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# On Finite Subgroups of the Plane Cremona Group

Elena Denisova

► **To cite this version:**

Elena Denisova. On Finite Subgroups of the Plane Cremona Group. Mathematics [math]. 2021. dumas-03650313

**HAL Id: dumas-03650313**

**<https://dumas.ccsd.cnrs.fr/dumas-03650313v1>**

Submitted on 24 Apr 2022

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ALGANT Master Thesis in Mathematics

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On Finite Subgroups of the Plane Cremona Group

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Advised by Prof. Andrea Fanelli



Universiteit Leiden

université  
de **BORDEAUX**

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Academic Year 2020-2021

Version: 6 June 2021

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# Acknowledgements

I would like to thank my advisor Prof. Andrea Fanelli for his support, and willingness to quickly and thoroughly answer any questions about birational geometry, no matter how insignificant or obvious they might seem at first glance. I want to thank Prof. Yuri Bilu' who was my tutor at the university of Bordeaux and Prof. Peter Stevenhagen who was my tutor at the university of Leiden for the invaluable advice they gave me over the past two years. I also want to thank Prof. Dajano Tossici, the person who, in these difficult times of the coronavirus, gave his students the opportunity to come to Bordeaux and made our stay as happy and joyful as possible.

Finally and mostly, I am deeply grateful to my husband, my parents and my friends for constant support and faith in me.

# Chapter 1

## Introduction

### History of the Results

The projective space is one of the central objects in mathematics. That is why it is important to understand the structure of  $\text{Bir}(\mathbb{P}_{\mathbb{K}}^n)$  the group of birational transformations of the projective space  $\mathbb{P}_{\mathbb{K}}^n$  of dimension  $n$  over the field  $\mathbb{K}$ . This group was introduced by Cremona ([Cr63], [Cr65]) and it is called **the Cremona Group**.

For  $n = 1$  we have that the group  $\text{Bir}(\mathbb{P}_{\mathbb{K}}^1)$  is the group of birational transformations of the projective line  $\mathbb{P}_{\mathbb{K}}^1$  which is well-known and equals  $\text{PGL}(2, \mathbb{K})$ . In case  $n = 2$  the structure of  $\text{Bir}(\mathbb{P}_{\mathbb{K}}^2)$  is way more complicated and very few results are known on  $\text{Bir}(\mathbb{P}_{\mathbb{K}}^n)$  for  $n \geq 3$ . We will concentrate our attention on the plane Cremona group over the field  $\mathbb{C}$ .

The first way to understand the structure of the Cremona group is to study it group-theoretically. There exist works based on presentations of the group using the generators and relations (for example [Giz82], [Isk83], [Bla12b]), but they do not provide the full understanding of the algebraic properties of the group. For example, given an abstract group it is not possible to say if the given group is isomorphic to a subgroup of the Cremona group. In fact it is known that  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is not simple (i.e. it is not a *nontrivial group whose only normal subgroups are the trivial group and the group itself*, [Ca-La10]), not linear (i.e. it is not *conjugate to the subgroup of  $\text{GL}(n, \mathbb{K})$  for any  $n$  and any  $\mathbb{K}$* , [Cor13]) and not an algebraic group (i.e. it is not a *group which is an algebraic variety, such that the multiplication and inversion operations are given by regular maps on the variety*, [Bl-Fur13]).

The second way to understand the Cremona group is to study its subgroups. The history of these results lasts for more than one hundred years.

- The earliest work studying the Cremona Group is considered to be the Bertini's work on involutions ([Ber77]). In 1877 Bertini identified three types of conjugacy classes of involutions which are now known as de Jonquières, Geiser and Bertini involutions<sup>1</sup>.
- Later on S. Kantor ([Kan95]) and A. Wiman ([Wim96]) described finite subgroups of the Cremona group. However, we cannot say whether a given finite group is isomorphic to a subgroup of the Cremona group using this description.
- A lot of Russian mathematicians successfully worked in this area of study (especially M.K. Gizatullin, V.A. Iskovskikh and Y. Manin). The result which we are particularly interested in is the classification

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<sup>1</sup>there were some corrections in Bertini's proof of classification of conjugacy classes ([Ba-Be00])

of minimal  $G$ -surfaces (rational surfaces with the regular action of some group  $G$  on them) provided by Yu. Manin in the abelian case [Man67] and V.A. Iskovskikh in the general case [Isk79].

• Finally, it is considered that the modern approach has started with the work of L. Bayle and A. Beauville on involutions ([Ba-Be00]). This was the first work which gave the example of a clear description of conjugacy classes: using their results it is possible to conclude whether the two given involutions are conjugate or not. This approach was developed by J. Blanc in his thesis ([Bla06]).

## Goals and Structure of this Project

The goal of this project is to study the basic theory of algebraic surfaces and some finite subgroups of the plane Cremona Group to understand its structure. In this section we will discuss the structure this project.

We first note that for a rational surface  $X$  (i.e.  $X$  birational to  $\mathbb{P}_{\mathbb{C}}^2$ ) we have  $\text{Bir}(X) \cong \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . Let  $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  be a finite group. We can regularize its action: there exists a smooth projective rational surface  $X$  on which  $G$  acts biragularly (10.1.2). Now we give the definition which will help us in later discussion:

**Definition.** Suppose  $X$  is a smooth projective rational surface,  $G \subset \text{Aut}(X)$  is a group acting biregularly on  $X$ . We call a pair  $(G, X)$  a  $G$ -surface.

Now there we have two options:

- 1  $G = id$ . This case corresponds to the birational theory of projective surfaces.
- 2  $G \neq id$ . In this case we we apply the birational theory of projective surfaces to study of finite subgroups of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .

1 We start with the case: **G = id**.

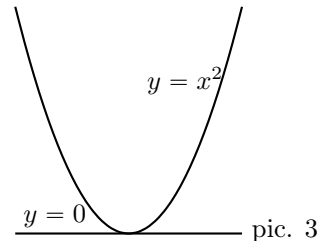
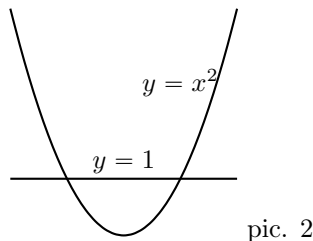
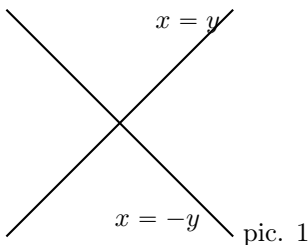
We go through some tools we need to develop in this project:

### ① Theory of Intersections.

The first thing we are interested in is to learn to compute the number of points where two curves on  $X$  intersect. To do this for different irreducible curves  $C, C'$  we look at the intersection locally at point  $x \in X$  and define  $\text{mult}_x(C, C') := \dim_{\mathbb{C}} \mathcal{O}_{X,x}/(f, g)$  where  $f, g \in \mathcal{O}_{X,x}$  are local equations for  $C$  and  $C'$  resp. at  $x$ ) and then define **intersection multiplicity** by taking the sum:

$$C \cdot C' := \sum_{x \in C \cap C'} \text{mult}_x(C, C')$$

Note that this description is very natural. To see it let's consider some examples:



Two lines on pic. 1 intersect at 1 point so the intersection number is 1; a line and a parabola on pic. 2 intersect at 2 points each of multiplicity 1 so the intersection number is 2; a line and a parabola on pic. 3 intersect at 1 point but it has multiplicity 2 so the intersection number is also 2.

It turns out (3.2.5) that we can define a symmetric bilinear form:

$$\text{Pic}(X) \times \text{Pic}(X) \rightarrow \mathbb{Z}, (D, D') \mapsto D \cdot D'$$

which respects this definition.

## ② Blow-Up Construction.

The second important thing is the blow-up construction which is described in 4.1:

**Theorem.** *For a smooth projective surface  $X$  and point  $p$  on  $X$  there exists a smooth projective surface  $\tilde{X}$  and a morphism  $\pi : \tilde{X} \rightarrow X$  which are unique up to isomorphism such that:*

1.  $\pi|_{\pi^{-1}(X \setminus \{p\})} : \pi^{-1}(X \setminus \{p\}) \rightarrow X \setminus \{p\}$  is an isomorphism.
2.  $\pi^{-1}(p) = E \cong \mathbb{P}_{\mathbb{C}}^1$ .

$$\begin{array}{c} \mathbb{P}_{\mathbb{C}}^1 \cong E \subset \tilde{X} \\ \downarrow \pi \\ p \in X \end{array}$$

*the blow-up*

$\pi$  is called the **blow-up of  $X$  at point  $p$** , and  $E$  is called the **exceptional divisor**.

## ③ Castelnuovo's contractibility Theorem and Minimal Surfaces.

The third thing we want to understand is when it is possible to contract curves. Castelnuovo's contractibility criterion (5.5.1) gives the answer:

**Theorem.** *Suppose  $X$  is a smooth projective surface, there is  $E \subset X$  such that  $E \cong \mathbb{P}_{\mathbb{C}}^1$  and  $E^2 = -1$ . Then there exists a morphism  $f : X \rightarrow X'$  to a smooth projective surface  $X'$  and point  $P \in X'$  such that  $X$  is isomorphic via  $f$  to the blow up of  $X'$  at point  $P$  and  $E$  is the exceptional curve.*

Using the blow-ups it is possible to define the order on the set of smooth projective surfaces. Moreover, if we consider a smooth projective rational surface  $X$  then we can find a minimal rational surface  $Y$ <sup>2</sup> after we contract all the exceptional curves using Castelnuovo's contractibility criterion. For rational surfaces we have (9.3.3):

**Theorem.** *Suppose  $X$  is a smooth projective rational surface. Then the following are equivalent:*

$$X \text{ is minimal} \Leftrightarrow \text{there is no exceptional curve on } X \Leftrightarrow (X \cong \mathbb{P}_{\mathbb{C}}^2 \text{ or } X \cong \mathbb{F}_n \text{ for } n \neq 1).$$

② The second case is:  $\boxed{\mathbf{G} \neq \mathbf{id}}$ .

In this case we adapt the birational theory of rational surfaces to the  $G$ -equivariant setting.

## ④ Minimal $G$ -pair.

Suppose  $(G, X)$  is a  $G$ -surface. We say that a birational map  $\phi : X \dashrightarrow X'$  is  **$G$ -equivariant**

<sup>2</sup>i.e. for a surface  $Y$  we have that any birational morphism  $Y \rightarrow Z$  to a smooth projective surface  $Z$  is an isomorphism



if the  $G$ -action on  $X'$  induced by  $\phi$  is biregular and  $\phi$  itself is called a **birational map of  $G$ -surfaces**. We say that a pair  $(G, X)$  is **minimal** or  $G$  **acts minimally on  $X$**  if any  $G$ -equivariant birational morphism  $\phi : X \rightarrow X'$  is an isomorphism.

We prove that we can realize finite subgroup  $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  (10.1.2) as a subgroup of  $\text{Aut}(X)$  where  $X$  is a rational surface. After contracting orbits of disjoint exceptional curves we can assume that  $(X, G)$  is minimal and the study of all  $G$ -surfaces can be reduced to studying only minimal ones.

The classification of minimal  $G$ -surfaces was given by Yu. Manin in the abelian case [Man67] and V.A. Iskovskikh in the general case [Isk79]:

**Theorem.** *Suppose  $X$  is a rational surface,  $G \subset \text{Aut}(X)$  is finite,  $(G, X)$  is a minimal pair. Then exactly one of the following holds:*

1.  $\text{rk}(\text{Pic}(X)^G) = 1$  and  $X$  is a Del Pezzo surface.
2.  $\text{rk}(\text{Pic}(X)^G) = 2$  it is a conic bundle (with the structure is invariant by  $G$ ).

So to study the minimal  $G$ -pairs we need to study the groups of automorphisms of conic bundles and Del Pezzo surfaces.

The final goal of this project is to study the groups of automorphisms of some conic bundles and some del Pezzo surfaces to get information about the conjugacy classes of finite subgroups of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . We prove that:

**Theorem.** *Let  $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  be a finite group. Then  $G$  is conjugated to the subgroup of one of the following groups:*

1. De Jonquieres group which is isomorphic to  $\text{PGL}(2, \mathbb{C}(x)) \rtimes \text{PGL}(2, \mathbb{C})$ .
2.  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) \cong \text{PGL}(3, \mathbb{C})$ .
3.  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) = (\text{PGL}(2, \mathbb{C}))^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ .
4.  $\text{Aut}(X_6) \cong (\mathbb{C}^*)^2 \rtimes (S_3 \times \mathbb{Z}/2\mathbb{Z})$ , where  $X_6$  is the del Pezzo surface of degree 6.
5.  $\text{Aut}(X_5) \cong S_5$  where  $X_5$  is the del Pezzo surface of degree 5.
6.  $\text{Aut}(X_4) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \text{Aut}(X_4, \pi, 5)$ , where  $\text{Aut}(X_4, \pi, 5)$  is the lift of the group of automorphisms of  $\mathbb{P}_{\mathbb{C}}^2$  that leave the set  $\{p_1, \dots, p_5\}$  invariant.
7.  $\text{Aut}(X_k)$  where  $X_k$  is a del Pezzo surface of degree  $k$  for  $k = 1, 2, 3$ .

The detailed summary of the results obtained in this project can be found in Chapter 2.

## Notations

The word "surface" will mean a smooth projective surface over  $\mathbb{C}$ , i.e. a nonsingular projective integral separated scheme of finite type over  $\mathbb{C}$  of dimension 2 if not stated otherwise.

Suppose  $X$  is a surface;  $D, D'$  are the divisors on  $X$ . Then we write:

- $\text{Pic}(X)$  for the Picard group of  $X$ , i.e. group of divisors on  $X$  modulo linear equivalence  $' \sim'$  it is also a group of isomorphism classes of invertible sheaves
- $D \sim D'$  if  $D$  and  $D'$  represent the same class in  $\text{Pic}(X)$
- $\mathcal{O}_X(D)$  for the invertible sheaf corresponding to  $D$
- $\text{Pic}(X)^G$  for the part of the Picard group which is fixed by the group  $G \subset \text{Aut}(X)$
- $H^i(X, \mathcal{O}_X(D))$  for the  $i$ -th cohomology group of the sheaf  $\mathcal{O}_X(D)$
- $h^i(D) = h^i(X, \mathcal{O}_X(D)) = \dim(H^i(X, \mathcal{O}_X(D)))$
- $\chi(X, \mathcal{O}_X(D)) = h^0(D) - h^1(D) + h^2(D)$
- $|D| =$  for the set of effective divisors linearly equivalent to  $D$
- $K_X$  for the canonical divisor on  $X$ , i.e.  $\mathcal{O}_X(K_X) = \Omega_X^2$
- $\text{Aut}(X)$  for group of automorphisms of  $X$
- $\text{Bir}(X)$  for group of birational maps  $X \dashrightarrow X$
- $\text{GL}(n) = \text{GL}(n, \mathbb{C})$ , general linear group over  $\mathbb{C}$
- $\text{PGL}(n) = \text{GL}(n, \mathbb{C})/\mathbb{C}^*$ , general projective linear group over  $\mathbb{C}$
- $S_n$  for the permutation group of degree  $n$
- $A_n$  for the alternating group of degree  $n$
- $D_{2n}$  for the dihedral group of order  $2n$
- $\mathbb{F}_n$  for the  $n$ -th Hirzebruch surface

# Chapter 2

## Summary

In **Chapter 3** we define *the intersection product* on surfaces and prove the results which we use in later discussion: *Riemann-Roch for surfaces* (here  $X$  is a surface and  $C$  is a divisor on  $X$ ):

$$\chi(X, C) = \chi(X, \mathcal{O}_X) + 1/2(C^2 - C \cdot K_X)$$

and *genus formula* (here  $X$  is a surface and  $C$  is a smooth curve on  $X$ ):

$$g(C) = 1 + 1/2(C^2 + C \cdot K_X)$$

In **Chapter 4** we define and give the construction of *the blow-up* of the surface  $X$  at point  $x \in X$  and prove the important properties of the blow-up  $\pi : \tilde{X} \rightarrow X$  with  $\pi^{-1}(x) = E \cong \mathbb{P}_{\mathbb{C}}^1$ ,  $\tilde{C}$  is a strict transform of irreducible curve  $C$ ,  $D, D'$  are divisors on  $X$ ,  $F$  divisor on  $\tilde{X}$ :

$$\pi^*C = \tilde{C} + m_x(C) \cdot E, \quad (\pi^*D) \cdot (\pi^*D') = D \cdot D', \quad E \cdot (\pi^*D) = 0, \quad E^2 = -1$$

$$\text{Pic}(X) \oplus \mathbb{Z}[E] \cong \text{Pic}(\tilde{X}), \quad \pi^*D \cdot F = D \cdot \pi_*F, \quad K_{\tilde{X}} = \pi^*K_X + E \text{ and } K_{\tilde{X}}^2 = K_X^2 - 1$$

In **Chapter 5** we prove that if we have surfaces  $X, X'$  and a birational map  $\phi : X \dashrightarrow X'$ . Then we can find a surface  $\tilde{X}$  together with morphisms  $f, g$  which are composites of blow-ups and isomorphisms such that the diagram commutes:

$$\begin{array}{ccc} & \tilde{X} & \\ f \swarrow & & \searrow g \\ X & \overset{\phi}{\dashrightarrow} & X' \end{array}$$

we also prove the *Castelnuovo's contractibility criterion*: suppose we have a surface  $X$  and there is  $E \subset X$  such that  $E \cong \mathbb{P}_{\mathbb{C}}^1$  and  $E^2 = -1$ . Then there exists a morphism  $f : X \rightarrow X_0$  to a smooth projective surface  $X_0$  and point  $P \in X_0$  such that  $X$  is isomorphic via  $f$  to the blow up of  $X_0$  at point  $P$  and  $E$  is the exceptional curve.

In **Chapter 6** we give the definition of *bubble spaces* and prove that for irreducible curves  $C, D$  on the surface  $X$  we have  $C \cdot D = \sum_{p \in B(X)} m_p(C) \cdot m_p(D)$ .

In **Chapter 7** we study *the del Pezzo surfaces*.

In **Chapter 8** we define and construct *Hirzebruch surfaces*  $\mathbb{F}_n$ . We also prove the important properties of  $\mathbb{F}_n$  (here  $E_n = \{([1 : 0 : 0], [u : v]) \mid [u : v] \in \mathbb{P}_{\mathbb{C}}^1\}$ ,  $f$  is the class of the fiber of  $\mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$ )

$$\text{Pic}(\mathbb{F}_n) = \mathbb{Z}[f] \oplus \mathbb{Z}[E_n], \quad f^2 = 0, \quad (E_n)^2 = -n, \quad E_n \cdot f = 1$$

$$K_{\mathbb{F}_n} = -2E_n - (2+n)f, \quad \mathbb{F}_n \cong \mathbb{F}_m \Leftrightarrow m = n$$

We also prove that  $\text{Aut}(\mathbb{F}_n) = \mathbb{C}^{n+1} \rtimes \text{GL}(2, \mathbb{C}) / \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \mid \mu^n = 1 \right\}$ .

In **Chapter 9** we define *the elementary links* and prove that for the pair of surfaces  $X_1, X_2$  such that  $X_i \in \{\mathbb{P}_{\mathbb{C}}^2, \mathbb{F}_n \mid n \geq 0\}$  for  $i = 1, 2$ . We have that any birational map  $\phi : X_1 \dashrightarrow X_2$  decomposes into automorphisms and elementary links. We also prove that

**Theorem.** *Suppose  $X$  is a rational surface. Then*

$$X \text{ is minimal} \Leftrightarrow \text{there is no } (-1)\text{-curve on } X \Leftrightarrow (X \cong \mathbb{P}_{\mathbb{C}}^2 \text{ or } X \cong \mathbb{F}_n \text{ for } n \neq 1).$$

In **Chapter 10** we observe that a birational map  $\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  gives an isomorphism between  $\text{Bir}(X)$  and  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . So a finite group  $G \subset \text{Aut}(X)$  (for  $X$  a surface) corresponds to a subgroup of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ : we choose a birational map  $\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  and get  $\phi G \phi^{-1} \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . We also give the definitions of a *G-surface*, *G-equivariant birational map*, *minimal pair*, *a conic bundle* and state an important result:

**Theorem.** *Suppose  $X$  is a rational surface,  $G \subset \text{Aut}(X)$  is finite,  $(G, X)$  is a minimal pair. Then exactly one of the following holds:*

1.  $\text{rk}(\text{Pic}(X)^G) = 1$  and  $X$  is a Del Pezzo surface.
2.  $\text{rk}(\text{Pic}(X)^G) = 2$  it is a conic bundle (and the structure is invariant by  $G$ ).

In the first case the fixed part of the Picard group is generated by the canonical divisor, in the second case the fixed part of the Picard group is generated by the canonical divisor and the class of a fiber. We prove that if  $X$  is a rational surface,  $G \subset \text{Aut}(X)$  is a finite group and  $\text{rkPic}(X)^G = 1$  then  $X$  is a Del Pezzo surface and

$$\text{Pic}(X)^G = \begin{cases} \frac{1}{3}\mathbb{Z}K_X, & \text{if } X \cong \mathbb{P}_{\mathbb{C}}^2 \\ \frac{1}{2}\mathbb{Z}K_X, & \text{if } X \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\ \mathbb{Z}K_X, & \text{otherwise} \end{cases}$$

In **Chapter 11** we discuss the tools which may help to understand if two groups are conjugate in the Cremona group and give the definition of *G-elementary links*.

