact that genus is invariant under isomorphisms and translation is an isomorphism.  $\square$

Before going to the final proof of Theorem 4.1.1, we prove a small lemma on spreading out closed immersions from generic fibers to the base.

**Lemma 4.10.3.** *Suppose  $f: X \rightarrow Y$  be a morphism between integral schemes of finite presentation over  $S = \operatorname{Spec}(A)$  where  $A$  is an integral domain and let  $K = \operatorname{Frac}(A)$ . Suppose the induced morphism  $f_K: X_K \rightarrow Y_K$  between the generic fibers is a closed immersion. Then there exists an open set  $U$  of  $S$  such that the induced morphism*

$$f_U: X \times_S U \rightarrow Y \times_S U$$

*is a closed immersion.*

*Proof.* For a proof we refer to Theorem 8.10.5 of [Gro66]. □

*Proof of Theorem 4.1.1.* Note first that by excluding finitely many points from  $S$ , we can assume that  $S$  is affine. We consider the generic fiber  $\mathcal{C}_\eta$ . Note that by hypothesis of Theorem 4.1.1 we have that each special fiber  $\mathcal{C}_s$  is irreducible. Now by Lemma 37.22.8 of [Staa] we can pass to an open subset of  $S$  (i.e. exclude finitely many points) and then pass to finite affine irreducible non-singular curve  $S'$  such that all irreducible components of  $(\mathcal{C}_\eta)_{K'}$  are geometrically irreducible where  $K' = k(S')$ . But note for all  $s' \in S'(k)$ , the fiber  $(\mathcal{C} \times_S S')_{s'}$  is isomorphic to the fiber  $\mathcal{C}_s$  where  $s$  is the image of  $s'$  in  $S(k)$  and this fiber is irreducible by hypothesis. Hence by Lemma 37.25.1 of [Stab] we get that  $(\mathcal{C}_\eta)_{K'}$  is irreducible and hence geometrically irreducible. We can also choose  $S'$  such that  $(\mathcal{C}_\eta)_{K'}$  has a  $K'$ -rational point. Hence we can embed  $\mathcal{C}'_\eta$  into its Jacobian  $A$  which is an abelian variety defined over  $K'$ . Note that as  $S'$  is affine, we can define a Neron model of  $A$  over  $S'$  which we call  $\mathcal{A}$ . By the Neron mapping property, we obtain a morphism  $\mathcal{C}' = \mathcal{C} \times_S S' \rightarrow \mathcal{A}$  whose base change to the generic fiber is exactly the Abel-Jacobi map embedding  $\mathcal{C}'_\eta$  into  $A$ . Note that this map is not necessarily a closed immersion but thanks to Lemma 4.10.3, after passing to an open subset of  $S'$  (excluding finitely many points) we can assume that  $\mathcal{C}' \rightarrow \mathcal{A}$  is a closed immersion. Note that the claim of Theorem 4.1.1 follows almost immediately from the claim of Theorem 4.1.2 except that we just need to check that  $\mathcal{A}$  is actually an abelian scheme. We need to show that  $\mathcal{A} \rightarrow S'$  is smooth, proper and the fibers are irreducible. By definition the Neron model is smooth and each fiber is smooth commutative group variety. Hence we just need to show that the fibers are proper and irreducible. This might not be true over all of  $S'$  but as the generic fiber  $A$  of  $\mathcal{A}$  is proper and irreducible, the same is true by passing to an open sub set of  $S'$  by "spreading out" similar to Lemma 4.10.3 (we refer to Section 8.10 in [Gro66]). Hence finally we prove that (after possibly shrinking  $S'$ ) that  $\mathcal{A}$  is an abelian scheme and it finishes the proof.

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