In **Chapter 12** we give the definition of a *minimal conic bundle* and prove the following theorem:

**Theorem.** *Suppose we have a conic bundle  $(X, \pi)$ . Then the following are equivalent:*

$$(X, \pi) \text{ is a minimal conic bundle} \Leftrightarrow \pi \text{ is a smooth fibration} \Leftrightarrow X = \mathbb{F}_n \text{ for some } n.$$

We give the definition of *de Jonquieres group* which is isomorphic to  $\text{PGL}(2, \mathbb{C}(x)) \times \text{PGL}(2, \mathbb{C})$ . We prove the theorem:

**Theorem.** • *Suppose  $X$  is a surface,  $\Lambda$  is a pencil of rational curves on  $X$ . Then there exists a birational map*

$$\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$$

*such that  $\phi(\Lambda)$  is the pencil of fibers of  $\pi_1 : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  and  $\phi \text{Cr}(X, \Lambda) \phi^{-1} = \text{Cr}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \pi_1)$*

- *Suppose  $G$  is a finite subgroup of  $\text{Cr}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \pi_1)$ . Then there exist a conic bundle  $(X, \pi)$  together with a birational map of conic bundles  $\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  such that  $\phi G \phi^{-1} \subset \text{Aut}(X, \pi)$ .*
- *Suppose  $G$  be a finite subgroup of automorphisms  $\text{Aut}(X, \pi)$  of a conic bundle  $(X, \pi)$ . Then  $G$  is conjugate to a subgroup of  $\text{Cr}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \pi_1)$  by a birational map of conic bundles.*

This implies that the group  $Cr(X, \pi)$  is birationally conjugate to the de Jonquières group thus, the study of finite groups of automorphisms of conic bundles is equivalent to the study of finite subgroups of the de Jonquieres group. We also prove that:

**Theorem.** *If  $(X, \pi)$  is a conic bundle with at least one singular fiber, and there exists a finite group  $G \subset \text{Aut}(X, \pi)$  such that a pair  $(G, X)$  is minimal. Then, one of the following occurs:*

1.  $(X, \pi)$  is the exceptional conic bundle
2.  $G' = \{\text{elements which preserve any fiber}\} \subset G$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  or  $(\mathbb{Z}/2\mathbb{Z})^2$ .

In **Chapter 13** we discuss the groups of automorphisms of the del Pezzo surfaces. Recall that

$$\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3, \mathbb{C}), \quad \text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) = (\text{PGL}(2, \mathbb{C}))^2 \rtimes \mathbb{Z}/2\mathbb{Z}$$

and compute

$$\text{Aut}(X_6) \cong (\mathbb{C}^2) \rtimes (S_3 \times \mathbb{Z}/2\mathbb{Z}), \quad \text{Aut}(X_5) \cong S_5, \quad \text{Aut}(X_4) = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \text{Aut}(X_4, \pi, 5)$$

We also say some words about group of automorphisms  $X_k$  for  $k = 1, 2, 3$ .

In **Chapter 14** we consider the examples of some finite subgroups of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . We classify finite abelian subgroups of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  and finite abelian subgroups of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$ .

# Chapter 3

## Intersection Theory on Surfaces

Suppose  $X$  is a smooth projective surface over  $\mathbb{C}$ . In this case prime divisors on  $X$  are irreducible curves and Weil divisors are the formal combinations of irreducible curves with coefficients in  $\mathbb{Z}$  such that only finitely many of these coefficients are non-zero. We denote by  $P(X)$  - prime divisors of  $X$ . Consider a surface  $Y$  and a finite morphism  $f : X \rightarrow Y$ . Suppose  $C \in P(X)$  is an irreducible curve and define:

$$f_*(C) := \begin{cases} 0, & \text{if } f(C) \text{ is a point} \\ d \cdot f(C), & \text{where } d \text{ is the degree of the induced map } C \rightarrow f(C) \end{cases}$$

By linearity we define  $f_*(D)$  for any divisor  $D$ .

### 3.1 Intersection Multiplicity

**Definition 3.1.1.** Suppose  $C, C' \in P(X)$  are two distinct irreducible curves on  $X$  and  $x \in C \cap C'$ . Suppose that  $C$  and  $C'$  are locally defined at  $x$  by  $f, g \in \mathcal{O}_{X,x}$  respectively. We define:

$$\text{mult}_x(C, C') := \dim_{\mathbb{C}} \mathcal{O}_{X,x}/(f, g)$$

We say that  $C$  and  $C'$  are **transverse at  $x$**  if  $\text{mult}_x(C, C') = 1$ .

We define **intersection multiplicity** as:

$$C \cdot C' := \sum_{x \in C \cap C'} \text{mult}_x(C, C')$$

**Example 3.1.2.** Since intersection multiplicity is a local property it is enough to consider it on an affine neighborhood. Suppose  $X = \mathbb{A}_{\mathbb{C}}^2$ . Let's compute the intersection multiplicity at the origin of:

1.  $f(x, y) = x^2 - y, g(x, y) = x$

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2, (0,0)}}{(x^2 - y, x)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]_{(x,y)}}{(x^2 - y, x)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]_{(x,y)}}{(y, x)} = \dim_{\mathbb{C}}(\mathbb{C}) = 1$$

2.  $f(x, y) = x^2 - y, g(x, y) = xy$

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2, (0,0)}}{(x^2 - y, xy)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]_{(x,y)}}{(x^2 - y, xy)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, x^2]_{(x,x^2)}}{(x^3)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x]_x}{(x^3)} = 3$$

$$3. f(x, y) = y^2, g(x, y) = (x - y)^2$$

$$\begin{aligned} \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2, (0,0)}}{(y^2, (x-y)^2)} &= \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]_{(x,y)}}{(y^2, (x-y)^2)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x, y]_{(x,y)}}{(y^2, x^2 - 2xy)} = \\ &= \dim_{\mathbb{C}} \left( \frac{\mathbb{C}[x, y]_{(x,y)}}{(y^2, x(x-2y))} \right) = \dim_{\mathbb{C}} \left( \frac{\mathbb{C}[x, y]_{(x,y)}}{(y^2, x)} \times \frac{\mathbb{C}[x, y]_{(x,y)}}{(y^2, (x-2y))} \right) = \\ &= \dim_{\mathbb{C}} \left( \frac{\mathbb{C}[x, y]_{(x,y)}}{(y^2, x)} \right) + \dim_{\mathbb{C}} \left( \frac{\mathbb{C}[x, y]_{(x,y)}}{(y^2, (x-2y))} \right) = \dim_{\mathbb{C}} \frac{\mathbb{C}[y]_y}{(y^2)} + \dim_{\mathbb{C}} \frac{\mathbb{C}[y]_y}{(y^2)} = 4 \end{aligned}$$

**Proposition 3.1.3.** *Suppose  $X$  is a surface over  $\mathbb{C}$ , and  $C$  a smooth irreducible curve on  $X$ ,  $i : C \rightarrow X$  is the associated closed immersion. Then we have an exact sequence of sheaves:*

$$0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \xrightarrow{i^\#} i_*\mathcal{O}_C \rightarrow 0$$

*Proof.*  $C$  is an effective Cartier divisor (we can assume that because  $C$  is given by the equation) which is given by the collection  $\{(U_i, f_i)\}_{i \in I}$ ,  $f_i \in \mathcal{M}_X^*(U_i)$ ,  $\phi_{ij} = f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$ . Note that for open  $U$  we have:

$$\mathcal{O}_X(-C)(U) = \{f \in \mathcal{M}_X(U) \mid f/f_i \in \mathcal{O}_X(U \cap U_i) \ \forall i \in I \text{ such that } U \cap U_i \neq \emptyset\}$$

Thus if  $f \in \mathcal{O}_X(-C)(U) \Rightarrow f \in \mathcal{O}_X(U)$  and the map  $\mathcal{O}_X(-C) \rightarrow \mathcal{O}_X$  is defined this way.

Now to check that the sequence of sheaves is exact it's enough to prove that it is exact on stalks. Suppose  $x \in X$  so we need to prove that:

$$0 \rightarrow (\mathcal{O}_X(-C))_x \rightarrow \mathcal{O}_{X,x} \xrightarrow{i_x^\#} (i_*\mathcal{O}_C)_x \rightarrow 0$$

is exact and if  $x \in U_i$  we write it as:

$$0 \rightarrow f_i\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x} \xrightarrow{i_x^\#} (i_*\mathcal{O}_C)_x \rightarrow 0$$

1. The first map is injective by the construction.
2. The last map is surjective since  $i^\#$  is a closed immersion by assumption.
3. Suppose that  $x \notin C$  then  $(i_*\mathcal{O}_C)_x = 0$ . Moreover, the  $f_i$ -s are invertible in some neighborhood of  $x$ . Thus  $f_i\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$  and the desired sequence is exact.
4. Suppose that  $x \in C$  and suppose that  $C$  is locally defined by  $f_i$ . The image of  $f_i\mathcal{O}_{X,x}$  in  $(i_*\mathcal{O}_C)_x$  is 0 since  $f_i$  is 0 in all the points of  $C$  around  $x$ . Moreover, all the points in  $\mathcal{O}_{X,x}$  which are 0 are defined locally by  $f_i$  and the desired sequence is exact.

□

We write  $\mathcal{O}_C$  instead of  $i_*\mathcal{O}_C$  in this sequence. So we get the important exact sequence:

$$\boxed{0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \xrightarrow{i^\#} \mathcal{O}_C \rightarrow 0}$$

Suppose  $C, C'$  are two distinct irreducible curves. Consider a sheaf supported on  $C \cap C'$ :

$$\mathcal{O}_{C \cap C'} := \mathcal{O}_X / (\mathcal{O}_X(-C) + \mathcal{O}_X(-C'))$$

Note that for  $x \in C \cap C'$  the stalks are given by:

$$(\mathcal{O}_{C \cap C'})_x = \mathcal{O}_{X,x} / (f, g)$$

where  $f$  and  $g$  are local equations at  $x$  for  $C$  and  $C'$  respectively so we get that:

$$C \cdot C' = h^0(X, \mathcal{O}_{C \cap C'})$$

## 3.2 Intersection Product

Define the **Euler Characteristic** of a line bundle  $\mathcal{L}$  as:

$$\chi(X, \mathcal{L}) := \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, \mathcal{L})$$

For a sheaf  $\mathcal{F}$  which has 0-dimension support we have  $H^i(X, \mathcal{F}) = 0$  for  $i \geq 1$  so

$$\chi(X, \mathcal{F}) = h^0(X, \mathcal{F}) \Rightarrow (C \cdot C') = \chi(X, \mathcal{O}_{C \cap C'})$$

**Definition 3.2.1.** Suppose  $\mathcal{L}, \mathcal{L}'$  are line bundles on  $X$ .

Define the **intersection product**:

$$\mathcal{L} \cdot \mathcal{L}' := \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}^{-1}) - \chi(X, \mathcal{L}'^{-1}) + \chi(X, \mathcal{L}^{-1} \otimes \mathcal{L}'^{-1})$$

Recall that for a smooth projective surface  $X$  we have  $C \rightarrow \mathcal{O}_X(C)$  is a one to one correspondence between divisors and invertible subsheaves of  $\mathcal{M}_X$  so for  $\mathcal{L} = \mathcal{O}_X(C)$ ,  $\mathcal{L}' = \mathcal{O}_X(C')$  we can write it as:

$$\mathcal{O}(C) \cdot \mathcal{O}(C') := \chi(X, \mathcal{O}_X) - \chi(X, -C) - \chi(X, -C') + \chi(X, -C - C')$$

**Proposition 3.2.2.** Suppose  $C, C'$  are two distinct irreducible curves on  $X$  then:

$$\mathcal{O}_X(C) \cdot \mathcal{O}_X(C') = C \cdot C'$$

*Proof.* Consider the sequence:

$$0 \rightarrow \mathcal{O}_X(-C - C') \xrightarrow{\text{inclusion}} \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C') \xrightarrow{(s-s')} \mathcal{O}_X \rightarrow \mathcal{O}_{C \cap C'} \rightarrow 0$$

To prove that it is exact it is enough to prove it on stalks:

$$0 \rightarrow \mathcal{O}_X(-C - C')_x \rightarrow \mathcal{O}_X(-C)_x \oplus \mathcal{O}_X(-C')_x \rightarrow \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{C \cap C',x} \rightarrow 0$$

Which is equivalent to the following sequence where  $f, g$  are local expressions of  $C$  and  $C'$  respectively:

$$0 \rightarrow fg\mathcal{O}_{X,x} \rightarrow f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{C \cap C',x} \rightarrow 0$$

Where the map  $fg\mathcal{O}_{X,x} \rightarrow f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x}$  is given by  $fg\alpha \rightarrow (f \cdot g\alpha, g \cdot f\alpha)$ , the map  $f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$  is given by  $(f \cdot \alpha, g \cdot \beta) \rightarrow f \cdot \alpha - g \cdot \beta$  for  $\alpha, \beta \in \mathcal{O}_{X,x}$ .

Suppose  $x \notin C$  and  $x \notin C'$  then  $f, g$  are invertible in  $\mathcal{O}_{X,x}$  so  $fg\mathcal{O}_{X,x} = f\mathcal{O}_{X,x} = g\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ ;  $\mathcal{O}_{C \cap C',x} = 0$  and we get the exact sequence:

$$0 \rightarrow fg\mathcal{O}_{X,x} \rightarrow f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x} \rightarrow 0$$

since equivalent to exact sequence:

$$0 \rightarrow \mathcal{O}_{X,x} \xrightarrow{\text{inclusion}} \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x} \xrightarrow{(s-s')} \mathcal{O}_{X,x} \rightarrow 0$$

Suppose  $x \notin C$  and  $x \in C'$  (the opposite case is proved similarly) then  $f$  are invertible in  $\mathcal{O}_{X,x}$  so  $fg\mathcal{O}_{X,x} = f\mathcal{O}_{X,x}$ ,  $g\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ ;  $\mathcal{O}_{C \cap C',x} = 0$  and we get the exact sequence:

$$0 \rightarrow fg\mathcal{O}_{X,x} \rightarrow f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x} \rightarrow 0$$



since equivalent to exact sequence:

$$0 \rightarrow f\mathcal{O}_{X,x} \xrightarrow{\text{inclusion}} f\mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x} \xrightarrow{(s-s')} \mathcal{O}_{X,x} \rightarrow 0$$

Suppose now  $x \in C \cap C'$ , in this case the sequence on the stalks is:

$$0 \rightarrow fg\mathcal{O}_{X,x} \xrightarrow{\text{inclusion}} f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \xrightarrow{(s-s')} \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}/(f, g) \rightarrow 0$$

1. The first map is clearly injective.
2. The last map  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}/(f, g)$  is surjective as projection.
3. The image of  $f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$  is the ideal  $(f, g)$  which is the kernel of  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}/(f, g)$ .
4. We have that  $afg \rightarrow (f \cdot ag, g \cdot af) \rightarrow afg - afg = 0$  thus:

$$\text{image}(fg\mathcal{O}_{X,x} \rightarrow f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x}) \subset \text{kernel}(f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x})$$

5. Suppose  $(a, b) \in \mathcal{O}_{X,x}^2$  such that  $ag - bf = 0$ , i.e.  $(a, b) \in \ker(f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x})$ . Then since  $f$  and  $g$  are coprime there exists  $l \in \mathcal{O}_{X,x}$  such that  $a = lg, b = lf$ . So we get that:  $l \rightarrow (lg, lf) = (a, b)$ .

Thus:

$$\text{kernel}(f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}) \subset \text{image}(fg\mathcal{O}_{X,x} \rightarrow f\mathcal{O}_{X,x} \oplus g\mathcal{O}_{X,x})$$

and the desired sequence is exact, so we get:

$$\begin{aligned} C \cdot C' &= h^0(X, \mathcal{O}_{C \cap C'}) \stackrel{(*)}{=} \chi(X, \mathcal{O}_{C \cap C'}) = \\ &= \chi(X, \mathcal{O}_X) - \chi(X, -C) - \chi(X, -C') + \chi(X, -C - C') = \mathcal{O}_X(C) \cdot \mathcal{O}_X(C') \end{aligned}$$

(\*) holds since  $C \cap C'$  has 0-dimension support. □

Let's recall the Riemann-Roch theorem for curves ([Har77] Theorem I.1.3):

**Theorem 3.2.3.** (Riemann-Roch for curves) Suppose  $C$  is a smooth curve (i.e. a projective integral scheme of dimension 1 over  $\mathbb{C}$ ) and  $D$  is a divisor on  $C$  then we have:

$$\begin{aligned} \chi(C, D) &= \deg(D) + \chi(C, \mathcal{O}_C) = \\ &= \deg(D) + h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = \deg(D) + 1 - g(C) \end{aligned}$$

where  $g(C) = h^1(C, \mathcal{O}_C)$ .

**Proposition 3.2.4.** Suppose  $C$  is a smooth curve on  $X$ ,  $\mathcal{L} \in \text{Pic}(X)$  then

$$\boxed{\mathcal{O}_X(C) \cdot \mathcal{L} = \deg(\mathcal{L}|_C), \text{ where } \mathcal{L}|_C = \mathcal{L} \otimes \mathcal{O}_C}$$

*Proof.* Consider the exact sequence:

$$0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$$

Twisting by  $\mathcal{L}^{-1}$  we get the exact sequence (since  $\mathcal{L}^{-1}$  is flat):

$$0 \rightarrow \mathcal{L}^{-1} \otimes \mathcal{O}_X(-C) \rightarrow \mathcal{L}^{-1} \rightarrow \mathcal{L}^{-1} \otimes \mathcal{O}_C = \mathcal{L}^{-1}|_C \rightarrow 0$$

So we get:

$$\chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-C)) = \chi(X, \mathcal{O}_C)$$

and

$$\chi(X, \mathcal{L}^{-1}) - \chi(X, \mathcal{L}^{-1} \otimes \mathcal{O}_X(-C)) = \chi(X, \mathcal{L}^{-1}|_C)$$

By definition:

$$\begin{aligned} \mathcal{O}_X(C) \cdot \mathcal{L} &= \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-C)) - \chi(X, \mathcal{L}^{-1}) + \chi(X, \mathcal{O}_X(-C) \otimes \mathcal{L}^{-1}) = \\ &= \chi(X, \mathcal{O}_C) - \chi(X, \mathcal{L}^{-1}|_C) \stackrel{(\star)}{=} 1 - g(C) - (\deg(\mathcal{L}|_C) + 1 - g(C)) = \deg(\mathcal{L}|_C) \end{aligned}$$

( $\star$ ) by Riemann-Roch for curves. □

**Proposition 3.2.5.** *The intersection product*

$$\mathcal{L} \cdot \mathcal{L}' := \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}^{-1}) - \chi(X, \mathcal{L}'^{-1}) + \chi(X, \mathcal{L}^{-1} \otimes \mathcal{L}'^{-1})$$

is a symmetric bilinear form

$$\text{Pic}(X) \times \text{Pic}(X) \rightarrow \mathbb{Z}$$

such that we have the equality:

$$\mathcal{O}_X(C) \cdot \mathcal{O}_X(C') = C \cdot C'$$

for two distinct irreducible curves  $C, C'$  on  $X$ .

*Proof.* 1). The intersection product is symmetric:

$$\begin{aligned} \mathcal{L} \cdot \mathcal{L}' &= \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}^{-1}) - \chi(X, \mathcal{L}'^{-1}) + \chi(X, \mathcal{L}^{-1} \otimes \mathcal{L}'^{-1}) = \\ &= \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}'^{-1}) - \chi(X, \mathcal{L}^{-1}) + \chi(X, \mathcal{L}^{-1} \otimes \mathcal{L}'^{-1}) = \mathcal{L}' \cdot \mathcal{L} \end{aligned}$$

2). The intersection product is bilinear, i.e for  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in \text{Pic}(X)$  we have

$$\mathcal{L}_1 \cdot (\mathcal{L}_2 \otimes \mathcal{L}_3) = \mathcal{L}_1 \cdot \mathcal{L}_2 + \mathcal{L}_1 \cdot \mathcal{L}_3$$

$$(\mathcal{L}_1 \otimes \mathcal{L}_2) \cdot \mathcal{L}_3 = \mathcal{L}_1 \cdot \mathcal{L}_3 + \mathcal{L}_2 \cdot \mathcal{L}_3$$

Let's prove the first one (the proof of the second one is similar). Suppose  $\mathcal{L}_1 = \mathcal{O}_X(C)$  for  $C$  smooth. Then:

$$\begin{aligned} \mathcal{L}_1 \cdot (\mathcal{L}_2 \otimes \mathcal{L}_3) &= \mathcal{O}_X(C) \cdot (\mathcal{L}_2 \otimes \mathcal{L}_3) \stackrel{3.2.4}{=} \deg(\mathcal{L}_2 \otimes \mathcal{L}_3)|_C = \\ &= \deg(\mathcal{L}_2)_C + \deg(\mathcal{L}_3)_C = \mathcal{L}_1 \cdot \mathcal{L}_2 + \mathcal{L}_1 \cdot \mathcal{L}_3 \end{aligned}$$

Now we need to prove a general case. We can write  $D \sim A - B$ , where  $A, B$  are smooth curves (for sufficiently large  $n$  we have  $\mathcal{O}_X(D) \otimes \mathcal{O}_X(n), \mathcal{O}_X(n)$  very ample and we take  $B \in |\mathcal{O}_X(n)|, A \in |\mathcal{O}_X(D) \otimes \mathcal{O}_X(n)|$  and we can chose them to be smooth by the Bertini Theorem [Har77] III.10.9). Now suppose  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in \text{Pic}(X), \mathcal{L}_1 = \mathcal{O}_X(D) = \mathcal{O}_X(A - B) = \mathcal{O}_X(A) \otimes \mathcal{O}_X(-B)$ :

$$\begin{aligned} \mathcal{L}_1 \cdot (\mathcal{L}_2 \otimes \mathcal{L}_3) &= (\mathcal{O}_X(A) \otimes \mathcal{O}_X(-B)) \cdot (\mathcal{L}_2 \otimes \mathcal{L}_3) = \\ &= \mathcal{O}_X(A) \cdot (\mathcal{L}_2 \otimes \mathcal{L}_3) - \mathcal{O}_X(B) \cdot (\mathcal{L}_2 \otimes \mathcal{L}_3) = \\ &= \deg(\mathcal{L}_2)_A + \deg(\mathcal{L}_3)_A - \deg(\mathcal{L}_2)_B - \deg(\mathcal{L}_3)_B = \\ &= \mathcal{O}_X(A) \cdot \mathcal{L}_2 + \mathcal{O}_X(A) \cdot \mathcal{L}_3 - \mathcal{O}_X(B) \cdot \mathcal{L}_2 - \mathcal{O}_X(B) \cdot \mathcal{L}_3 = \mathcal{O}_X(A - B) \cdot \mathcal{L}_2 + \mathcal{O}_X(A - B) \cdot \mathcal{L}_3 = \\ &= \mathcal{O}_X(D) \cdot \mathcal{L}_2 + \mathcal{O}_X(D) \cdot \mathcal{L}_3 = \mathcal{L}_1 \cdot \mathcal{L}_2 + \mathcal{L}_1 \cdot \mathcal{L}_3 \end{aligned}$$

3) By Proposition 3.2.2 we know that for distinct irreducible curves  $C, C'$  we have:

$$\mathcal{O}_X(C) \cdot \mathcal{O}_X(C') = C \cdot C'$$

□

**Definition 3.2.6.** Suppose  $D, D'$  are the divisors on  $X$ . Define:

$$D \cdot D' := \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')$$

**Example 3.2.7.** Suppose  $X = \mathbb{P}_{\mathbb{C}}^2$ . We know that  $\text{Pic}(X) = \mathbb{Z}$  and any line  $L$  is a generator. Note that we can find lines  $L_1, L_2$  such that they meet transversely. Then:

$$L^2 = L_1 \cdot L_2 = 1$$

The intersection product is linear. If  $C$  is the curve of degree  $d$ ,  $C'$  is the curve of degree  $d'$ , then  $C \sim dL$ ,  $C' \sim d'L$ . Then we get the Bezout's Theorem:

$$C \cdot C' = (dL) \cdot (d'L) = (d \cdot d')L^2 = d \cdot d'$$

### 3.3 Riemann-Roch for Surfaces

Recall that **the canonical divisor**  $K_X \in \text{Pic}(X)$  on an algebraic variety  $X$  is given by the divisor of a 2-differential form. It is well-defined since two distinct differential forms give linearly equivalent divisors.

**Proposition 3.3.1.** (Riemann-Roch for surfaces)

For a smooth projective surface  $X$  and  $\mathcal{L} \in \text{Pic}(X)$  we have:

$$\chi(X, \mathcal{L}) = \chi(X, \mathcal{O}_X) + 1/2(\mathcal{L}^2 - \mathcal{L} \cdot K_X)$$

Recall that for a smooth projective surface  $X$  we have that  $C \mapsto \mathcal{O}_X(C)$  is a one to one correspondence between divisors and invertible subsheaves of  $\mathcal{M}_X$  so for  $\mathcal{L} = \mathcal{O}_X(C)$  we can write it as:

$$\chi(X, C) = \chi(X, \mathcal{O}_X) + 1/2(C^2 - C \cdot K_X)$$

*Proof.* First note that by Serre's Duality we have:

$$\chi(X, \mathcal{O}_X) = \chi(X, \mathcal{O}_X(K_X)) \quad \text{and} \quad \chi(X, \mathcal{L}) = \chi(X, \mathcal{L}^{-1} \otimes \mathcal{O}_X(K_X))$$

From definition we deduce:

$$\begin{aligned} \mathcal{L}^{-1} \cdot (\mathcal{L} \otimes \mathcal{O}_X(-K_X)) &= \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L}) - \chi(X, \mathcal{L}^{-1} \otimes \mathcal{O}_X(K_X)) + \chi(X, \mathcal{O}_X(K_X)) = \\ &= 2(\chi(X, \mathcal{O}_X) - \chi(X, \mathcal{L})) \Rightarrow \chi(X, \mathcal{L}) = \chi(X, \mathcal{O}_X) - 1/2\mathcal{L}^{-1} \cdot (\mathcal{L} \otimes \mathcal{O}_X(-K_X)) \stackrel{\text{bilinearity}}{=} \\ &= \chi(X, \mathcal{O}_X) + 1/2(\mathcal{L}^2 - \mathcal{L} \cdot \mathcal{O}_X(K_X)) \end{aligned}$$

□

**Example 3.3.2.** Suppose  $X = \mathbb{P}_{\mathbb{C}}^2$ . In this case  $\chi(\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}) = 1$ ,  $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(K_{\mathbb{P}_{\mathbb{C}}^2}) = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(-3)$ . Then by the Riemann-Roch for surfaces we obtain:

$$\chi(\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(d)) = 1 + 1/2(d^2 + 3d)$$

### 3.4 Genus Formula

**Proposition 3.4.1.** (*Genus formula*)

For a smooth projective surface  $X$  and an irreducible curve  $C$  on  $X$  the genus of  $C$  is:

$$g(C) = 1 + 1/2(C^2 + C \cdot K_X)$$

*Proof.* Consider the exact sequence:

$$0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$$

So we get:

$$\chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-C)) = \chi(X, \mathcal{O}_C)$$

By Riemann-Roch for  $X$ :

$$\chi(X, \mathcal{O}_X(-C)) = \chi(X, \mathcal{O}_X) + 1/2(C^2 + C \cdot K_X)$$

Thus, we obtain:

$$1 - g(C) \stackrel{(\star)}{=} \chi(X, \mathcal{O}_C) = -1/2(C^2 + C \cdot K_X) \Rightarrow g(C) = 1 + 1/2(C^2 + C \cdot K_X)$$

( $\star$ ) by Riemann-Roch for curves. □

**Example 3.4.2.** Suppose  $X = \mathbb{P}_{\mathbb{C}}^2$  and  $C$  is an irreducible curve of degree  $d$ , i.e.  $C \sim dL$  where  $L$  is a line. Then:

$$g(C) = 1 + 1/2(d^2 - 3d) = \frac{(d-1)(d-2)}{2}$$

# Chapter 4

## Blow-up

Suppose  $X$  is a smooth projective surface over  $\mathbb{C}$ .

### 4.1 Definition and Construction

**Proposition 4.1.1.** *For a surface  $X$  and point  $p$  on  $X$  there exists a surface  $\tilde{X}$  and a morphism  $\pi : \tilde{X} \rightarrow X$  which are unique up to isomorphism such that:*

1.  $\pi|_{\pi^{-1}(X \setminus \{p\})} : \pi^{-1}(X \setminus \{p\}) \rightarrow X \setminus \{p\}$  is an isomorphism.
2.  $\pi^{-1}(p) = E \cong \mathbb{P}_{\mathbb{C}}^1$ .

$$\begin{array}{c} \mathbb{P}_{\mathbb{C}}^1 \cong E \subset \tilde{X} \\ \text{the blow-up} \downarrow \pi \\ p \in X \end{array}$$

$\pi$  is called **the blow-up of  $X$  at point  $p$** ,  $E$  is called **the exceptional divisor/exceptional curve**. We will denote the blow-up of  $X$  at point  $p$  either  $\text{Bl}_p(X)$  or  $\tilde{X}$ .

**Construction:** First let's construct the blow-up  $\tilde{\mathbb{A}}_{\mathbb{C}}^2$  of  $\mathbb{A}_{\mathbb{C}}^2$  at point  $(0, 0)$ . It is given by:

$$\tilde{\mathbb{A}}_{\mathbb{C}}^2 = \text{Bl}_{(0,0)}(\mathbb{A}_{\mathbb{C}}^2) := \{((x_1, x_2), [y_0 : y_1]) \mid x_1 y_1 = x_2 y_0\} \subset \mathbb{A}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^1$$

Note that for  $(x_1, x_2) \neq (0, 0)$  (without loss of generality  $x_1 \neq 0$ ). We obtain

$$x_1 y_1 = x_2 y_0 \Rightarrow y_1 = (x_2/x_1)y_0 \Rightarrow [y_0 : y_1] = [1 : x_2/x_1]$$

so we see that  $\pi^{-1}((x_1, x_2))$  consists of one point. For  $(0, 0)$  we have  $\pi^{-1}((0, 0)) = \mathbb{P}_{\mathbb{C}}^1$ . Now consider a point  $p$  on surface  $X$ , the affine neighborhood  $U$  of  $p$  with local coordinates  $x_1, x_2$ . Define  $\tilde{U}$  as above:

$$\tilde{U} := \{((x_1, x_2), [y_0 : y_1]) \mid x_1 y_1 = x_2 y_0\} \subset U \times \mathbb{P}_{\mathbb{C}}^1$$

It's clear that  $U \setminus \{p\} \cong \tilde{U} \setminus \{\pi^{-1}(p)\}$ . So we can get  $\tilde{X}$  by passing  $\tilde{U}$  and  $X \setminus \{p\}$  along this intersection.

**Proposition 4.1.2.**  *$\tilde{X}$  is a smooth projective surface with  $E \cong \mathbb{P}_{\mathbb{C}}^1$  as above.*

*Proof.* Note that  $\tilde{X}$  is given by  $x_1 = x_2 y_0$  on  $\mathbb{A}_{\mathbb{C}}^2 \times U_1$  and by  $y_1 x_1 = x_2$  on  $\mathbb{A}_{\mathbb{C}}^2 \times U_0$  so it is smooth on the each part of the cover of the neighborhood of  $E$ , so  $\tilde{X}$  is itself smooth.  $\square$

Suppose  $\pi : \tilde{X} \rightarrow X$  is a blow up of  $X$  at point  $p \in X$ ,  $\pi^{-1}(p) = E$ . Note that we can identify the points of  $E = \pi^{-1}(x)$  with the tangent directions at point  $x$  on  $X$ . Suppose  $C$  is an irreducible curve on  $X$  and  $C$  passes through  $x$  with multiplicity  $m$ . Here the **multiplicity** is defined as follows: if  $f \in \mathfrak{m}_p^k \setminus \mathfrak{m}_p^{k+1}$  ( $\mathfrak{m}_p$  is a maximal ideal of  $\mathcal{O}_{X,p}$ ) then  $m = k$ . Suppose we can choose an open affine neighborhood  $p \in U \subset X$  ( $U \subset \mathbb{A}_{\mathbb{C}}^2$ ), such that locally we have  $p = (0,0)$ , and  $C$  is given by  $C : \sum_i f_i(\alpha, \beta) = 0$ , where  $f_i$  are homogeneous polynomials of degree  $i$  in  $\alpha, \beta$ . The multiplicity  $m_p(C)$  is equal to  $\min \{ i \mid f_i(\alpha, \beta) \neq 0 \}$ . We have:

- (1)  $m_p(C) \geq 0$       (2)  $m_p(C) = 0 \Leftrightarrow p \notin C$       (3)  $m_p(C) = 1 \Leftrightarrow p$  is a smooth point of  $C$   
(4)  $m_p(C) \geq 2 \Leftrightarrow p$  is a singular point of  $C$

Then the closure of  $\pi^{-1}(C \setminus \{p\})$  is an irreducible curve  $\tilde{C}$  on  $\tilde{X}$  called **the strict transform of  $C$** .

Recall that for  $\phi : X \rightarrow Y$  a dominant morphism of surfaces it is possible to show that there exists a group morphism

$$\phi^* : \text{Div}(Y) \rightarrow \text{Div}(X)$$

(the dominant hypothesis is not necessary but we assume it for simplicity). Such morphism gives is a field immersion  $\phi^* : \mathbb{C}(Y) \rightarrow \mathbb{C}(X)$ , where  $\mathbb{C}(-)$  denotes the field of rational functions. Consider a Cartier divisor  $D$  on  $Y$ . It is given by a collection  $\{(U_i, f_i)\}_{i \in I}$  where  $\{U_i\}_{i \in I}$  is an open cover of  $Y$  and  $f_i \in \mathcal{M}_X^*(U_i)$ ,  $\phi_{ij} = f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$  for all  $i, j \in I$ . We define

$$\phi^*D \in \text{Div}(X) \text{ which is given by } \{(\phi^{-1}(U_i)_{i \in I}, \phi^*f_i)\}_{i \in I}$$

The fact that it is a Cartier divisor on  $X$  can be proved straightforward.

**Example 4.1.3.** Consider a curve

$$C : x_1^2 - x_2^3 = 0 \subset \mathbb{A}_{\mathbb{C}}^2$$

and its pull-back  $\pi^*C$  under the blow-up  $\pi : \tilde{\mathbb{A}}_{\mathbb{C}}^2 \rightarrow \mathbb{A}_{\mathbb{C}}^2$  at  $(0,0)$

Suppose  $y_0 \neq 0 \Rightarrow$  from  $x_1y_1 = x_2y_0$  we obtain  $x_2 = x_1(y_1/y_0) =: x_1\alpha$ . So  $\pi^*C$  on affine standard open  $\mathbb{A}_{\mathbb{C}}^2 \times U_0$  is defined by

$$0 = x_1^2 - x_2^3 = x_1^2 - x_1^3\alpha^3 = x_1^2(1 - x_1\alpha^3)$$

and  $\tilde{C}$  is given by  $(1 - x_1\alpha^3)$ .

Suppose  $y_1 \neq 0 \Rightarrow$  from  $x_1y_1 = x_2y_0$  we obtain  $x_1 = x_2(y_0/y_1) =: x_2\beta$ . So  $\pi^*C$  on affine standard open  $\mathbb{A}_{\mathbb{C}}^2 \times U_1$  is defined by

$$0 = x_1^2 - x_2^3 = x_2^2\beta^2 - x_2^3 = x_2^2(\beta^2 - x_2)$$

and  $\tilde{C}$  is given by  $(\beta^2 - x_2)$ .

**Example 4.1.4.** Consider a curve

$$C : x_2^2 - x_1^2 - x_1^3 = 0 \subset \mathbb{A}_{\mathbb{C}}^2$$

and its pull-back  $\pi^*C$  under the blow-up  $\pi : \tilde{\mathbb{A}}_{\mathbb{C}}^2 \rightarrow \mathbb{A}_{\mathbb{C}}^2$  at  $(0,0)$ .

Suppose  $y_0 \neq 0 \Rightarrow$  from  $x_1y_1 = x_2y_0$  we obtain  $x_2 = x_1(y_1/y_0) =: x_1\alpha$ . So  $\pi^*C$  on affine standard open  $\mathbb{A}_{\mathbb{C}}^2 \times U_0$  is defined by:

$$0 = x_2^2 - x_1^2 - x_1^3 = x_1^2\alpha^2 - x_1^2 - x_1^3 = x_1^2(\alpha^2 - 1 - x_1)$$

and  $\tilde{C}$  is given by  $\alpha^2 - 1 - x_1$ .

Suppose  $y_1 \neq 0$  then similarly from  $x_1y_1 = x_2y_0$  we obtain  $x_1 = x_2(y_0/y_1) =: x_2\beta$ . So  $\pi^*C$  on affine standard open  $\mathbb{A}_{\mathbb{C}}^2 \times U_1$  is defined by

$$0 = x_2^2 - x_1^2 - x_1^3 = x_2^2 - x_2^2\beta^2 - x_2^3\beta^3 = x_2^2(1 - \beta^2 - x_2\beta^3)$$

and  $\tilde{C}$  is given by  $1 - \beta^2 - x_2\beta^3$ .

## 4.2 Properties of The Blow-up

Note that in the examples above we have that  $\pi^*C = \tilde{C} + 2E$  where  $2 = m_{(0,0)}(C)$ . Let's prove it in the general case.

**Proposition 4.2.1.** *Suppose  $C$  is an irreducible curve on  $X$  and  $C$  passes through point  $x$  with multiplicity  $m$ . Then:*

$$\boxed{\pi^*C = \tilde{C} + mE}$$

*Proof.* We see that there exists  $m' \in \mathbb{N}$  such that  $\pi^*C = \tilde{C} + m'E$  (since we have isomorphism  $\tilde{X} \setminus E \cong X \setminus \{p\}$  and  $\pi(\tilde{C}) = C$ ,  $\pi(E) = (0, 0)$ ). Now consider local coordinates at point  $P = (0, 0) \in U$ -neighborhood given by  $\alpha$  and  $\beta$ . The equation of  $C$  is given locally at  $\mathcal{O}_{X,P}$  by:

$$f = f_m(\alpha, \beta) + f_{m+1}(\alpha, \beta) + \dots, \text{ where } f_m(\alpha, \beta) \text{ is homogeneous of degree } m$$

note that  $f_m(\alpha, \beta) \neq 0$  since  $m$  is a multiplicity at  $P$  by assumption.

Now consider  $\tilde{U}$  as in the construction above. Points on  $E \cong \mathbb{P}_\mathbb{C}^1$  are given by  $\alpha v = u\beta$  consider it on  $u \neq 0$  so we can write:

$$\pi^*f = f(\alpha, \alpha(v/u)) = \alpha^m(f_m(1, v/u) + f_{m+1}(1, v/u) + \dots)$$

Since a local equation for  $E$  is given by  $\alpha = 0$  we obtain that the local multiplicity of  $E$  equals  $m$ , which means  $\pi^*C = \tilde{C} + mE$ .  $\square$

**Proposition 4.2.2.** *Suppose  $D, D'$  are divisors on  $X$ . Then:*

$$(\pi^*D) \cdot (\pi^*D') = D \cdot D', \quad E \cdot (\pi^*D) = 0, \quad E^2 = -1$$

*Proof.* • Note that similarly to the proof 3.2.5 we can find very ample divisors  $D_1, D_2, D'_1, D'_2$  such that  $D \sim D_1 - D_2, D' \sim D'_1 - D'_2$ . The divisors  $D_1, D_2, D'_1, D'_2$  separate points so we can find the divisors  $A_1 \sim D_1, A_2 \sim D_2, A'_1 \sim D'_1, A'_2 \sim D'_2$  such that  $P \notin \text{Supp}(A_i) \cup \text{Supp}(A'_i)$  for  $i = 1, 2$ . Then:

$$\begin{aligned} (\pi^*D) \cdot (\pi^*D') &= (\pi^*(D_1 - D_2)) \cdot (\pi^*(D'_1 - D'_2)) = \\ &= (\pi^*(A_1 - A_2)) \cdot (\pi^*(A'_1 - A'_2)) = (A_1 - A_2) \cdot (A'_1 - A'_2) = D \cdot D' \end{aligned}$$

• take the divisor  $A \sim D$  such that  $P \notin \text{Supp}(A)$ , Note that in this case  $\tilde{A} \cap E = 0$ . Then:

$$E \cdot (\pi^*D) = E \cdot (\pi^*A) = E \cdot \tilde{A} = 0$$

• Suppose  $C$  is a smooth curve passing through  $P$  such that  $\pi^*C = \tilde{C} + E$  (i.e with multiplicity 1 at point  $P$ ).  $\tilde{C}$  meets  $E$  transversally. Thus:

$$0 = E \cdot (\pi^*C) = E \cdot (\tilde{C} + E) = E \cdot \tilde{C} + E^2 = 1 + E^2 \Rightarrow E^2 = -1$$

$\square$

**Proposition 4.2.3.** *There are isomorphisms:*

$$\boxed{\text{Pic}(X) \oplus \mathbb{Z}[E] \cong \text{Pic}(\tilde{X}), \text{ where } (D, m) \rightarrow \pi^*D + mE}$$

*Proof.* consider an irreducible curve  $\tilde{C} \neq E$  on  $\tilde{X}$ . Note that  $\pi^*(\pi(\tilde{C})) = \tilde{C} + mE$  for some  $m \in \mathbb{Z}$  so we can define a surjective map:

$$\text{Pic}(X) \oplus \mathbb{Z} \rightarrow \text{Pic}(\tilde{X}), \text{ where } (D, m) \rightarrow \pi^*D + mE$$

Suppose  $\pi^*D + mE \sim 0$ . Note that:

$$0 = (\pi^*D + mE) \cdot E = \pi^*D \cdot E + mE \cdot E \stackrel{4.2.2}{=} 0 - m \cdot 1 \Rightarrow m = 0$$

Thus we obtain:

$$\mathcal{O}_X \cong^{(*)} \pi_*\mathcal{O}_{\tilde{X}} \cong \pi_*(\mathcal{O}_{\tilde{X}}(\pi^*(D + mE))) = \pi_*(\mathcal{O}_{\tilde{X}}(\pi^*D)) \cong \pi_*\mathcal{O}_{\tilde{X}}(D) \cong \mathcal{O}_X(D) \Rightarrow D \sim 0$$

the proof of  $(*)$  can found in [Har77] Corollary III.11.4. Thus, our map is an isomorphism.  $\square$

**Corollary 4.2.4.** *Suppose  $C \in \text{Pic}(X)$ ,  $D \in \text{Pic}(\tilde{X})$ , then*

$$\boxed{\pi^*C \cdot D = C \cdot \pi_*D}$$

**Definition 4.2.5.** *Note that the curve obtained by blowing-up is isomorphic to  $\mathbb{P}_{\mathbb{C}}^1$  and has self intersection  $-1$ . We call it a  $(-1)$ -**curve**.*

*Proof.* Suppose  $C \sim A - B$ , where  $A, B$  are curves not containing  $P$ . Then:

$$\pi^*C \cdot D = \pi^*(A - B) \cdot D = \pi^*A \cdot D - \pi^*B \cdot D = A \cdot \pi_*D - B \cdot \pi_*D = C \cdot \pi_*D$$

$\square$

**Proposition 4.2.6.**

$$\boxed{K_{\tilde{X}} = \pi^*K_X + E \text{ and } K_{\tilde{X}}^2 = K_X^2 - 1}$$

*Proof.* We know that  $\tilde{X} \setminus E \cong X \setminus P$  thus the canonical divisor  $K_{\tilde{X}}$  on  $\tilde{X} \setminus E$  coincides with the canonical divisor  $K_X$  on  $X \setminus P$ . So  $K_{\tilde{X}}$  is of the form  $K_{\tilde{X}} = \pi^*K_X + nE$  for some  $n \in \mathbb{Z}$ . By genus formula 3.4.1 for  $E \cong \mathbb{P}_{\mathbb{C}}^1$  (so  $g(E) = 0$ ) we have:

$$\begin{aligned} -2 &= 2g(E) - 2 = E \cdot (E + K_{\tilde{X}}) = E^2 + E \cdot K_{\tilde{X}} = -1 + E \cdot K_{\tilde{X}} \Rightarrow \\ &\Rightarrow E \cdot K_{\tilde{X}} = -1 \Rightarrow E \cdot (\pi^*K_X + nE) = -1 \Rightarrow -n = -1 \Rightarrow n = 1 \end{aligned}$$

Now:

$$K_{\tilde{X}}^2 = (\pi^*K_X + E)^2 = \pi^*K_X^2 + 2\pi^*K_X \cdot E + E^2 = K_X^2 - 1$$

$\square$



## Chapter 5

# Resolution of Indeterminacies

Suppose  $X$  is a smooth projective surface over  $\mathbb{C}$ ,  $\pi : \tilde{X} \rightarrow X$  is a blow up of  $X$  at point  $x \in X$ ,  $\pi^{-1}(x) = E$ .

### 5.1 Smooth Strict Transform of an Irreducible Curve

**Proposition 5.1.1.** *Suppose  $C$  is an irreducible curve on  $X$ ,  $\pi^*C = \tilde{C} + rE$ . Then  $\tilde{C} \cdot E = r$  and :*

$$\boxed{g(\tilde{C}) = g(C) - \frac{1}{2}r(r-1)}$$

where  $\tilde{C}$  is the strict transform of  $C$ .

*Proof.* Note that we have that  $\tilde{C} = \pi^*C - rE$ , where  $r = \tilde{C} \cdot E$ . By genus formula 3.4.1 we have:

$$\begin{aligned} 2g(\tilde{C}) - 2 &= \tilde{C} \cdot (\tilde{C} + K_{\tilde{X}}) = (\pi^*C - rE)(\pi^*C - rE + \pi^*K_X + E) = \\ &= (\pi^*C)^2 + \pi^*C \cdot E(-r + 1 - r) + r(r-1)E^2 + \pi^*C \cdot \pi^*K_X - rE \cdot \pi^*K_X = \\ &= C^2 - r(r-1) + C \cdot K_X = 2g(C) - 2 - r(r-1) \end{aligned}$$

Thus:

$$g(\tilde{C}) = g(C) - \frac{1}{2}r(r-1)$$

Thus, the blow-up of a singular point reduces the genus. □

**Corollary 5.1.2.** *Suppose  $C$  is an irreducible curve on  $X$ . Then there exists a sequence of blow-ups:*

$$X_n \rightarrow X_{n-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = X$$

such that the strict transform  $C_n$  of  $C$  on  $X_n$  is smooth.

*Proof.* We just blow-up singular points on the curve and using the previous proposition we conclude that the process will terminate since on each iteration the arithmetic genus decreases. □

## 5.2 Linear Systems

**Definition 5.2.1.** Suppose  $X, Y$  are projective varieties over  $\mathbb{C}$ . Then we call a map of the form

$$\phi : X \dashrightarrow Y, \text{ where } \phi = [f_0 : \dots : f_n]$$

such that  $f_0, \dots, f_n \in \mathbb{C}(X)$  have the property that for any point  $p \in X$  such that all  $f_0, \dots, f_n$  are defined we have that:

$$\phi(p) = [f_0(p) : \dots : f_n(p)] \in Y$$

**a rational map.** We call this map a **birational map** if we can find a rational map  $\phi^{-1} : Y \dashrightarrow X$  such that  $\phi \circ \phi^{-1}$  and  $\phi^{-1} \circ \phi$  are the identities as rational maps.

**Definition 5.2.2.** The plane Cremona group over the field  $\mathbb{C}$  denoted by  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is the group of birational transformations of the projective plane  $\mathbb{P}_{\mathbb{C}}^2$  of dimension 2 over the field  $\mathbb{C}$ .

**Proposition 5.2.3.**  $X$  is a surface,  $\phi : X \dashrightarrow Y$  is a rational map from the definition above then we have that  $X \setminus U = F$  is a finite set.

*Proof.* Locally we can write a rational map  $\phi$  as

$$\phi = [f_0 : \dots : f_n] = \left[ 1 : \frac{g_1}{h_1} : \dots : \frac{g_n}{h_n} \right],$$

where  $f_i$  are rational functions,  $g_i, h_i$  are holomorphic and relatively prime. Consider  $H$  the least common multiple of  $h_i$ -s. Then  $\phi$  is given by:

$$\phi = [H : f_1 H : \dots : f_n H] =: [\tilde{f}_0 : \tilde{f}_1 : \dots : \tilde{f}_n]$$

the functions  $\tilde{f}_i = f_i H$  are holomorphic and well-defined away from  $\cap V(\tilde{f}_i)$ . Suppose  $s$  is a common factor of  $\tilde{f}_i$ . Then if  $s^m | H$  for some  $m \in \mathbb{Z}_{\geq 1}$  and  $s^{m+1} \nmid H$ . Then we have that there exists  $i$  such that  $s^m | h_i$ . But in this case we have that

$$s \nmid H \cdot \frac{g_i}{h_i} = H f_i = \tilde{f}_i \Rightarrow V(\tilde{f}_i) \text{ does not contain divisors}$$

So we have that  $X \setminus U$  consists of at most points and since  $X \setminus U$  is a closed subset of compact space we conclude that  $X \setminus U$  consists of a finite number of points.  $\square$

Let's make some remarks:

- We can set  $\phi(C) = \overline{\phi(C \setminus F)}$  to be **the image of  $\phi$**  for a smooth projective curve  $C$  (since  $\phi$  is defined on  $C \setminus F$ ).
- We have the isomorphism  $\text{Pic}(X) \cong \text{Pic}(X \setminus F)$  ([Har77] Proposition II.6.5).
- We call  $|D|$  **a complete linear system** and a linear subspace  $\mathcal{S}$  of  $|D|$  **a linear system**. We may look at  $|D|$  as:

$$|D| = \{ \text{zeros of } s \in H^0(X, \mathcal{O}_X(D)) \} / \mathbb{C}^\times$$

Thus  $|D|$  can be seen as the projective space associated to the vector space  $H^0(X, \mathcal{O}_X(D))$  and  $\mathcal{S} \subset |D|$  as a subspace of this projective space. We say that the dimension of  $\mathcal{S}$  equals the dimension of the corresponding subspace.

- If  $\dim(\mathcal{S}) = 1$  then we call it **a pencil**.
- We say that  $F$  is **a fixed part of the linear system  $\mathcal{S}$**  if  $F$  is contained in any element of  $\mathcal{S}$ . We call a point  $P$  **a fixed point (base point) of  $\mathcal{S}$**  if any element of  $\mathcal{S}$  contains  $P$ .

- For any  $D' \in \mathcal{S}$  we have that  $|D' - F|$  has no fixed part.
- If  $\mathcal{S}$  has no fixed part then the number of fixed points equals  $b$  which satisfies  $0 \leq b \leq D^2$ <sup>1</sup>.

**Proposition 5.2.4.** *Suppose that  $X$  is a smooth projective surface. Then there is a one-to-one correspondence between:*

1. Rational maps  $\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^m$  such that no hyperplane contains  $\phi(X)$ .
2. Linear systems of dimension  $m$  without fixed part on  $X$ .

*Proof.* • Consider a rational map  $\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^m$  and associate  $\phi^*|H|$  to it. Here  $|H|$  is a complete linear system of hyperplanes on  $\mathbb{P}_{\mathbb{C}}^m$ . Note that by assumption  $\phi(X)$  is contained in no hyperplane, so we get that  $|H|$  has no fixed part, which means that  $\phi^*|H|$  has no fixed part.

- Suppose we have a linear system  $\mathcal{S}$  on  $X$  with no fixed part,  $\mathcal{S}^\vee$  is dual to  $\mathcal{S}$ . Define a rational map  $\phi : X \dashrightarrow \mathcal{S}^\vee$  where a point  $P$  on  $X$  is sent to the hyperplane of divisors in  $\mathcal{S}$  containing  $P$ . This map is not defined only at fixed points. □

Note that we can define the map  $\phi$  above using sections. Let  $|D|$  is a complete linear system, and  $(s_0, \dots, s_n)$  is a basis of  $H^0(X, \mathcal{O}_X(D))$  then we have a rational map

$$\phi_{|D|} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(D))^\vee), \text{ where } x \rightarrow [s_0(x) : \dots : s_n(x)]$$

A linear system  $|D|$  is **very ample** if  $\phi_{|D|}$  is an embedding.

### 5.3 Embeddings Given by Linear Systems on $\mathbb{P}_{\mathbb{C}}^2$

**Definition 5.3.1.** *The surface  $X$  is called **rational** if it is birational to  $\mathbb{P}_{\mathbb{C}}^2$ .*

Suppose that we have a rational surface  $X \subset \mathbb{P}_{\mathbb{C}}^n$ .  $X$  is birationally equivalent to  $\mathbb{P}_{\mathbb{C}}^2$  so we obtain a rational map  $\phi : \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^N$ . Suppose that  $X$  is contained in no hyperplane of  $\mathbb{P}_{\mathbb{C}}^n$ . Thus by 5.2.4 we can associate to it a linear system  $\mathcal{S}$  on  $\mathbb{P}_{\mathbb{C}}^2$  with no fixed component.

Suppose that  $\mathcal{S}$  is such linear system and  $\phi : \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathcal{S}^\vee \cong \mathbb{P}_{\mathbb{C}}^N$ . Note that  $\phi$  is not always a morphism. Assume that for  $\tilde{X}$  which is a blow-up of  $\mathbb{P}_{\mathbb{C}}^2$  ( $\pi : \tilde{X} \rightarrow \mathbb{P}_{\mathbb{C}}^2$ ) in distinct base points  $p_1, \dots, p_r$  of linear system  $\mathcal{S}$  and we have that  $f = \phi \circ \pi$  is a morphism:

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow \pi & \searrow f = \phi \circ \pi & \\ \mathbb{P}_{\mathbb{C}}^2 & \xrightarrow{\phi} & \mathbb{P}_{\mathbb{C}}^N \end{array}$$

Note that if  $E_i = \pi^{-1}(p_i)$ ,  $m_i$  the minimal multiplicity of members of  $\mathcal{S}$  at  $p_i$ ,  $d$  is the degree of members of  $\mathcal{S}$ ,  $l \subset \mathbb{P}_{\mathbb{C}}^2$  is a line (so  $\mathcal{S} \subset |dl|$ ),  $\mathcal{L} = \pi^*l$ . Then we get:

$$\tilde{\mathcal{S}} \subset |dL - \sum_i m_i E_i|$$

$f$  is an embedding when  $\tilde{\mathcal{S}}$  separates points and tangent directions. The first condition means that for any  $x, y \in X$  we can find a member of  $\tilde{\mathcal{S}}$  which passes through  $x$  and not through  $y$ . The second condition means that for any  $x \in X$  and any tangent direction at  $x$ , there exists a member of  $\tilde{\mathcal{S}}$  which

<sup>1</sup> $D$  is effective so we can only have  $D^2 < 0$  if  $h^0(D) = 1$ , but it's not possible since  $|D|$  defines a birational map, so  $D^2 \geq 0$

passes through  $x$  and has not the given direction at  $x$ .

Suppose that  $f$  is an embedding, i.e. its image  $f(\tilde{X}) = X \subset \mathbb{P}_{\mathbb{C}}^N$  is the smooth rational surface. Then from proposition 4.2.3 and the fact that  $l$  is the generator of  $\text{Pic}(\mathbb{P}_{\mathbb{C}}^2) = \mathbb{Z}$ ,  $\pi^*l = L$  we get that:

$$\text{Pic}(X') = \bigoplus_i \mathbb{Z}[E_i] \oplus \mathbb{Z}[L], \text{ where } L^2 = 1, E_i^2 = -1$$

We can write a **hyperplane divisor** and compute **the degree of  $X$** :

$$H = dL - \sum_i m_i E_i, \quad \text{deg}(X) = H^2 = d^2 - \sum_i m_i^2$$

**Lines** on  $X'$  are given by curves  $C$  such that  $C \cdot H = 1$ . **Conics** on  $X$  are given by curves  $C'$  such that  $C' \cdot H = 2$  etc.

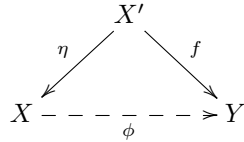
**Example 5.3.2.** Consider the complete linear system of quadrics in  $\mathbb{P}_{\mathbb{C}}^2$  then  $\dim(H^0(\mathbb{P}_{\mathbb{C}}^2, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(2))) = 6$ . A basis is given by the monomials  $x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2$ . The linear system is very ample and we have an embedding

$$v : \mathbb{P}_{\mathbb{C}}^2 \hookrightarrow \mathbb{P}_{\mathbb{C}}^5$$

which is called a **Veronese embedding**.

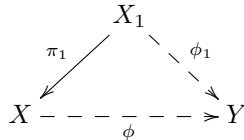
## 5.4 Resolution of Rational Maps

**Theorem 5.4.1.** Suppose  $X$  is a smooth projective surface,  $Y$  is a projective variety,  $\phi : X \dashrightarrow Y$  is a rational map. Then there exists a smooth projective surface  $X'$  together with a morphism  $f : X' \rightarrow Y$  and a morphism  $\eta : X' \rightarrow X$  which can be decomposed as a finite number of blow-ups such that  $\phi \circ \eta = f$ :



*Proof.* By assumption  $Y$  is a projective variety so we can view  $Y$  as a subset of  $\mathbb{P}_{\mathbb{C}}^m$ . Thus, we can assume  $Y = \mathbb{P}_{\mathbb{C}}^m$  and no hyperplane contains  $\phi(X)$ . By proposition 5.2.4 we have that  $\phi$  corresponds to a linear system  $\mathcal{S} \subset |D|$  of dimension  $m$  without fixed part on  $X$ .

Suppose  $\mathcal{S}$  has a base point  $P \in X$  (otherwise  $\phi$  is trivially a morphism). Now we consider a blow-up of  $X$  at point  $P$ :  $\pi_1 : X_1 \rightarrow X$ . Consider  $\pi_1^* \mathcal{S} \subset |\pi_1^* D|$ . Note that if  $F_1$  is a fixed part of  $\pi_1^* \mathcal{S}$ . Then  $F_1$  is supported on  $E \subset \pi_1^* \mathcal{S}$  with positive multiplicity  $k_1$ . Now we consider the corresponding system  $\mathcal{S}^1 \subset |\pi_1^* D - k_1 E|$  where  $|\pi_1^* D - k_1 E| = |D_1|$  has no fixed component, so it gives a rational map  $\phi_1 : X_1 \dashrightarrow \mathbb{P}_{\mathbb{C}}^m$  such that  $\phi_1 = \phi \circ \pi_1$ :



Now we proceed the process by induction (obtaining a blow-up  $\pi_n : X_n \rightarrow X_{n-1}$ , a linear system  $\pi_n^* \mathcal{S}^n \subset |D_n|$  and a rational map  $\phi_n : X_n \dashrightarrow \mathbb{P}_{\mathbb{C}}^m$ ). Note, that  $D_n = \pi_n^* D_{n-1} - k_n E_n$ . Then we have:

$$0 \leq D_n^2 = (\pi_n^* D_{n-1} - k_n E_n)^2 = (\pi_n^* D_{n-1})^2 - 2k_n D_{n-1} \cdot E_n + k_n^2 E_n^2 = D_{n-1}^2 - k_n^2$$

so the process terminates. Thus, at the end we obtain  $\mathcal{S}^n$  with no base points so  $f := \phi : X_n \rightarrow \mathbb{P}_{\mathbb{C}}^m$  is a morphism as desired.  $\square$

**Proposition 5.4.2.** *Suppose  $X$  is an irreducible possibly singular projective surface,  $X'$  is a smooth projective surface,  $f : X \rightarrow X'$  is a birational morphism such that  $f^{-1}$  is not defined at  $P \in X'$ :*

$$P \in X' \xrightarrow{f^{-1}} X \supset C$$

Then  $f^{-1}(P)$  is a curve  $C$  on  $X$ .

*Proof.* By looking at the problem locally we suppose that  $X$  is affine so we have the embedding  $i : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$ . Now consider a rational map  $i \circ f^{-1} : X \dashrightarrow \mathbb{A}_{\mathbb{C}}^n$  defined by  $i \circ f^{-1}(x') = (g_1, \dots, g_n)$  where  $g_i$  is a rational function for any  $i$ . Without loss of generality we can suppose that  $g_1 = u/v$  is not defined at  $P \in X'$ , i.e. we take  $u, v \in \mathcal{O}_{X', P}$  coprime,  $v(P) = 0$ . Suppose  $C \subset X$  is a curve defined by  $f^*v = 0$ . Then  $f^*u = x_1 f^*v$ , ( $x_1$  is the first coordinate function on  $X \subset \mathbb{A}_{\mathbb{C}}^n$ ). So  $f^*u = f^*v = 0$  on  $C$ . Thus  $C = f^{-1}(V(u = 0, v = 0))$ . We assumed that  $v, u$  are coprime  $\Rightarrow V$  is finite, so we can shrink the neighborhood and assume that it contains only one point  $P$  and we get  $f^{-1}(P) = C$ .  $\square$

**Proposition 5.4.3.** *Suppose  $X, X'$  are smooth projective surfaces,  $\phi : X \dashrightarrow X'$  is a rational map.  $\phi^{-1}$  is not defined at  $P \in X'$ . Then there exists  $C$  (a curve on  $X$ ) such that  $\phi(C) = P$ .*

$$C \subset X \xrightarrow{\phi} X' \ni P$$

*Proof.* By assumption  $\phi : X \dashrightarrow X'$  is a rational map so it corresponds to a morphism  $f : U \rightarrow X'$  where  $U \subset X$  is open. Consider:

$$\Gamma_f = \{(u, f(u)) \mid u \in U\} \subset U \times X' \subset X \times X'$$

$$X_1 = \overline{\Gamma}_f \subset X \times X'$$

Note that  $X_1$  together with projection morphism  $q, q'$  form a commutative diagram:

$$\begin{array}{ccc} & X_1 & \\ q \swarrow & & \searrow q' \\ X & \xrightarrow{\phi} & X' \end{array}$$

Note that by assumption  $\phi^{-1}$  is not defined at point  $P \in X'$ , so  $(q')^{-1}$  is not defined at point  $P \in X'$  and we see that  $(q')^{-1} : X' \dashrightarrow X_1$  satisfy the conditions of proposition 5.4.2. So there exists a curve  $C_1 \subset X_1$  such that  $q'(C_1) = P$ . Moreover, since  $X_1 \subset X \times X'$  we have  $q(C_1) = C \subset X$  a curve on  $X$  such that  $\phi(C) = P$ .  $\square$

**Proposition 5.4.4.** *Suppose  $X, Y$  are smooth projective surfaces,  $f : X \rightarrow Y$  is a birational morphism,  $f^{-1}$  is not defined at  $P \in Y$ . Then there exists a birational morphism  $g : X \rightarrow \tilde{Y}$  such that  $f = \pi \circ g$ , where  $\pi : \tilde{Y} \rightarrow Y$  is a blow-up of  $Y$  at  $P$ .*

$$\begin{array}{ccc} & & \tilde{Y} \\ & g \nearrow & \downarrow \pi \\ X & \xrightarrow{f^{-1}} & Y \ni P \\ & \searrow f & \end{array}$$

*Proof.* Consider a birational map  $g = \pi^{-1} \circ f$ . We need to prove that it is a birational morphism. Suppose the opposite, i.e that  $g$  is not defined at point  $Q \in X$ . Applying proposition 5.4.3 we get a

curve  $\tilde{C} \subset \tilde{Y}$  such that  $g^{-1}(\tilde{C}) = Q$ . Note that  $f(Q) = \pi(\tilde{C})$  and the only curve which has a point as its image via  $\pi$  is  $E$ . Thus,  $\tilde{C} = E$  and  $f(Q) = P$ .

$$\begin{array}{ccc}
 & \tilde{Y} \supset E = \tilde{C} & \\
 & \nearrow g & \downarrow \pi \\
 Q \in X & \xleftarrow{f^{-1}} Y \ni P & \\
 & \xrightarrow{f} & 
 \end{array}$$

Consider a maximal ideal  $\mathfrak{m}_Q \subset \mathcal{O}_{X,Q}$  and suppose we have local coordinates  $(x, y)$  such that  $f^*y \in \mathfrak{m}_Q^2$  in the neighborhood of point  $P \in Y$ . We can find such coordinates since in case  $g^*y \notin \mathfrak{m}_Q^2$ , i.e.  $g^*y$  vanishes at  $g^{-1}(P)$  with multiplicity 1 and so gives a local equation for  $g^{-1}(P)$ . So we can find some  $u \in \mathcal{O}_{X,Q}$  such that  $g^*x = ug^*y$ . Now we choose a new  $y$  by setting  $y' := x - u(Q)y$ , we see that  $g^*y' = (u - u(Q))g^*y \in \mathfrak{m}_Q^2$ .

Now consider a point  $P' \in E$  such that the map  $g^{-1}$  is defined at  $P'$ . Then for almost all points on  $E$  we have that  $(g^{-1})^*g^*y = \pi^*y \in \mathfrak{m}_{P'}^2$ . By construction of the blow-up morphism  $\pi^*y$  is a local coordinate for all points in  $E$  except one - contradiction.  $\square$

**Theorem 5.4.5.** *Suppose  $X, X_0$  are smooth projective surfaces,  $f : X \rightarrow X_0$  is a birational morphism. Then there exists a sequence of  $\pi_k : X_k \rightarrow X_{k-1}$ , for  $k \in \{1, \dots, n\}$ , where each  $\pi_i$  is a blow-up,  $u : X \xrightarrow{\cong} X_n$  and  $f = \pi_1 \circ \dots \circ \pi_n \circ u$ :*

$$\begin{array}{ccccccc}
 & & X & & & & \\
 & u & \swarrow & & \searrow & f & \\
 X_n & \xrightarrow{\pi_n} & X_{n-1} & \xrightarrow{\pi_{n-1}} & \dots & \xrightarrow{\pi_2} & X_1 \xrightarrow{\pi_1} X_0
 \end{array}$$

*Proof.* • In case  $f$  is an isomorphism there is nothing to prove.

• Suppose  $f$  is not an isomorphism. Then there is a point  $P_1$  on  $X_0$  where  $f^{-1}$  is not defined and we can apply proposition 5.4.4 to get the factorization  $f = \pi_1 \circ f_1$ , where  $\pi_1 : X_1 \rightarrow X_0$  is the blow up of  $X_0$  at point  $P_1$  and  $f_1 : X \rightarrow X_1$  is a birational morphism. By induction we proceed the process. Let's prove that it terminates. Suppose that it does not. Then there exist infinite sequences of blow-ups  $\{\pi_i : X_i \rightarrow X_{i-1}\}$  and birational morphisms  $\{f_i : X \rightarrow X_i\}$  such that for any  $i$  they satisfy the relations  $\pi_i \circ f_i = f_{i-1}$ , i.e. the following diagram is commutative:

$$\begin{array}{ccc}
 & X & \\
 f_i \swarrow & & \searrow f_{i-1} \\
 X_i & \xrightarrow{\pi_i} & X_{i-1} \ni P_i
 \end{array}$$

Suppose the number of curves on  $X$  contracted by  $f_i$  equals  $n_i$ . This number is well-defined since  $n_i$  is finite: suppose  $f_i^{-1}$  is not defined at point  $Q \in X_i$  then  $f_i^{-1}(Q) \subset X$  is a closed subset of  $X$  which means that it can be seen as a finite number of irreducible components and there are only finitely many such  $Q \in X_i$ . From commutative diagram we obtain that if a curve  $C$  is contracted by  $f_i$  then it is contracted by  $f_{i-1}$  and there is an irreducible curve  $C_i \subset X$  such that  $f_i(C_i) = E_i \subset X_i$  is the exceptional curve for  $\pi_i : X_i \rightarrow X_{i-1}$ .

$$\begin{array}{ccc}
 & C_i \subset X & \\
 f_i \swarrow & & \searrow f_{i-1} \\
 E_i \subset X_i & \xrightarrow{\pi_i} & X_{i-1} \ni P_i
 \end{array}$$

We see that  $f_i(C_i) = E_i$  and  $f_{i-1}(C_i) = P_i$  thus  $n_i < n_{i-1}$ . So the process terminates.  $\square$

From theorems 5.4.1 and 5.4.5 we obtain a corollary:

**Corollary 5.4.6.** *Suppose  $X, X'$  are smooth projective surfaces,  $\phi : X \dashrightarrow X'$  is a birational map. Then there exists  $\tilde{X}$  smooth projective surfaces such that the following diagram is commutative:*

$$\begin{array}{ccc} & \tilde{X} & \\ f \swarrow & & \searrow g \\ X & \dashrightarrow \phi \dashrightarrow & X' \end{array}$$

where  $f, g$  are composites of blow-ups and isomorphisms.

## 5.5 Castelnuovo's Contractibility Criterion

We first state Nakai-Moishezon Criterion [Har77], Theorem V.1.10:

**Theorem.** *A divisor  $D$  on a smooth projective surface  $X$  is ample if and only if  $D^2 > 0$  and  $D \cdot C > 0$  for any (irreducible) curve  $C \subset X$ .*

**Theorem 5.5.1.** *(Castelnuovo's contractibility criterion)*

*Suppose  $X$  is a smooth projective surface, there is  $E \subset X$  such that  $E \cong \mathbb{P}_{\mathbb{C}}^1$  and  $E^2 = -1$ . Then there exists a morphism  $f : X \rightarrow X_0$  to a smooth projective surface  $X_0$  and point  $P \in X_0$  such that  $X$  is isomorphic via  $f$  to the blow up of  $X_0$  at point  $P$  and  $E$  is the exceptional curve.*

*Proof.* • Consider a very ample divisor  $H$  on  $X$  such that  $H^1(X, \mathcal{O}_X(H)) = 0$ . Suppose  $k = H \cdot E$ ,  $H' = H + kE$  (note that  $k > 0$  by Nakai-Moishezon Criterion since  $H$  is ample and  $E$  is effective). Note that  $E \cong \mathbb{P}_{\mathbb{C}}^1$  and line bundles on  $\mathbb{P}_{\mathbb{C}}^1$  are parametrized by their degree. Thus:

$$\mathcal{O}_X(H)|_E \cong \mathcal{O}_E(k) \quad \mathcal{O}_X(E)|_E \cong \mathcal{O}_E(-1) \quad \mathcal{O}_X(H')|_E \cong \mathcal{O}_E$$

For  $i \in \{1, \dots, k\}$  consider the exact sequence:

$$0 \rightarrow \mathcal{O}_X(H + (i-1)E) \rightarrow \mathcal{O}_X(H + iE) \rightarrow \mathcal{O}_E \otimes \mathcal{O}_X(H + iE) = \mathcal{O}_E(k-i) \rightarrow 0$$

which gives rise to the part long exact sequence:

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}_X(H + (i-1)E)) \rightarrow H^0(X, \mathcal{O}_X(H + iE)) \xrightarrow{r_i} H^0(E, \mathcal{O}_E(k-i)) \rightarrow \\ \rightarrow H^1(X, \mathcal{O}_X(H + (i-1)E)) \rightarrow H^1(X, \mathcal{O}_X(H + iE)) \rightarrow 0 \end{aligned}$$

(since  $H^1(E, \mathcal{O}_E(k-i)) = H^1(\mathbb{P}_{\mathbb{C}}^1, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(k-i))^{k \geq i} = 0$ ). Suppose  $i = 1$ :

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}_X(H)) \rightarrow H^0(X, \mathcal{O}_X(H + E)) \xrightarrow{r_1} H^0(E, \mathcal{O}_E(k-1)) \rightarrow \\ \rightarrow H^1(X, \mathcal{O}_X(H)) = 0 \rightarrow H^1(X, \mathcal{O}_X(H + E)) \rightarrow 0 \Rightarrow H^1(X, \mathcal{O}_X(H + E)) = 0 \end{aligned}$$

Thus, by induction on  $i$  we have that  $H^1(X, \mathcal{O}_X(H + iE)) = 0$  for  $i \in \{1, \dots, k\}$  and  $r_i$ -s are surjective. Suppose  $s \in H^0(X, \mathcal{O}_X(E))$  for  $V(s) = E$ , the restrictions of  $(a_{i,0}, \dots, a_{i,k-i}) \in H^0(X, \mathcal{O}_X(H + iE))$  give a basis in  $H^0(E, \mathcal{O}_E(k-i)) = H^0(E, \mathcal{O}_E(k-i))$ ,  $(e_0, \dots, e_n)$  is the basis of  $H^0(X, \mathcal{O}_X(H))$ . The basis for  $H^0(X, \mathcal{O}_X(H')) = H^0(X, \mathcal{O}_X(H + kE))$  is given by:

$$(e_0 s^k, \dots, e_n s^k, a_{1,0} s^{k-1}, \dots, a_{1,k-1} s^{k-1}, \dots, a_{k-1,1} s, a_{k,0})$$

Now using the corresponding linear system we can define a rational map  $\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^N$  (for some  $N \in \mathbb{Z}$ ) to the projective space such that  $\phi_{X \setminus E} : X \setminus E \rightarrow \mathbb{P}_{\mathbb{C}}^N$  is an embedding since  $(e_0, \dots, e_n)$  defines an embedding. Looking at the first  $n + 1$  terms, we see that there are no basepoints away from  $V(s = 0) = E$ . For a point on  $E$ , all the terms are non-zero except the last one. Thus, we have a basepoint free map, that sends  $E$  to  $P = [0 : \dots : 0 : 1]$ . Denote  $\phi(X) = X'$ . We obtained a morphism  $\epsilon : X \rightarrow X'$  such that  $X \setminus E \cong X' \setminus P$ ,  $\epsilon(E) = P$ .

• To prove that  $X'$  is smooth it is enough to show that  $X'$  is smooth at  $P$ . Consider  $U = X \setminus V(a_{k,0})$  an open neighborhood of  $E$ . Consider  $x = \frac{a_{k-1,0}}{a_{k,0}}, y = \frac{a_{k-1,1}}{a_{k,0}} \in H^0(U, \mathcal{O}_U(-E))$  such that their restrictions give the basis of  $H^0(E, \mathcal{O}_E(1))$  and for some smaller neighborhood  $U'$  we have that  $x, y$  do not vanish at the same time. Thus we can define a morphism  $h_2 : U' \rightarrow \mathbb{P}_{\mathbb{C}}^1 : (\alpha, \beta) \rightarrow [x(\alpha, \beta) : y(\alpha, \beta)]$  and a morphism  $h_1 : U' \rightarrow \mathbb{A}_{\mathbb{C}}^2 : (\alpha, \beta) \rightarrow (s\alpha(\alpha, \beta), sy(\alpha, \beta))$ . Now we obtain a morphism:

$$(h_1, h_2) : U' \rightarrow \mathbb{A}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^1$$

Since  $V(s) = E$  then this morphism factors through  $\tilde{\mathbb{A}}_{\mathbb{C}}^2 \subset \mathbb{A}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^1$  (since  $x \cdot sy = y \cdot sx$ ) and moreover  $E \cong E_{\mathbb{A}}$  where  $E_{\mathbb{A}}$  is the exceptional curve on  $\tilde{\mathbb{A}}_{\mathbb{C}}^2$ . Now we recall that

$$\tilde{\mathbb{A}}_{\mathbb{C}}^2 = \{((x_1, x_2), [y_0 : y_1]) \mid x_1 y_1 = x_2 y_0\} \subset \mathbb{A}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^1 = \{((x_1, x_2), [y_0 : y_1])\}$$

Suppose  $Q \in E$ . We have that  $x, y$  do not vanish at the same time thus we may assume that  $x(Q) = 0, y(Q) = 1$ . In this case we have that  $h(Q) = ((0, 0), [0 : 1])$ . Consider  $(x_2, y_0/y_1)$  as local coordinates at  $h(Q)$ . Note that  $h^*x_2 = sy$  (has order of vanishing 1 at  $E$ ),  $h^*(y_0/y_1) = x/y$  (local coordinate at  $Q$  on  $E$ ). Thus, for any point  $Q$  on  $E$  the local coordinates at  $h(Q)$  under the inverse image of  $h$  give local coordinates at  $Q$ . We see that  $h$  is étale.

Let's now view this situation in the analytic topology and consider a commutative diagram:

$$\begin{array}{ccc} E \subset U & \xrightarrow{h} & \tilde{\mathbb{A}}_{\mathbb{C}}^2 \supset E_{\mathbb{A}} \\ \downarrow \epsilon & & \downarrow \pi \\ P \in \epsilon(U) & \xrightarrow{\bar{h}} & \mathbb{A}_{\mathbb{C}}^2 \ni (0, 0) \end{array}$$

Note that in this case we have a local homeomorphism  $h : U \rightarrow X = h(U)$  (i.e. for any point on  $E$  we have that  $h$  is a homeomorphism near this point). Thus, there exist neighbourhood  $U'$  of  $E$  such that  $h : U' \cong V' = h(U')$ . So we get that  $\bar{h} : \epsilon(U') \cong \eta(V')$  and  $\epsilon(U')$  is smooth since  $\eta(V') \subset \mathbb{A}_{\mathbb{C}}^2$  is smooth.  $\square$



# Chapter 6

## Bubble Spaces

### 6.1 Basic Definitions

**Definition 6.1.1.** Suppose  $X$  is a smooth projective surface, consider the bubble space of  $X$ :

$$\mathcal{B}(X) = \left\{ (y, Y, \pi) \mid \begin{array}{l} y \text{ is a point on smooth projective surface } Y \text{ and} \\ \pi : Y \rightarrow X \text{ is a birational morphism} \end{array} \right\} / \sim$$

where  $(y, Y, \pi) \sim (y', Y', \pi')$  if and only if there is an open neighborhood of  $y$  such that the birational map  $(\pi')^{-1} \circ \pi : Y \dashrightarrow Y'$  restricts to an isomorphism of neighborhoods  $y \in U_y \subset Y$ ,  $y' \in U_{y'} \subset Y'$  which sends  $y \rightarrow y'$ , i.e. the following diagram is commutative:

$$\begin{array}{ccc} y \in Y & \xrightarrow{(\pi')^{-1} \circ \pi} & y' \in Y' \\ \pi \searrow & & \swarrow \pi' \\ & X & \end{array}, \quad (\pi')^{-1} \circ \pi(y) = y' \text{ and } U_y \cong U_{y'} \text{ via } (\pi')^{-1} \circ \pi$$

- If the point is equivalent to  $(x, X, id_X)$  then we call it a **proper point**:

$$\begin{array}{ccc} y \in Y & \xrightarrow{\pi} & x \in X \\ \pi \searrow & & \swarrow id_X \\ & X & \end{array}, \quad \pi(y) = x \text{ and } U_y \cong U_x \text{ via } \pi$$

otherwise we call it an **infinitely near point**.

- Suppose now we have  $y, y' \in \mathcal{B}(X)$  corresponding to  $(y, Y, \pi)$ ,  $(y', Y', \pi')$  respectively we say that  $y$  is **infinitely near** to  $y'$  if  $(\pi')^{-1} \circ \pi : Y \rightarrow Y'$  is the morphism such that a curve  $C$  containing  $y$  is contracted to  $y'$ :

$$\begin{array}{ccc} y \in C \subset Y & \xrightarrow{(\pi')^{-1} \circ \pi} & y' \in Y' \\ \pi \searrow & & \swarrow \pi' \\ & X & \end{array}, \quad (\pi')^{-1} \circ \pi(C) = y'$$

- Suppose now we have  $y, y' \in \mathcal{B}(X)$  corresponding to  $(y, Y, \pi)$ ,  $(y', Y', \pi')$  respectively. We say that  $y$  is **proximate** to  $y'$  if  $y$  lies on the strict transform of the curve given by the blow-up of  $y'$  (notation:  $y > y'$ ). If we also have that there exists a neighborhood in which  $(\pi')^{-1} \circ \pi$  is the blow-up

of  $y'$ , we say that  $y$  belongs to the first neighborhood of  $y'$ ; we say that  $y$  belongs to the  $n$ -th neighborhood of  $y'$  if there exists a sequence of points  $y_1, \dots, y_n$  ( $y_n = y$ ) such that for all  $i \geq 2$  we have that  $y_i$  is in the first neighborhood of  $y_{i-1}$  and  $y_1$  is in the first neighborhood of  $y'$ .

Note that since birational morphisms are represented by the sequence of blow-ups (up to isomorphism) then we have that any point  $y \in \mathcal{B}(X)$  is either a proper point or is in the  $n$ -th neighborhood of a unique point  $x \in X$  ( $n \geq 1$ ). If  $n > 1$  then by induction we have that  $y$  is in the first neighborhood of a unique point  $y' \in \mathcal{B}(X)$  which is in the  $(n-1)$ -th neighbourhood of  $x$ .

Suppose we have a birational morphism  $\pi : Y \rightarrow X$ . So we may consider

$$\mathcal{B}(\pi^{-1}) = \{ \text{base points of } \pi^{-1} \}$$

We know that birational morphisms are represented by the sequence of blow-ups so we write  $\pi = \pi_1 \circ \dots \circ \pi_k$  where  $\pi_i : X_i \rightarrow X_{i-1}$  is the blow-up of a point  $p_i \in X_{i-1}$  for  $i \in \{1, \dots, k\}$ . Then:

$$\mathcal{B}(\pi^{-1}) = \bigcup_i \{p_i \mid i \in \{1, \dots, k\}\} = \bigcup_i \{(p_i, X_{i-1}, \pi_1 \circ \dots \circ \pi_{i-1}) \mid i \in \{1, \dots, k\}\}$$

This set does not depend on the factorization since the blow-ups are local and we can exchange the order. Moreover, if we have birational morphisms  $\pi : Y \rightarrow X$ ,  $\pi' : Y' \rightarrow X$  such that  $\mathcal{B}(\pi^{-1}) = \mathcal{B}((\pi')^{-1})$  then there exists an isomorphism  $\tau : Y \rightarrow Y'$  with  $\pi' \circ \tau = \pi$ .

Suppose  $C$  is an irreducible curve on a smooth projective surface  $X$ , and  $p \in \mathcal{B}(X)$ , so that  $p$  corresponds to  $(p, \tilde{X}, \pi)$ , where  $\pi : \tilde{X} \rightarrow X$  is a birational morphism. Then we define  $m_p(C)$  as the multiplicity of the strict transform  $\tilde{C}$  of  $C$  at  $p \in \tilde{X}$ .

**Proposition 6.1.2.** *Suppose  $C, D$  are irreducible curves on the smooth projective surface  $X$ . In this case we have that:*

$$C \cdot D = \sum_{p \in \mathcal{B}(X)} m_p(C) \cdot m_p(D)$$

*Proof.* For  $C \cap D = \emptyset$  we have then  $C \cdot D = 0$  and the result follows. Otherwise, suppose we have a point  $p \in C \cap D$ , and let  $\pi : \tilde{X} \rightarrow X$  be the blow-up of  $X$  at  $p$ . We have that  $\tilde{C} \sim \pi^*(C) - m_p(C)E$ ,  $\tilde{D} \sim \pi^*(D) - m_p(D)E$ . Thus:

$$\tilde{C} \cdot \tilde{D} = C \cdot D - m_p(C) \cdot m_p(D)$$

We managed to reduce  $\tilde{C} \cdot \tilde{D}$  so we get the result by induction on integer  $C \cdot D$  which is always  $\geq 0$ .  $\square$

# Chapter 7

## Del Pezzo Surfaces

Recall that **the canonical divisor**  $K_X \in \text{Pic}(X)$  on an algebraic variety  $X$  is given by the divisor of a differential form. It is well-defined since two distinct 2-differential forms give linearly equivalent divisors. Moreover for the blow-up  $\pi : \tilde{X} \rightarrow X$  at point  $p$  by Proposition 4.2.6 we have  $K_{\tilde{X}} = \pi^*K_X + E$  where  $E = \pi^{-1}(p)$ . We state the Nakai-Moishezon Criterion ([Har77], V.1.10) mentioned in the previous chapters:

**Theorem.** *A divisor  $D$  on a smooth projective surface  $X$  is ample if and only if  $D^2 > 0$  and  $D \cdot C > 0$  for any (irreducible) curve  $C \subset X$ .*

### 7.1 Definition of del Pezzo Surfaces and Equivalent Conditions

**Definition 7.1.1.** *We call a smooth projective surface  $X$  a **del Pezzo surface** if its anti-canonical divisor  $-K_X$  is ample.*

**Proposition 7.1.2.** *Suppose  $X$  is a smooth projective surface and we have a birational morphism  $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^2$ . Then the following conditions are equivalent:*

1.  $X$  is a del Pezzo surface;
2.  $\pi$  is the blow-up of  $0 \leq k \leq 8$  points of  $\mathbb{P}_{\mathbb{C}}^2$  (not infinitely near points) such that no 3 are colinear, no 6 belong to the same conic, and no 8 lie on a cubic having a double point at one of them;
3.  $K_X^2 \geq 1$  and for any irreducible curve  $C$  of  $X$  we have  $C^2 \geq -1$  ;
4.  $C \cdot (-K_X) > 0$  for an effective divisor  $C$

*Proof.* Suppose we blow-up a point  $x$  which infinitely near to a point  $x'$ . Then  $\pi$  blows-up  $x'$ . Now we blow up  $x'$  and see that the strict transform of the curve obtained after this blow-up is a smooth curve  $C \cong \mathbb{P}_{\mathbb{C}}^1 \subset X$  such that  $C^2 \leq -2$  on  $X$ . By genus formula we have

$$C \cdot K_X = -2 + 2g(C) - C^2 \geq -2 - (-2) \geq 0$$

We see that in this case all the statements are false and thus we may that  $\pi$  is the blow-up of  $k$  points which are not infinitely near:  $x_1, \dots, x_k \in \mathbb{P}_{\mathbb{C}}^2$ .  $\text{Pic}(X)$  is generated by  $L = \pi^*l$ , where  $l$  is the line on  $\mathbb{P}_{\mathbb{C}}^2$ , and  $E_i = \pi^{-1}(x_i)$  for  $i \in \{1, \dots, k\}$ , such that

$$E_i^2 = -1, L \cdot E_i = E_i \cdot E_j = 0 \text{ for } i \neq j$$

(1) $\Rightarrow$ (4) By definition of del Pezzo surface we have that  $-K_X$  is ample. Thus, for sufficiently large  $m$  we have that  $-mK_X$  is very ample, so the linear system  $| -mK_X |$  defines an embedding into a projective space. Then the degree of an effective divisor  $C$  in this embedding is given by

$$-mK_X \cdot C > 0 \Rightarrow C \cdot (-K_X) > 0$$

(4) $\Rightarrow$ (2),(3) Suppose we have an irreducible curve  $C$  on  $X$  such that  $C^2 \leq -2$ . Then the genus formula gives:

$$C \cdot (C + K_X) = -2 + 2g(C) \geq 2 \Rightarrow C \cdot -K_X \leq 2 + C^2 \leq 0$$

-contradiction with (4). So it follows from (4) that for any irreducible curve  $C$  on  $X$  we have  $C^2 \geq -1$ .

- Suppose  $x_1, x_2, x_3$  belong to the same line  $L_1$  then  $\tilde{L}_1 \sim L - \sum E_i$ . So we have

$$\tilde{L}_1^2 = (L - \sum E_i)^2 \stackrel{(\star)}{\leq} 1 - 3 = -2$$

( $\star$ ) is true since we have  $\sum E_i^2 \leq -3$ . So we have a contradiction with (4) as was mentioned above.

- Suppose  $x_1, x_2, x_3, x_4, x_5, x_6$  belong to the same conic  $Q_2$  we obtain:

$$\tilde{Q}_2^2 = (2L - \sum E_i)^2 = 4 - 6 = -2$$

- contradiction with (4) as was mentioned above.

- Suppose  $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$  belong to the same cubic  $Q_3$  and  $Q_3$  is singular at  $x_1$  we obtain:

$$\tilde{Q}_3^2 = (3L - 2E_1 - \sum_2^8 E_i)^2 = 9 - 4 - 7 = -2$$

- contradiction with (4) as was mentioned above. So we have (4) $\Rightarrow$ (2).

- Suppose that  $k \geq 9$  then we can consider an irreducible (points in general position) cubic  $Q$  through 9 of them (so we have  $g(Q) = 1$ ). Then we have

$$\tilde{Q}^2 = (3L - \sum E_i)^2 \leq 9 - 9 = 0$$

Thus, from the genus formula  $\tilde{Q}^2 + K_X \cdot \tilde{Q} = -2 + 2g(\tilde{Q}) \Rightarrow K_X \cdot \tilde{Q} \geq 0$ - contradiction with (4). Thus

$$k \leq 8 \text{ and } K_X^2 = (\pi^* K_{\mathbb{P}^2} + \sum_{i=1}^k E_i)^2 = 9 - k \geq 1$$

(3) $\Rightarrow$ (2) we know that  $K_X^2 = (\pi^* K_{\mathbb{P}^2} + \sum_{i=1}^k E_i)^2 = 9 - k \geq 1$  so  $k \leq 8$ . Moreover, since for irreducible curve  $C$  we have  $C^2 \geq -1$  then we have  $x_1, \dots, x_k$  ( $k \in \{1, \dots, 8\}$ ) satisfy the conditions: no 3 are colinear, no 6 are on the same conic, and no 8 are on a cubic having a double point at one of them, since otherwise as in the previous part we can find an example with  $C^2 \leq -2$ .

(2) $\Rightarrow$ (1) Note that  $k \leq 8$  thus  $K_X^2 = (\pi^* K_{\mathbb{P}^2} + \sum_{i=1}^k E_i)^2 = 9 - k \geq 1$ . By Nakai-Moishezon Criterion to prove that  $-K_X$  is ample it is left to prove that for any irreducible curve  $C \subset X$  we have  $-K_X \cdot C > 0$ . Note that if  $C = E_i$  then  $-K_X \cdot C = -K_X \cdot E_i = 1 > 0$ . Suppose  $C \neq E_i$  for  $i \in \{1, \dots, k\}$ . In this case we can write

$$C \sim dL - \sum_{i=1}^k (C \cdot E_i) E_i \text{ for } d > 0$$

$\mathbf{d} = \mathbf{1}$  : then  $C$  is a strict transform of the line, so it contains at most two points out of  $x_1, \dots, x_k$ . So

$$\begin{aligned} -K_X \cdot C &= -\left(\pi^* K_{\mathbb{P}_C^2} + \sum_{i=1}^k E_i\right) \cdot \left(\pi^* l - \sum_{i=1}^k (C \cdot E_i) E_i\right) = \\ &= \left(3L - \sum_{i=1}^k E_i\right) \cdot \left(L - \sum_{i=1}^k (C \cdot E_i) E_i\right) \geq 3 - 2 \geq 1 \end{aligned}$$

$\mathbf{d} = \mathbf{2}$  : then  $C$  is a strict transform of the conic, so it contains at most five points out of  $x_1, \dots, x_k$ . So

$$\begin{aligned} -K_X \cdot C &= -\left(\pi^* K_{\mathbb{P}_C^2} + \sum_{i=1}^k E_i\right) \cdot \left(2\pi^* l - \sum_{i=1}^k (C \cdot E_i) E_i\right) = \\ &= \left(3L - \sum_{i=1}^k E_i\right) \cdot \left(2L - \sum_{i=1}^k (C \cdot E_i) E_i\right) \geq 6 - 5 \geq 1 \end{aligned}$$

$\mathbf{d} = \mathbf{3}$  : then  $C$  is a strict transform of the cubic, so it contains at most eight points out of  $x_1, \dots, x_k$ . So

$$\begin{aligned} -K_X \cdot C &= -\left(\pi^* K_{\mathbb{P}_C^2} + \sum_{i=1}^k E_i\right) \cdot \left(3\pi^* l - \sum_{i=1}^k (C \cdot E_i) E_i\right) = \\ &= \left(3L - \sum_{i=1}^k E_i\right) \cdot \left(3L - \sum_{i=1}^k (C \cdot E_i) E_i\right) \geq 9 - 8 \geq 1 \end{aligned}$$

$\mathbf{d} \geq \mathbf{4}$  : consider a cubic  $Q$  passing through  $x_1, \dots, x_k$  (we can find such cubic since  $k \leq 8$ ) now we have that  $\tilde{Q} \sim 3L - \sum_{i=1}^k E_i \sim -K_X$ . Thus,  $-K_X$  is an effective divisor and  $-K_X \cdot C \geq 0$ , so its enough to prove that  $-K_X \cdot C \neq 0$ . Suppose  $-K_X \cdot C = 0$ ,  $C \sim dL - \sum_{i=1}^k (C \cdot E_i) E_i =: dL - \sum_{i=1}^k \alpha_i E_i$  for  $d \geq 4$ . We have:

$$\begin{aligned} -2 + 2g(C) &= C^2 + K_X \cdot C = C^2 = \left(dL - \sum_{i=1}^k \alpha_i E_i\right)^2 = d^2 - \sum_{i=1}^k \alpha_i^2 \Rightarrow \sum_{i=1}^k \alpha_i^2 = d^2 + 2 - 2g(C) \\ 0 &= C \cdot K_X = \left(dL - \sum_{i=1}^k \alpha_i E_i\right) \left(3L - \sum_{i=1}^k E_i\right) = 3d - \sum_{i=1}^k \alpha_i \Rightarrow 3d = \sum_{i=1}^k \alpha_i \end{aligned}$$

Thus:

$$9d^2 = \left(\sum_{i=1}^k \alpha_i\right)^2 \leq k \sum_{i=1}^k \alpha_i^2 \leq 8 \sum_{i=1}^8 \alpha_i^2 \Rightarrow d^2 \leq 8(2 - 2g(C))$$

For  $d \geq 4$  we have the solution  $d = 4$ ,  $g(C) = 0$  and  $\alpha_i = \alpha_j$  for all  $i, j$ . But then the equality  $3d = \sum_{i=1}^k \alpha_i$  fails.  $\square$

# Chapter 8

## Hirzebruch Surfaces

### 8.1 Definition and Construction

**Definition 8.1.1.** We call

$$\mathbb{F}_n = \{([x : y : z], [u : v]) \mid yv^n = zu^n\} \subset \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^1$$

the  $n$ -th Hirzebruch surface  $\mathbb{F}_n$ , and  $\pi_n : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is the projection on the second factor. Remark that then  $\mathbb{F}_0 \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  via  $([x : y : z], [u : v]) \rightarrow ([x : y], [u : v])$ .

**Proposition 8.1.2.** The maps:

$$\rho_{1,n} : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{F}_n, \text{ where } ([\alpha : \beta], t) \rightarrow ([\alpha : \beta : \beta t^n], [1 : t])$$

$$\rho_{2,n} : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{F}_n, \text{ where } ([\alpha : \beta], t) \rightarrow ([\alpha : \beta t^n : \beta], [t : 1])$$

are open embeddings with  $\rho_{1,n}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1) = U_{1,n}$ ,  $\rho_{2,n}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1) = U_{2,n}$  such that  $\{U_{i,n}\}_{i \in \{1,2\}}$  an open cover of  $\mathbb{F}_n$  so that  $\mathbb{F}_n$  is smooth and  $\pi : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is a  $\mathbb{P}_{\mathbb{C}}^1$ -bundle. Moreover, we have a birational map:

$$\rho_{1,n} \circ (\rho_{2,n})^{-1} : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1 \dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1, \text{ where } ([\alpha : \beta], t) \rightarrow ([\alpha : \beta t^n], 1/t)$$

*Proof.* They are clearly injective. Moreover, for example  $U_{1,n}$  (similarly for  $U_{2,n}$ ) is open since:

$$\mathbb{F}_n \setminus U_{1,n} = \{([x : y : z], [u : v]) \mid yv^n = zu^n, u = 0, v = 1\} = \{([x : 0 : z], [0 : 1]) \mid \} \cong \mathbb{P}_{\mathbb{C}}^1$$

□

### 8.2 Birational Map between $\mathbb{F}_n$ and $\mathbb{F}_{n+1}$

**Proposition 8.2.1.** Let  $E_n = \{([1 : 0 : 0], [u : v])\} \subset \mathbb{F}_n$ . For  $n \geq 0$  the map  $\rho_{1,n+1} \circ \rho_{1,n}^{-1}$  gives a birational map  $\phi_n : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$  such that we have the commutative diagram:

$$\begin{array}{ccc} & Y & \\ \eta_n \swarrow & & \searrow \eta_{n+1} \\ \mathbb{F}_n & \dashrightarrow \phi_n \dashrightarrow & \mathbb{F}_{n+1} \end{array}$$

where  $\eta_n, \eta_{n+1}$  are birational morphisms such that:

1.  $\eta_n$  is the blow-up of

$$p = ([1 : 0 : 0], [0 : 1]) = \rho_{2,n}([1 : 0], 0) \in E_n \subset \mathbb{F}_n$$

2.  $\eta_{n+1}$  is the blow-up of

$$q = ([0 : 0 : 1], [0 : 1]) = \rho_{2,n+1}([0 : 1], 0) \in \mathbb{F}_{n+1}$$

3. the curve contracted by  $\eta_{n+1}$  on  $q$  is the strict transform by  $\eta_n$  of the fiber of  $\pi_n$  containing  $p$

4. the curve contracted by  $\eta_n$  on  $p$  is the strict transform by  $\eta_{n+1}$  of the fiber of  $\pi_{n+1}$  containing  $q$ .

*Proof.* In the notation of the proposition above we have an isomorphism

$$\rho_{1,n+1} \circ \rho_{1,n}^{-1}|_{U_{1,n}} : U_{1,n} \xrightarrow{\cong} \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1 \xrightarrow{\cong} U_{1,n+1}$$

So we get the birational map  $\phi_n : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$ . Now we have the commutative diagram (here  $U_{i,k}$  are identified with  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1$  for  $i \in \{1, 2\}$ ,  $k \in \{n, n+1\}$ ):

$$\begin{array}{ccc} U_{1,n} \xrightarrow{\rho_{1,n} \circ (\rho_{2,n})^{-1}} U_{2,n} & & ([\alpha : \beta], t) \longrightarrow ([\alpha : \beta t^n], 1/t) \\ \phi_n|_{U_{1,n}} \downarrow & & \downarrow \\ U_{1,n+1} \xrightarrow{\rho_{1,n+1} \circ (\rho_{2,n+1})^{-1}} U_{2,n+1} & & ([\alpha : \beta], t) \longrightarrow ([\alpha : \beta t^{n+1}], 1/t) \end{array}$$

Thus, the birational map  $U_{2,n} \dashrightarrow U_{2,n+1}$  is given by:  $([\alpha : \beta], 1/t) \rightarrow ([\alpha : \beta t], 1/t)$  which is equivalent to  $([\alpha : \beta], u) \rightarrow ([u\alpha : \beta], u)$ , where  $u = 1/t$ . Now consider

$$Y' = \{([a : b], [c : d], t) \mid tad = bc\} \subset \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1$$

with morphisms  $\eta'_n : Y' \rightarrow U_{2,n}$  and  $\eta'_{n+1} : Y' \rightarrow U_{2,n+1}$ :

$$\begin{array}{ccc} & Y' & \\ \eta'_n \swarrow & & \searrow \eta'_{n+1} \\ U_{2,n} & \xrightarrow{\phi_n|_{U_{2,n}}} & U_{2,n+1} \end{array} \quad \begin{array}{ccc} & ([a : b], [c : d], t) & \\ & \swarrow & \searrow \\ ([a : b], t) & \xrightarrow{\quad \quad \quad} & ([c : d], t) \end{array}$$

$\eta'_n$  is the blow-up of  $p = ([1 : 0], 0) \in U_{2,n}$ ,  $\eta'_{n+1}$  is the blow-up of  $q = ([0 : 1], 0) \in U_{2,n+1}$ . The morphism  $\eta'_{n+1}$  sends the curve contracted by  $\eta'_n$  onto the  $\{t = 0\}$ , which corresponds to the fiber of  $\pi_{n+1}$  containing  $q$ . The same way,  $\eta'_n$  sends the curve contracted by  $\eta'_{n+1}$  onto  $\{t = 0\}$ , which corresponds to the fiber of  $\pi_n$  containing  $p$ . Now we just extend morphisms to get the desired construction.  $\square$

### 8.3 The Picard Group of Hirzebruch Surfaces

**Proposition 8.3.1.** Recall that  $E_n = \{([1 : 0 : 0], [u : v]) \mid [u : v] \in \mathbb{P}_{\mathbb{C}}^1\}$ .

Then for any  $n \geq 0$ , we have:

- (1)  $\text{Pic}(\mathbb{F}_n) = \mathbb{Z}[f] \oplus \mathbb{Z}[E_n]$ , where  $f$  is the class of a fiber
- (2)  $f^2 = 0$ ,  $(E_n)^2 = -n$ ,  $E_n \cdot f = 1$
- (3)  $K_{\mathbb{F}_n} = -2E_n - (2+n)f$

*Proof.* (1), (2) Let's prove by induction that  $\text{Pic}(\mathbb{F}_n) = \mathbb{Z}[f] \oplus \mathbb{Z}[E_n]$  and  $(E_n)^2 = -n$ :  
 $\mathbf{n} = \mathbf{0}$  : in this case we have  $\mathbb{F}_0 \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  and the  $E_0$  and  $f$  are fibers of the projections. Thus

$$\boxed{\text{Pic}(\mathbb{F}_0) = \mathbb{Z}[f] \oplus \mathbb{Z}[E_0] \text{ and } E_0^2 = 0}$$

$\mathbf{n} \rightarrow \mathbf{n} + 1$  : Consider the birational map  $\phi_n : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$  together with the morphisms  $\eta_n, \eta_{n+1}$  which we constructed in the proposition 8.2.1:

$$\begin{array}{ccc} & Y & \\ \eta_n \swarrow & & \searrow \eta_{n+1} \\ \mathbb{F}_n & \xrightarrow{\phi_n} & \mathbb{F}_{n+1} \\ \pi_n \downarrow & & \downarrow \pi_{n+1} \\ \mathbb{P}_{\mathbb{C}}^1 & & \mathbb{P}_{\mathbb{C}}^1 \end{array}$$

In the notation of this proposition we have that  $p \in E_n$  and  $\eta_n$  is the blow-up at  $p$ . Now suppose  $f_p$  is the fiber of  $\pi_n$  over  $p$ . Then we have that  $\tilde{f}_p$  is contracted by  $\eta_{n+1}$  onto  $q \in \mathbb{F}_{n+1}$ . Now suppose  $f_q$  is the fiber of  $\pi_{n+1}$ . Then we have that  $\tilde{f}_q$  is contracted by  $\eta_n$  onto  $p \in \mathbb{F}_n$ . Moreover,  $\phi_n$  maps  $E_n$  onto  $E_{n+1}$ . By induction we have that  $E_n^2 = -n$ . Now since  $p \in E_n$  then for  $\tilde{E}_n \subset Y$  we have that  $\tilde{E}_n^2 = -n - 1 = -(n + 1)$ . We have that  $\tilde{E}_n \cap \tilde{f}_p = \emptyset$  and  $\eta_{n+1}$  acts by contraction of  $\tilde{f}_p$  so  $(\eta_{n+1}(\tilde{E}_n))^2 = -(n + 1)$ . Thus:

$$\boxed{E_{n+1}^2 = -(n + 1)}$$

Now we need to show that  $\text{Pic}(\mathbb{F}_{n+1}) = \mathbb{Z}[f_q] \oplus \mathbb{Z}[E_{n+1}]$ : note that  $\eta_{n+1}$  is the blow-up of  $q$  which contracts  $\tilde{f}_p$ . Thus, we have that:

$$\text{Pic}(Y) = \eta_{n+1}^*(\text{Pic}(\mathbb{F}_{n+1})) \oplus \mathbb{Z}[\tilde{f}_p]$$

Similarly,  $\eta_n$  is the blow-up of  $p$  which contracts  $\tilde{f}_q$ . Thus, we have that:

$$\text{Pic}(Y) = \eta_n^*(\text{Pic}(\mathbb{F}_n)) \oplus \mathbb{Z}[\tilde{f}_q]$$

By induction we have that  $\text{Pic}(\mathbb{F}_{n+1}) = \mathbb{Z}[f_p] \oplus \mathbb{Z}[E_n]$ . So  $\text{Pic}(Y)$  is generated by  $\eta_n^*(f_p) = \tilde{f}_p + \tilde{f}_q$ ,  $\eta_n^*(E_n) = \tilde{E}_n$  and  $\tilde{f}_q$ . We see that

$$\eta_{n+1}^*(f_q) = \tilde{f}_p + \tilde{f}_q, \quad \eta_{n+1}^*(E_{n+1}) = \tilde{E}_n$$

Thus,  $\boxed{\text{Pic}(\mathbb{F}_{n+1}) = \mathbb{Z}[f_q] \oplus \mathbb{Z}[E_{n+1}]}$ .

Note that  $E_n$  is a section of the  $\mathbb{P}_{\mathbb{C}}^1$ -bundle  $\pi_n : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$ . Thus we have that  $\boxed{E_n \cdot f = 1}$ ,  $\boxed{f^2 = 0}$  since for points  $p \neq p'$  on  $\mathbb{P}_{\mathbb{C}}^1$  we have  $X_p \cap X_{p'} = \emptyset$

(3). We can write  $K_{\mathbb{F}_n} = aE_n + bf$  and by genus formula we have that

$$\begin{aligned} f \cdot (f + K_{\mathbb{F}_n}) &= -2 + 2g(f) \Rightarrow -2 = f \cdot K_{\mathbb{F}_n} = f \cdot (aE_n + bf) = a \\ E_n \cdot (E_n + K_{\mathbb{F}_n}) &= -2 + 2g(E_n) \Rightarrow -2 + n = E_n \cdot K_{\mathbb{F}_n} = \\ &= E_n \cdot (aE_n + bf) = -an + b = 2n + b \Rightarrow b = -2 - n \end{aligned}$$

Thus,  $\boxed{K_{\mathbb{F}_n} = -2E_n - (2 + n)f}$ . □



**Proposition 8.3.2.**  $\mathbb{F}_n$  satisfy the following properties:

1. Suppose that we have a curve  $C \subset \mathbb{F}_n$  then there exist  $a, b \geq 0$  such that

$$C \sim aE_n + bf$$

2. Suppose  $s \neq E_n$  is a section of  $\pi_n : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$  then

$$s \sim E_n + (n+k)f \text{ for } k \geq 0, s^2 = n + 2k \text{ and } s \cdot E_n = k$$

3. Suppose we have a morphism  $\pi : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$  such that a general fiber is isomorphic to  $\mathbb{P}_{\mathbb{C}}^1$ . Then there exists  $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^1)$  such that either  $\pi = \alpha \circ \pi_n$  or  $n = 0$  and  $\pi = \alpha \circ \pi'_0$ , where  $\pi'_0 : \mathbb{F}_0 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is given by  $([x : y : x], [u, v]) \rightarrow [x : y]$ .

4. Suppose we have a morphism  $\pi : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$  such that a general fiber is isomorphic to  $\mathbb{P}_{\mathbb{C}}^1$ . Then minimum of the self-intersections of the sections of  $\pi$  is  $-n$ .

5.  $\mathbb{F}_n \cong \mathbb{F}_m \Leftrightarrow m = n$ .

*Proof.* (1) Note that since  $\text{Pic}(\mathbb{F}_n) = \mathbb{Z}[f] \oplus \mathbb{Z}[E_n]$  then we can write a curve  $C$  as  $C = aE_n + bf$  for some  $a, b \in \mathbb{Z}$ . Suppose  $C = f$  then  $a = 0, b = 1$ . Suppose  $C = E_n$  then  $a = 1, b = 0$ . Suppose  $C \notin \{f, E_n\}$  then

$$-na + b = C \cdot E_n \geq 0 \Rightarrow b \geq na, a = C \cdot f \geq 0 \Rightarrow b \geq na \geq a \geq 0$$

(2) Suppose  $C \neq E_n$  is the section of  $\pi_n$ . By the previous part we can write  $C$  as  $C \sim aE_n + bf$ . Then

$$1 = C \cdot f = a, b - n = C \cdot E_n \geq 0 \Rightarrow C = E_n + (n+k)f$$

Moreover:

$$C^2 = (E_n + (n+k)f)^2 = E_n^2 + 2(n+k)E_n \cdot f + (n+k)^2 f^2 = -n + 2(n+k) = n + 2k,$$

$$C \cdot E_n = (E_n + (n+k)f) \cdot E_n = E_n^2 + (n+k)f \cdot E_n = -n + n + k = k$$

(3) Suppose for a morphism  $\pi : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$  we have that a general fiber  $f'$  is isomorphic to  $\mathbb{P}_{\mathbb{C}}^1$ . Note that in case fibers of  $\pi$  and  $\pi_n$  coincide we can find  $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^1)$  such that  $\pi = \alpha \circ \pi_n$ . Assume that  $f' \neq f$ , where  $f$  is a fiber of  $\pi_n$ . We can write  $f'$  as  $f' \sim aE_n + bf$  such that  $a > 0, b \geq 0$ . Now we have that

$$0 = (f')^2 = (aE_n + bf)^2 = aE_n^2 + 2abE_n \cdot f + b^2 f^2 = -na^2 + 2ab = a(2b - na) \Rightarrow 2b = na$$

$$0 \leq E_n \cdot f' = E_n \cdot (aE_n + bf) = -an + b = -2b + b = -b \Rightarrow b = 0 \Rightarrow f' = E_n$$

$0 = (f')^2 = E_n^2 = -n \Rightarrow n = 0$  and the fibers of  $\pi$  are equal to the fibers of  $\pi'_0$  so we can find  $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^1)$  such that  $\pi = \alpha \circ \pi'_0$

(4), (5) We conclude from (2) that the minimum value of self-intersections of the sections of  $\pi_n$  equals  $-n$ . We have that  $\pi'_0 = \pi_0 \tau$  for  $\tau \in \text{Aut}(\mathbb{F}_0)$  where  $\tau$  exchanges the factors, thus we obtain that the minimum of the self-intersections of the sections of  $\pi'_0$  also equal 0. Thus, by (3) we obtain statement (4) which gives (5).  $\square$

## 8.4 Automorphisms of Hirzebruch Surfaces

Recall that  $\mathrm{GL}(2, \mathbb{C})$  acts linearly on  $\mathbb{C}^2$  thus we have a linear action on polynomials in 2 variables. This action preserves the subspace of homogeneous polynomials of degree  $n$  and gives the group  $\mathbb{C}^{n+1} \rtimes \mathrm{GL}(2, \mathbb{C})$  with elements:

$$\left( a_0 Y^n + a_1 XY^{n-1} + \dots + a_n X^n, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in \mathbb{C}^{n+1} \rtimes \mathrm{GL}(2, \mathbb{C})$$

$\mathbb{C}^{n+1} \rtimes \mathrm{GL}(2, \mathbb{C})$  acts on  $\mathbb{F}_n$  in a following way:

$$\begin{aligned} & \left( a_0 Y^n + a_1 XY^{n-1} + \dots + a_n X^n, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) ([x : y : z], [u : v]) = \\ = & \begin{cases} \text{for } u \neq 0 : \\ ([xu^n + y(a_0 v^n + a_1 uv^{n-1} + \dots + a_n u^n) : y(au + bv)^n : y(cu + dv)^n], [au + bv : cu + dv]) \\ \text{for } v \neq 0 : \\ ([xv^n + z(a_0 v^n + a_1 uv^{n-1} + \dots + a_n u^n) : z(au + bv)^n : z(cu + dv)^n], [au + bv : cu + dv]) \end{cases} \end{aligned}$$

**Proposition 8.4.1.** *The kernel of this action consists of matrices of the form  $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$  where  $\mu^n = 1$ .*

*Proof.* Suppose we take some element in kernel and apply it to  $([x : y : z], [u : v])$ , where  $u, v \neq 0$ :

$$\begin{aligned} & \left( a_0, \dots, a_n, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) ([x : y : z], [u : v]) = ([x : y : z], [u : v]) \Rightarrow \\ & \Rightarrow ([xv^n + z(a_0 v^n + a_1 uv^{n-1} + \dots + a_n u^n) : z(au + bv)^n : z(cu + dv)^n], [au + bv : cu + dv]) = \\ & = ([x + z(a_0 + a_1(u/v) + \dots + a_n(u/v)^n) : y(a + b(v/u))^n : z(c(u/v) + d)^n], [au + bv : cu + dv]) = \\ & = ([x : y : z], [u : v]) \end{aligned}$$

we see that we should have  $a_0 = \dots = a_n = 0$  (we get it from points where  $z \neq 0$ ). From the point  $[u : v] = [0 : 1]$  we get that  $b = 0$ ; doing similar calculations for point  $[u : v] = [1 : 0]$  we get  $c = 0$  thus  $[au : dv] = [u : v] \Rightarrow a = d$  and we have:

$$([x : ya^n : zd^n], [u : v]) = ([x : y : z], [u : v]) \Rightarrow a^n = d^n = 1 \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ where } a^n = 1$$

Moreover, all matrices of the form  $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$  where  $\mu^n = 1$  are in the kernel.  $\square$

So we have that  $\mathbb{C}^{n+1}$  is the subgroup of  $\mathrm{Aut}(\mathbb{F}_n)$ ;  $(a_0, \dots, a_n)$  sends  $([x : y : z], [u : v]) \in \mathbb{F}_n$  to

$$\begin{cases} ([xu^n + y(a_0 v^n + a_1 uv^{n-1} + \dots + a_n u^n) : yu^n : yv^n], [u : v]) \text{ for } u \neq 0 \\ ([xv^n + z(a_0 v^n + a_1 uv^{n-1} + \dots + a_n u^n) : zu^n : zv^n], [u : v]) \text{ for } v \neq 0 \end{cases}$$

Suppose  $f_0$  is the fiber over  $[1 : 0]$  which consists of points with  $v = 0$ . We see that the open set  $\mathbb{F}_n \setminus (E_n \cup f_0)$  is invariant by the action of  $\mathbb{C}^{n+1}$ . Moreover, we have the isomorphism:

$$\mathbb{A}_{\mathbb{C}}^2 \xrightarrow{\cong} \mathbb{F}_n \setminus (E_n \cup f_0), \text{ where } (x, y) \rightarrow ([x : y^n : 1], [y : 1])$$

The action of  $\mathbb{C}^n$  on this subset corresponds to  $(x, y) \rightarrow (x + a_0 + a_1 y + a_2 y^2 + \dots + a_n y^n, y)$ .

**Proposition 8.4.2.**  $\text{Aut}(\mathbb{F}_0)$  acts transitively on  $\mathbb{F}_0$ . For  $n \geq 1$   $\text{Aut}(\mathbb{F}_n)$  has 2 orbits:  $E_n$  and  $\mathbb{F}_n \setminus E_n$ .

*Proof.*  $\mathbf{n} = \mathbf{0}$  :  $\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})$  acts transitively on  $\mathbb{P}_0 = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ .

$\mathbf{n} \geq \mathbf{1}$  : Note that there is a unique curve  $E_n \subset \mathbb{F}_n$  such that  $E_n^2 = -n$ . Thus  $E_n$  is invariant by action of  $\text{Aut}(\mathbb{F}_n)$ , by the same reason  $\mathbb{F}_n \setminus E_n$  is invariant by action of  $\text{Aut}(\mathbb{F}_n)$ .  $\text{GL}(2, \mathbb{C})$  acts transitively on fibers and  $\mathbb{C}^{n+1}$  acts transitively on  $f \setminus E_n$ , for  $f$  a fiber. Thus,  $\mathbb{C}^{n+1} \rtimes \text{GL}(2, \mathbb{C})$  acts transitively on  $E_n$  and  $\mathbb{F}_n \setminus E_n$ .  $\square$

**Proposition 8.4.3.** For  $n \geq 1$  we have that:

$$\boxed{\text{Aut}(\mathbb{F}_n) = \mathbb{C}^{n+1} \rtimes \text{GL}(2, \mathbb{C}) \Big/ \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \mid \mu^n = 1 \right\}}$$

*Proof.* Let's prove by induction:

$\mathbf{n} = \mathbf{1}$  : in this case  $\text{Aut}(\mathbb{F}_1) = \text{Aut}(\tilde{\mathbb{P}}_{\mathbb{C}}^2)$ , i.e the automorphisms of  $\mathbb{P}_{\mathbb{C}}^2$  with one point fixed. Without loss of generality we can assume this point to be  $p = [1 : 0 : 0]$ .

Now suppose  $g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \text{PGL}(3, \mathbb{C}) = \text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  and  $g(p) = p \Rightarrow a_{21} = a_{31} = 0$  so we can

fix  $a_{11} = 1$  (since we look up to linear equivalence), moreover  $\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \in \text{GL}(2, \mathbb{C})$ ,  $a_{12}, a_{13} \in \mathbb{C}$ .

Thus

$$\text{Aut}(\mathbb{F}_1) = \mathbb{C}^2 \rtimes \text{GL}(2, \mathbb{C})$$

$\mathbf{n} \rightarrow \mathbf{n} + \mathbf{1}$  : by induction we know that

$$\text{Aut}(\mathbb{F}_n) = \mathbb{C}^{n+1} \rtimes \text{GL}(2, \mathbb{C}) \Big/ \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \mid \mu^n = 1 \right\}$$

Moreover, the automorphisms of  $\mathbb{F}_{n+1}$  that fix  $q = ([0 : 0 : 1], [0 : 1]) \in \mathbb{F}_{n+1}$  correspond to automorphisms of  $\mathbb{F}_n$  fixing the base-point  $p = ([1 : 0 : 0], [0 : 1])$  of  $\phi_n$ . We see that the automorphisms  $\left( (a_0, \dots, a_n), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$  which fix  $p$  of  $\mathbb{F}_n$  are in one-to-one correspondence with the automorphisms  $\left( (b_0, \dots, b_{n+1}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = ((0, a_0, \dots, a_n), \begin{pmatrix} a & b \\ c & d \end{pmatrix})$  that fix  $q$  of  $\mathbb{F}_{n+1}$ . We know that

$$G = \mathbb{C}^{n+2} \rtimes \text{GL}(2, \mathbb{C}) \Big/ \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \mid \mu^{n+1} = 1 \right\} \subset \text{Aut}(\mathbb{F}_{n+1})$$

(note that it is reasonable to take the quotient modulo the kernel of the action). Suppose that it is not the equality (i.e.  $G \subsetneq \text{Aut}(\mathbb{F}_{n+1})$ ) and take  $f \in \text{Aut}(\mathbb{F}_{n+1}) \setminus G$  such that  $f(q) = q'$  ( $q' \in \mathbb{F}_{n+1} \setminus E_{n+1}$  since  $q \notin E_{n+1}$  and by the previous proposition  $E_{n+1}$  and  $\mathbb{F}_{n+1} \setminus E_{n+1}$  are different orbits of  $\text{Aut}(\mathbb{F}_{n+1})$ ). Note that  $G$  acts transitively on  $\mathbb{F}_{n+1} \setminus E_{n+1}$  so we can find  $h \in G$  such that  $h(q') = q$ . Thus,  $h \circ f(q) = q$  and  $h \circ f$  is the automorphism of  $\mathbb{F}_{n+1}$  that fixes  $q$ , i.e. it comes from the automorphism that fixes  $p$ , i.e.  $h \circ f(q) \in G \Rightarrow f \in G$  and  $G = \text{Aut}(\mathbb{F}_{n+1})$ .  $\square$

# Chapter 9

## Elementary Links and Decomposition Theorem

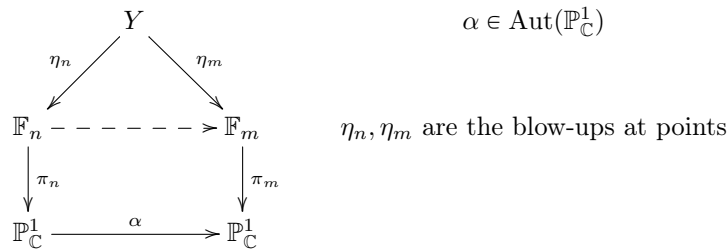
Recall that the **n-th Hirzebruch surface**  $\mathbb{F}_n$  is given by:

$$\mathbb{F}_n = \{([x : y : z], [u : v]) \mid yv^n = zu^n\} \subset \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^1$$

thus we naturally have the projection  $\mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^2$  which takes  $E_n$  to  $[1 : 0 : 0]$ . Note that for  $n = 1$  this is a blow-up of  $\mathbb{P}_{\mathbb{C}}^2$  at  $[1 : 0 : 0]$ . Moreover, we have that if  $X$  is the blow-up of  $\mathbb{P}_{\mathbb{C}}^2$  at any point then  $X \cong \mathbb{F}_1$ .

### 9.1 Elementary Links

Suppose  $\tau : \mathbb{F}_1 \rightarrow \mathbb{P}_{\mathbb{C}}^2$  is the blow-up of a point. We call a birational map  $\tau^{-1} : \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{F}_1$  an **elementary link of type I**. We call a birational map  $\phi : \mathbb{F}_n \dashrightarrow \mathbb{F}_m$  an **elementary link of type II** when there exists a commutative diagram:



We call a birational map  $\mathbb{F}_1 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  an **elementary link of type III** if it is the blow-up of a point of  $\mathbb{P}_{\mathbb{C}}^2$ . We call birational map  $\mathbb{F}_0 \dashrightarrow \mathbb{F}_0$  an **elementary link of type IV** if it is an automorphism which exchanges the fibers of the two projections.

### 9.2 Decomposition Theorem

**Proposition 9.2.1.** *Suppose we have a point  $p \in \mathbb{F}_n$ . Then we can find a unique (up to automorphism of  $\mathbb{F}_m$ ) elementary link  $\phi : \mathbb{F}_n \dashrightarrow \mathbb{F}_m$  with exactly one base point  $p$ . Moreover, if  $n \geq 1$  and  $p \notin E_n$  then  $m = n - 1$  and  $m = n + 1$  otherwise.*

*Proof.* By proposition 8.2.1 there exists an elementary link  $\phi_n : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$  where  $\phi_n$  has a base-point on  $E_n$  and the inverse  $\phi_n^{-1}$  has a base-point on  $\mathbb{F}_{n+1} \setminus E_{n+1}$ . Take  $\alpha_n \in \text{Aut}(\mathbb{F}_n)$  and  $\alpha_{n+1} \in \text{Aut}(\mathbb{F}_{n+1})$ . We have that the map  $\psi_n = \alpha_{n+1} \circ \phi_n \circ \alpha_n$  is an elementary link  $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$ :

$$\begin{array}{ccc} \mathbb{F}_n & \dashrightarrow & \mathbb{F}_{n+1} \\ \uparrow \alpha_n & & \uparrow \alpha_{n+1} \end{array}$$

Moreover, the Proposition 8.4.2 shows that we may take any point of  $E_n$  as base-point of  $\psi$  and any point of  $\mathbb{F}_{n+1} \setminus E_{n+1}$  as base-point of  $\psi_n^{-1}$ . The result follows.  $\square$

**Theorem 9.2.2.** *Suppose  $X = \mathbb{P}_{\mathbb{C}}^2$  or  $X = \mathbb{F}_n$  for some  $n \geq 0$ ;  $\eta : Y \rightarrow X$  is a birational morphism and  $E \subset Y$  is a  $(-1)$ -curve. Then:*

1. *There exists a birational morphism  $\tau : Y \rightarrow Z$ , where  $Z$  is a smooth projective surface such that  $\tau(E) = q$  ( $q$  is a point on  $Z$ ) and moreover  $\tau$  is the blow-up at  $q$ .*
2. *There exists a sequence of birational maps  $\phi_1, \dots, \phi_k$ , where each  $\phi_i : X_{i-1} \dashrightarrow X_i$  is an elementary link,  $X = X_0$ , and  $\psi = \phi_k \circ \dots \circ \phi_1 \circ \eta \circ \tau^{-1}$  is a birational morphism.*

$$\begin{array}{ccc} Y_m = Y & \xrightarrow{\tau} & Z \\ \eta \downarrow & & \downarrow \psi \\ X & \dashrightarrow^{\phi_1} X_1 \dashrightarrow^{\phi_2} \dots \dashrightarrow^{\phi_k} X_k \end{array}$$

*Proof.* By assumption  $\eta : Y \rightarrow X$  is a birational morphism. Thus, by the Theorem 5.4.5 it can be decomposed into the sequence of blow-ups  $\eta_i : Y_i \rightarrow Y_{i-1}$  of point  $p_i \in Y_{i-1}$  ( $Y_0 = X$ ,  $Y_m = Y$ ) where  $\mathcal{E}_i$  is the curve contracted by  $\eta_i$ :

$$\begin{array}{ccc} Y_m = Y & \xrightarrow{\tau} & Z \\ \eta_m \downarrow & \curvearrowright \eta & \downarrow \psi \\ Y_{m-1} & & \\ \eta_{m-1} \downarrow & & \\ \dots & & \\ \eta_1 \downarrow & & \\ Y_0 = X & \dashrightarrow^{\phi_1} X_1 \dashrightarrow^{\phi_2} \dots \dashrightarrow^{\phi_k} X_k \end{array}$$

Suppose  $E_i = (\eta_{i+1} \circ \dots \circ \eta_m)^*(\mathcal{E}_i)$ .

Suppose we have a smooth curve  $E \subset Y$  such that  $E \cong \mathbb{P}_{\mathbb{C}}^1$  and  $E^2 = -1$ . Note that if  $\eta$  contracts  $E$  then there exists  $E_i$  such that  $E_i = E$  and  $p_i$  is not infinitely near to any other  $p_j$ . In this case we can change the order and put  $i = m$  so that  $\tau = \eta_m$ ,  $Z = Y_{m-1}$ .

Now we suppose that  $\eta$  does not contract  $E$ . Suppose  $E_X = \eta(E) \subset X$ . We see that in this case  $E = \tilde{E}_X$  and  $E = \tilde{E}_X \sim \eta^* E_X - \sum m_{p_i}(E_X) E_i$ . Suppose that  $f_X \in \text{Pic}(X)$  is the fiber of  $\pi_n : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$  and  $f_Y = \eta^* f_X \in \text{Pic}(Y)$ . We may assume that  $X = \mathbb{F}_n$  for some  $n \geq 0$  (since  $\tilde{\mathbb{P}}_{\mathbb{C}}^2 = \mathbb{F}_1$ ) and denote  $d = E \cdot f_Y = E_X \cdot f_X$ . We also have that:

$$-1 = E^2 = (\eta^* E_X - \sum m_{p_i}(E_X) E_i)^2 = E_X^2 - \sum m_{p_i}(E_X)^2$$

Suppose  $\mathbf{d} = \mathbf{0}$  : in this case we have that  $d = E_X \cdot f_X = 0 \Rightarrow E_X$  is a fiber of  $\pi_n \Rightarrow E_X^2 = 0$  ( $\star$ )  $\Rightarrow \sum m_{p_i}(E_X) E_i = 1$  and without loss of generality we can assume that  $m_{p_1}(E_X) = 1$  and

$m_{p_i}(E_X) = 0$  for  $i \neq 1$ . Now we have that  $\eta_1 : Y_1 \rightarrow X = \mathbb{F}_n$  is the blow-up of a point so we can write  $\tau' : Y_1 \rightarrow \mathbb{F}_{n \pm 1} = \mathbb{F}_{n'}$  for contraction of the strict transform of  $E_X$  in  $Y_1$  onto some point  $q \in \mathbb{F}_{n'}$  (the existence follows from 9.2.1, 8.4.2). So we obtain an elementary link  $\phi : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n'}$  where  $\phi = \tau' \circ (\eta_1)^{-1}$ :

$$\begin{array}{ccc} & Y_1 & \\ \eta_1 \downarrow & \searrow \tau' & \\ X = \mathbb{F}_n & \xrightarrow{\phi} & \mathbb{F}_{n'} \end{array}$$

Now define  $\eta' : Y \rightarrow \mathbb{F}_{n'}$  as  $\eta' = \tau' \circ \eta_2 \circ \dots \circ \eta_m$ ,  $\eta'$  contracts  $E$  onto point  $q \in \mathbb{F}_{n'}$ . Since no infinitely near point is blown-up, we can decompose  $\eta'$  as a sequence of blow-ups where  $q$  will be the last point blown-up instead of begin the first one. We denote the last blow-up  $Y \rightarrow Z$  and  $\psi : Z \rightarrow \mathbb{F}_{n'}$  remains. Thus:

$$\begin{array}{ccc} Y & \xrightarrow[\tau]{\text{contracts } E} & Z \\ \downarrow & \searrow \eta' & \downarrow \psi \\ Y_1 & & \\ \eta_1 \downarrow & \searrow \tau' & \\ X = \mathbb{F}_n & \xrightarrow{\phi = \phi_1} & \mathbb{F}_{n'} \end{array} \quad \begin{array}{l} k = 1, X_1 = \mathbb{F}_{n'}, \phi = \phi_1 \\ \psi = \eta' \circ \tau^{-1} = \phi_1 \circ \eta \circ \tau^{-1} \end{array}$$

Suppose  $\mathbf{d} > \mathbf{0}$ . Let's prove the result by induction on  $d$ .

Suppose that we have a point  $p_i$  with  $2m_{p_i}(E_X) > d$ . By changing the order we can assume that  $2m_{p_1}(E_X) > d$ . Similarly to the previous case we denote  $\tau' : Y_1 \rightarrow \mathbb{F}_{n'}$  the contraction of  $\tilde{f}_1 \subset Y_1$  onto point  $q \in \mathbb{F}_{n'}$  where  $f_1$  is the fiber of  $\pi_n : \mathbb{F}_n \rightarrow \mathbb{P}_{\mathbb{C}}^1$  such that  $p \in f_1$  and obtain the elementary link  $\phi = \tau' \circ \eta_1^{-1} : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n'}$ . Note that replacing  $\eta$  by  $\eta' = \tau' \circ \eta_2 \circ \dots \circ \eta_m$  does not affect  $d$  (since a general fiber of  $\pi_n$  is sent onto a general fiber of  $\pi_{n'}$  by  $\phi$ ) and multiplicities  $m_{p_i}(E_X) = m_{p_i}(\eta'(E))$  for  $i \neq 1$ . Let's look at  $m_{p_1}(\eta'(E))$ :

$$m_{p_1}(\eta'(E)) = (\eta_2 \circ \dots \circ \eta_m)(E) \cdot \tilde{f}_1 = (\eta_2 \circ \dots \circ \eta_m)(E) \cdot (\eta_1^* f_X - \mathcal{E}_1) = d - m_{p_1}(E_X)$$

Since  $2m_{p_1}(E_X) > d$  and  $d = m_{p_1}(E_X) + m_{p_1}(\eta'(E))$  we obtain that  $m_{p_1}(\eta'(E)) < m_{p_1}(E_X)$ . Using this method we can assume that for all  $p_i$  we have  $2m_{p_i}(E_X) \leq d$ .

Note that  $\text{Pic}(X) = \text{Pic}(\mathbb{F}_n) = \mathbb{Z}[E_n] \oplus \mathbb{Z}[f_X]$ . Note that  $d = E_X \cdot f_X$  thus we can write  $E_X = dE_n + bf_X$  ( $b \in \mathbb{N}$ ).

Suppose  $\mathbf{E}_X \cdot \mathbf{E}_n < \mathbf{0}$ . We have that  $-1 = E^2 = E_X^2 - \sum m_{p_i}(E_X)^2 \Rightarrow E_X^2 \geq -1$  so

$$E_X^2 = -d^2n - 2db \geq -1 \Rightarrow d^2n + 2db \leq 1 \Rightarrow \begin{cases} b = 0, d = 1 \\ n = 1 \end{cases} \quad \text{for } d > 0, dn > b$$

$n = 1$  and  $E_X = E_1$ . Now we consider a map  $\mathbb{F}_1 \rightarrow \mathbb{P}_{\mathbb{C}}^2$  which contracts  $E_1$  and get the result:

$$\begin{array}{ccc} & Y & \\ \eta \downarrow & \searrow & \\ X = \mathbb{F}_1 & \xrightarrow{\phi} & \mathbb{P}_{\mathbb{C}}^2 \end{array}$$

Suppose  $\mathbf{E}_X \cdot \mathbf{E}_n = \mathbf{0}$ , then by similar calculations we obtain that either  $n = 1, b = 0, d = 1$  and we proceed as above or  $b = 0, n = 0$  and  $E_X = dE_0 \Rightarrow E_X^2 = 0$  and we proceed as in  $(\star)$ .

Suppose  $\mathbf{E}_X \cdot \mathbf{E}_n > \mathbf{0}$ , i.e  $b - dn > 0$  and so  $\epsilon = 2b - dn > b > dn$ . We have that:

$$E = \eta^* E_X - \sum m_{p_i}(E_X) E_i = d\eta^*(E_n) + b\eta^*(f_X) - \sum m_{p_i}(E_X) E_i$$

$$K_Y = \eta^*(K_X) + \sum E_i, \quad K_X \cdot E_n = -2 - n, \quad K_X \cdot f_X = -2$$

So:

$$-1 = E^2 = -d^2 n + 2bd - \sum (m_{p_i}(E_X))^2 = d\epsilon - \sum (m_{p_i}(E_X))^2$$

$$-1 = K_Y \cdot E = (-2 - n)d - 2b + \sum (m_{p_i}(E_X))^2 = -\epsilon - 2d + \sum m_{p_i}(E_X)$$

Thus:

$$\sum (m_{p_i}(E_X))^2 = d\epsilon + 1 \text{ and } \sum (m_{p_i}(E_X))^2 = \epsilon + 2d + 1$$

Moreover, since  $d > 2m_{p_i}(E_X) \geq 0$  we obtain:

$$\begin{aligned} 0 < \sum m_{p_i}(E_X)(d - 2m_{p_i}(E_X)) &= d \sum m_{p_i}(E_X) - 2 \sum (m_{p_i}(E_X))^2 = 2d^2 - d - \epsilon d - 2 \Rightarrow \\ &\Rightarrow 2d^2 - \epsilon d > d + 2 > 0 \Rightarrow 2d > \epsilon - 2b - nd > b > nd \Rightarrow n < 2 \end{aligned}$$

$\mathbf{n} = \mathbf{0}$  :  $2d > \epsilon = 2b - dn = 2b \Rightarrow d > b$ . We now apply the elementary link of type IV which exchanges  $E_0$  and  $f$  exchanging  $d$  and  $b$ , which decreases  $d$  and we can proceed by induction.

$\mathbf{n} = \mathbf{1}$  :  $b > nd = d$ . Suppose that for all  $i$  we have  $m_{p_i}(E_X) < b - d$ . Then we have:

$$\begin{aligned} 0 &\geq \sum (m_{p_i}(E_X))^2 - \sum m_{p_i}(E_X)(b - d) = 2bd - d^2 + 1 - (2b + d - 1)(b - d) = \\ &= b(3d - 2b) + (b - d) - 1 \Rightarrow^{(b-d)-1 \geq 0, b \geq 0} 3d - 2b \leq 0 - \text{contradiction with } 2d > \epsilon = 2b - d \end{aligned}$$

Thus there exists  $p_i$  such that  $m_{p_i}(E_X) \geq b - d$ , without loss of generality we set  $m_{p_1}(E_X) \geq b - d$ . Note that  $E_X \cdot E_1 = b - d \Rightarrow p_1 \notin E_1$  and we can assume that  $p_1$  is a proper point of  $X = \mathbb{F}_1$ . Now we contract  $E_1$  via blow-up  $\tau : \mathbb{F}_1 \rightarrow \mathbb{P}_\mathbb{C}^2$  and blowing-up  $p_1$   $\pi : \mathbb{F}_1 \rightarrow \mathbb{P}_\mathbb{C}^2$  with the exceptional curve  $F_1$  we have that:

$$\begin{aligned} \tau^*(\tau(E_X)) \cdot \tau^*(\tau(f_X)) &= (E_X + (E_X \cdot E_1)E_1) \cdot (f_X + (f_X \cdot E_1)E_1) = (E_X + (b - d)E_1) \cdot (f_X + E_1) = \\ &= E_X \cdot f_X + (b - d)E_1 \cdot f_X + E_1 \cdot E_X + (b - d)E_1^2 = d + (b - d) + (b - d) - (b - d) = b \end{aligned}$$

Since  $b = \tau^*(\tau(E_X)) \cdot \tau^*(\tau(f_X)) = \tau(E_X) \cdot \tau(f_X) = \pi^*(\tau(E_X)) \cdot \pi^*(\tau(f_X))$  we obtain:

$$\begin{aligned} b &= \pi^*(\tau(E_X)) \cdot \pi^*(\tau(f_X)) = (\widetilde{\tau(E_X)} + m_{p_1}(\tau(E_X)) \cdot F_1) \cdot \pi^*(\tau(f_X)) = \\ &= \widetilde{\tau(E_X)} \cdot \widetilde{\tau(f_X)} + m_{p_1}(\tau(E_X)) = \widetilde{\tau(E_X)} \cdot \widetilde{\tau(f_X)} + m_{p_1}(E_X) > \\ &> \widetilde{\tau(E_X)} \cdot \widetilde{\tau(f_X)} + (b - d) \Rightarrow b > d' + (b - d) \Rightarrow d' < d \end{aligned}$$

and we proceed by induction. □

**Corollary 9.2.3.** *Suppose  $X$  is a smooth projective rational surface. Then we can find a birational morphism  $\phi : X \rightarrow X'$ , where  $X' = \mathbb{P}_\mathbb{C}^2$  or  $X' = \mathbb{F}_n$  for  $n \neq 1$ .*

*Proof.* Note that if we find a morphism  $X \rightarrow \mathbb{F}_1$  then we find the desired morphism  $X \rightarrow \mathbb{F}_1 \rightarrow \mathbb{P}_{\mathbb{C}}^2$ . Now we have that  $X$  is a rational surface. By definition there is a birational map  $\phi : \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow X$ . By 5.4.6 we can find morphisms  $\pi, \eta$  (which are the compositions of blow-ups) such that the following diagram is commutative by :

$$\begin{array}{ccc}
 & Y & \\
 \pi \swarrow & & \searrow \eta \\
 \mathbb{P}_{\mathbb{C}}^2 & \overset{\phi}{\dashrightarrow} & X
 \end{array} \quad \phi = \eta \circ \pi^{-1}$$

In case  $\eta$  is an isomorphism we are done. Suppose  $\eta$  is NOT an isomorphism. Then  $\eta$  is the sequence of blow-ups (up to isomorphism). Suppose  $E \subset Y$  is the first curve contracted by this sequence. Then by the theorem 9.2.2 we have a morphism  $\tau : Y \rightarrow Y'$ , where  $Y'$  is a smooth projective surface and  $\tau$  contracts  $E$ . Moreover, from the same theorem we get a morphism  $\pi' : Y' \rightarrow X''$ , where  $X'' = \mathbb{P}_{\mathbb{C}}^2$  or  $\mathbb{F}_n$  for  $n \geq 0$  and the following diagram is commutative:

$$\begin{array}{ccc}
 Y & \xrightarrow{\tau} & Y' \\
 \pi \downarrow & & \downarrow \pi' \\
 \mathbb{P}_{\mathbb{C}}^2 & \dashrightarrow & X'' \xrightarrow{\phi'} X
 \end{array} \quad \eta'$$

We replace  $\phi$  with  $\phi'$ ,  $\pi$  with  $\pi'$ ,  $\eta$  with  $\eta'$  and decrease the number of contractions in  $\eta$  so eventually we will get the isomorphism.  $\square$

**Remark.** Note that from the results above we obtain the **Castelnuovo's contraction criterium for rational surfaces** which was already proved in general case as Theorem 5.5.1:

Suppose  $X$  is a smooth rational projective surface.  $E \subset X$  is a  $(-1)$ -curve. Then here exists a birational morphism  $\tau : X \rightarrow Y$  to a smooth projective surface  $Y$  contracting  $E$  onto some point, and which is the blow-up of this point.

*Proof.* By corollary 9.2.3 we can find a birational morphism  $\phi : X \rightarrow X'$ , where  $X' = \mathbb{P}_{\mathbb{C}}^2$  or  $X' = \mathbb{F}_n$  for  $n \neq 1$ . Thus, from theorem 9.2.2 we obtain the result.  $\square$

**Corollary 9.2.4.** Suppose  $X$  is a smooth rational projective surface with a point  $p \in X$ . Then there exists an open neighborhood  $U \ni p$  such that  $U \cong \mathbb{A}_{\mathbb{C}}^2$ .

*Proof.* For  $X = \mathbb{P}_{\mathbb{C}}^2$  we have the result. In case  $X = \mathbb{F}_n$  we have the result from the description in 8.1.2. Assume that we have the statement for  $X$  let's prove it for the blow-up  $\pi : \tilde{X} \rightarrow X$  at point  $q \in X$ . Consider an open subset  $U \subset X$  such that  $U \cong \mathbb{A}_{\mathbb{C}}^2$ . If  $q \notin U \Rightarrow \pi^{-1}(U) \cong U \cong \mathbb{A}_{\mathbb{C}}^2$ . If  $q \in U$  we move  $q$  to the origin  $\mathbb{A}_{\mathbb{C}}^2$ . We also have that  $\pi^{-1}(U) \cong \tilde{\mathbb{A}}_{\mathbb{C}}^2$ , which is covered by open subsets isomorphic to  $\mathbb{A}_{\mathbb{C}}^2$ . Thus, from 9.2.3 we get the result.  $\square$

**Corollary 9.2.5.** Suppose we have surfaces  $X_1, X_2$  such that  $X_i \in \{\mathbb{P}_{\mathbb{C}}^2, \mathbb{F}_n \mid n \geq 0\}$  for  $i = 1, 2$ . Then any birational map  $\phi : X_1 \dashrightarrow X_2$  decomposes into automorphisms and elementary links of type I, II, III, IV

*Proof.* Note that by 5.4.6 we can find morphisms  $\pi, \eta$  (which are the compositions of blow-ups) such that the following diagram is commutative:

$$\begin{array}{ccc}
 & Y & \\
 \pi \swarrow & & \searrow \eta \\
 X_1 & \overset{\phi}{\dashrightarrow} & X_2
 \end{array} \quad \phi = \eta \circ \pi^{-1}$$



in case  $\pi$  is an isomorphism  $\eta$  is a blow-up at one point so  $\phi$  is the link of type 1. Otherwise at in 9.2.3 we reduce the number of blow-ups and so get the result.  $\square$

**Corollary 9.2.6.**  *$\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is generated by  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  and by the birational maps preserving a given pencil of lines through one point.*

*Proof.* Note that the links of type I,II,III clearly preserve the pencil of lines. So by 9.2.5 its only enough to prove that we can get rid of links of type IV. Suppose we have a link of type IV:

$$\begin{array}{ccc} \mathbb{F}_0 = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 & \xrightarrow{\cong} & \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\ \text{proj}_1 \downarrow & & \downarrow \text{proj}_2 \\ \mathbb{P}_{\mathbb{C}}^1 & & \mathbb{P}_{\mathbb{C}}^1 \end{array}$$

Note that  $\text{proj}_2 = \pi_1 \circ \pi_q^{-1} \circ \pi \circ \phi_0$  where:  $\phi_0 : \mathbb{F}_0 \dashrightarrow \mathbb{F}_1$  is a link of type II (we take  $p$  and  $q$  from the notation in 8.2.1),  $\pi : \mathbb{F}_1 \rightarrow \mathbb{P}_{\mathbb{C}}^2$  is a blow-down of  $(-1)$ -curve on  $\mathbb{F}_1$  which is a link of type III,  $\pi_q^{-1} : \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{F}_1$  is a blow-up of  $\pi(q)$  which is a link of type I and  $\pi_1 : \mathbb{F}_1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is the fibration and so we are done.  $\square$

### 9.3 Description of Del Pezzo Surfaces

**Theorem 9.3.1.** *Suppose  $X$  is a smooth projective rational surface. Then the following conditions are equivalent:*

1.  $X$  is a del Pezzo surface
2.  $X \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ ,  $X \cong \mathbb{P}_{\mathbb{C}}^2$  or  $X$  is the blow-up of  $0 \leq k \leq 8$  points of  $\mathbb{P}_{\mathbb{C}}^2$  such that no 3 are colinear, no 6 belong to the same conic, and no 8 lie on a cubic having a double point at one of them
3.  $K_X^2 \geq 1$  and for any irreducible curve  $C$  of  $X$  we have  $C^2 \geq -1$
4. Suppose  $C$  is an effective divisor then  $C \cdot (-K_X) > 0$

*Proof.* Let's prove that  $\mathbb{F}_0 = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  satisfies (1), (3), (4):

We know that  $K_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1} = -2(f_1 + f_2)$  where  $f_1, f_2$  are the fibers of the projections by 8.3.1. Let  $D = -1/2K_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1}$ . Note that

$$D^2 = (f_1 + f_2)^2 = 2 > 0, D \cdot f_i = (f_1 + f_2) \cdot f_i = 1 > 0 \text{ for } i = 1, 2$$

Thus, by Nakai-Moishenson criteria  $D$  is ample (with any irreducible curve on  $\mathbb{F}_0$  is positive)  $\Rightarrow -K_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1}$  is ample  $\Rightarrow$  by definition  $\mathbb{F}_0 = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  is a del Pezzo surface. Moreover we can write any irreducible curve  $C$  as  $af_2 + bE_0$  and  $a, b \geq 0$ . So we see that  $C^2 \geq -1$ ,  $C \cdot (-K_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1}) > 0$ . So all the conditions are satisfied for  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ . In case we have a birational morphism  $X \rightarrow \mathbb{P}_{\mathbb{C}}^2$  we already proved all the equivalences in 7.1.2. Suppose we do not have such morphism. Then in any case by 9.2.3 we have  $\pi : X \rightarrow \mathbb{F}_n$  for  $n \neq 1$ . Suppose  $n = 0$ , then  $X \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ . In case  $\pi$  an isomorphism and we are done. Otherwise  $\pi$  factors through the blow-up  $\tau : Z \rightarrow \mathbb{F}_0$  at point  $p$ . An elementary link centered at  $p$  (described in 8.2.1) gives rise to a birational morphism  $Z \rightarrow \mathbb{F}_1$ , and then to a birational morphism  $X \rightarrow \mathbb{P}_{\mathbb{C}}^2$ :

$$\begin{array}{ccccc} & & Z & & \\ & \nearrow & \downarrow \tau & \searrow & \\ X & \xrightarrow{\pi} & \mathbb{F}_0 & \dashrightarrow & \mathbb{F}_1 \longrightarrow \mathbb{P}_{\mathbb{C}}^2 \end{array}$$

So we can assume that  $n \geq 2$ . In this case, the strict transform on  $X$  of  $E_n$  gives a smooth curve  $C \subset X$  such that  $C \cong \mathbb{P}_{\mathbb{C}}^1$  and  $C^2 \leq -2$ . By genus formula we have  $K_X \cdot C + C^2 = -2 \Rightarrow K_X \cdot C \leq 0$  - contradiction with (1), (3), (4).  $\square$

**Definition 9.3.2.** We call a smooth projective surface  $X$  **minimal** if any birational morphism  $X \rightarrow Y$  to a smooth projective surface  $Y$  is an isomorphism.

**Proposition 9.3.3.** Suppose  $X$  is a smooth projective rational surface. Then the following are equivalent:

$$X \text{ is minimal} \Leftrightarrow \text{there is no } (-1)\text{-curve on } X \Leftrightarrow ( X \cong \mathbb{P}_{\mathbb{C}}^2 \text{ or } X \cong \mathbb{F}_n \text{ for } n \neq 1 ).$$

*Proof.* (2) $\Rightarrow$ (3) By assumption there is no curve which can be contracted.

(1) $\Rightarrow$ (3) By 9.2.3 there exists a birational morphism  $X \rightarrow X'$  where  $X' = \mathbb{P}_{\mathbb{C}}^2$  or  $\mathbb{F}_n$  for  $n \neq 1$  which should be an isomorphism by definition of minimality.

(3) $\Rightarrow$ (2) there is no  $(-1)$ -curve on  $\mathbb{P}_{\mathbb{C}}^2$ . Suppose  $X = \mathbb{F}_n$  for  $n \neq 1$ . Suppose we have a curve  $C \subset \mathbb{F}_n$ ,  $C \sim aE_n + bf$  for  $a, b \geq 0$ . Then  $C^2 = -a^2n + 2ab = a(2b - an)$ . We see that  $C^2 = -1 \Rightarrow n = 1$  which is not the case.  $\square$

# Chapter 10

## Regulations of Finite Groups

We know that for the pair of smooth projective surfaces  $X, Y$  and a birational morphism  $\phi : X \dashrightarrow Y$  we can find a smooth projective surface  $Z$  together with morphisms  $\eta_X, \eta_Y$  (which are the sequence of blow-ups and no  $(-1)$ -curve in  $Z$  is contracted by both of them) such that the following diagram is commutative by 5.4.6:

$$\begin{array}{ccc}
 & Z & \\
 \eta_X \swarrow & & \searrow \eta_Y \\
 X & \dashrightarrow \phi \dashrightarrow & Y
 \end{array}
 \quad \phi = \eta_Y \circ \eta_X^{-1}$$

the base-points of  $\phi$  (resp. of  $\phi^{-1}$ ) are the points blown-up by  $\eta_X$  (resp.  $\eta_Y$ ). These points form two subsets of the bubble spaces  $\mathcal{B}(X)$  and  $\mathcal{B}(Y)$  respectively. We denote them  $\mathcal{B}(\phi) \subset \mathcal{B}(X)$  and  $\mathcal{B}(\phi^{-1}) \subset \mathcal{B}(Y)$ .

### 10.1 Properties of $\mathcal{B}(\phi)$

**Proposition 10.1.1.** 1. Suppose  $\pi : \tilde{X} \rightarrow X$  the blow-up of a point  $p \in X$ . Then  $\pi$  induces a bijection

$$\pi_{\bullet} : \mathcal{B}(\tilde{X}) \rightarrow \mathcal{B}(X) \setminus \{p\}$$

2. Suppose we have a pair of smooth projective surfaces  $X, Y$  and  $\phi : X \dashrightarrow Y$  is a birational map. Then  $\phi$  induces a bijection

$$\phi_{\bullet} : \mathcal{B}(X) \setminus \mathcal{B}(\phi) \rightarrow \mathcal{B}(Y) \setminus \mathcal{B}(\phi^{-1})$$

3. Suppose we have smooth projective surfaces  $X, Y, Z$  and birational map  $\phi : X \dashrightarrow Y, \psi : Y \dashrightarrow Z$  between them. Then for any point  $p \in \mathcal{B}(X) \setminus \mathcal{B}(\phi)$  such that  $\phi_{\bullet}(p) \in \mathcal{B}(Y) \setminus \mathcal{B}(\phi)$  we have:

$$\psi_{\bullet}(\phi_{\bullet}(p)) = (\psi\phi)_{\bullet}(p)$$

*Proof.*

1. The points in  $\mathcal{B}(\tilde{X})$  are given by equivalence classes of triplets  $(y, Y, \eta)$  such that  $y \in Y$  and  $\eta : Y \rightarrow \tilde{X}$  is a birational morphism. These points correspond to  $(y, Y, \pi \circ \eta)$  which are the elements of  $\mathcal{B}(X)$ . Note that all the points of  $\mathcal{B}(X) \setminus \{p\}$  are obtained this way.

2. We can find a smooth projective surface  $Z$  together with morphisms  $\eta_X, \eta_Y$  (which are the

sequences of blow-ups and no  $(-1)$ -curve in  $Z$  is contracted by both of them) such that the following diagram commutes by 5.4.6:

$$\begin{array}{ccc}
 & Z & \\
 \eta_X \swarrow & & \searrow \eta_Y \\
 X & \overset{\phi}{\dashrightarrow} & Y
 \end{array}
 \quad \phi = \eta_Y \circ \eta_X^{-1}$$

By (1) we see that  $\eta_X$  gives the bijection  $\mathcal{B}(Z) \rightarrow \mathcal{B}(X) \setminus \mathcal{B}(\phi)$  and  $\eta_Y$  gives the bijection  $\mathcal{B}(Z) \rightarrow \mathcal{B}(Y) \setminus \mathcal{B}(\phi^{-1})$ . Thus we obtain the desired bijection.

3. Follows from construction of  $\phi_\bullet$ . □

**Proposition 10.1.2.** *Suppose  $X$  is a smooth projective surface and  $G \subset \text{Bir}(X)$  is a finite subgroup. Define  $\mathcal{B}(G) = \bigcup_{g \in G} \mathcal{B}(g)$ . Then there exists a birational morphism  $\pi : \tilde{X} \rightarrow X$ , which is the blow-up of the points of  $\mathcal{B}(G)$  and  $\pi^{-1}G\pi \subset \text{Aut}(\tilde{X})$ .*

*Proof.* Note that  $\mathcal{B}(g)$  is finite (since resolution of intermediates of  $g$  has a finite number of blow-ups), and if  $p \in \mathcal{B}(g)$  is infinitely near to some point  $q$  then  $q \in \mathcal{B}(G)$  so we see that  $\mathcal{B}(G) = \bigcup_{g \in G} \mathcal{B}(g)$  is finite. Thus, we can blow-up  $X$  in points of  $\mathcal{B}(G)$  to get a smooth projective surface  $\tilde{X}$ . Let  $\pi : \tilde{X} \rightarrow X$  be the sequence of blow-ups.

Let us fix  $g \in G$ . Now we consider a minimal resolution of  $g$ :

$$\begin{array}{ccc}
 & Z & \\
 \eta_1 \swarrow & & \searrow \eta_2 \\
 X & \overset{g}{\dashrightarrow} & X
 \end{array}
 \quad g = \eta_2 \circ \eta_1^{-1}$$

Note that  $\mathcal{B}(g) \subset \mathcal{B}(G) \Rightarrow \tau_1 = \eta_1^{-1} \circ \pi : \tilde{X} \rightarrow Z$  is a birational morphism. Similarly, since  $\mathcal{B}(g^{-1}) \subset \mathcal{B}(G) \Rightarrow \tau_2 = \eta_2^{-1} \circ \pi : \tilde{X} \rightarrow Z$  is a birational morphism:

$$\begin{array}{ccccc}
 \tilde{X} & & & & \tilde{X} \\
 \searrow \tau_1 & & \swarrow \tau_2 & & \\
 & Z & & & \\
 \swarrow \eta_1 & & \searrow \eta_2 & & \\
 X & \overset{g}{\dashrightarrow} & X & & X
 \end{array}$$

By the previous proposition we have a bijection

$$g_\bullet : \mathcal{B}(X) \setminus \mathcal{B}(g) \rightarrow \mathcal{B}(X) \setminus \mathcal{B}(g^{-1})$$

This induces the bijection we want

$$g_\bullet : \mathcal{B}(G) \setminus \mathcal{B}(g) \rightarrow \mathcal{B}(G) \setminus \mathcal{B}(g^{-1})$$

since for  $p \in \mathcal{B}(G) \setminus \mathcal{B}(g)$  we can find  $h \in G$  such that  $p \in \mathcal{B}(h)$ . Since  $p$  is not a base-point of  $g$ , the image  $g_\bullet(p)$  is well-defined in  $\mathcal{B}(X)$  and is a base-point of  $h \circ g^{-1}$  and is not a base-point of  $g^{-1}$ . In particular  $g_\bullet(p) \in \mathcal{B}(G) \setminus \mathcal{B}(g^{-1})$ . We do the same thing with  $g^{-1}$  and get the result. □

**Definition 10.1.3.** *We say that two groups  $G \subset \text{Bir}(X)$  and  $G' \subset \text{Bir}(X')$  are **birationally conjugate** if there is a birational map  $\phi : X \dashrightarrow X'$  such that  $\phi G \phi^{-1} = G'$ , i.e two groups represent the same conjugacy class in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .*

## 10.2 Definition of $G$ -surfaces

**Definition 10.2.1.** Suppose  $X$  is a smooth projective rational surface,  $G \subset \text{Aut}(X)$  is a group acting biregularly on  $X$ . We call a pair  $(G, X)$  a  **$G$ -surface**. We call a rational map  $\phi : (G, X) \dashrightarrow (G, X)$   **$G$ -rational** if for any element  $g \in G$  we can find  $g' \in G$  such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ g \downarrow & & \downarrow \exists g' \\ X & \xrightarrow{\phi} & X' \end{array}$$

Suppose  $(G, X)$  is a  $G$ -surface. We say that a birational map  $\phi : X \dashrightarrow X'$  is  **$G$ -equivariant** if the  $G$ -action on  $X'$  induced by  $\phi$  is biregular and  $\phi$  itself is called a **birational map of  $G$ -surfaces**. We say that a pair  $(G, X)$  is **minimal** or  **$G$  acts minimally on  $X$**  if any  $G$ -equivariant birational morphism  $\phi : X \rightarrow X'$  is an isomorphism.

## 10.3 Classification of $G$ -surfaces

We proved in 10.1.2 that we can realize a finite subgroup  $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  as a subgroup of  $\text{Aut}(X)$  where  $X$  is a rational surface. After contracting orbits of disjoint exceptional curves we can assume that  $(X, G)$  is minimal and the study of all  $G$ -surfaces can be reduced to studying only minimal ones. But before stating a very powerful result we need to give a definition:

**Definition 10.3.1.**

1. Suppose we have a rational surface  $X$  together with a morphism  $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ . Then we call a pair  $(X, \pi)$  a **conic bundle** if:
  - all but finitely many fibers are isomorphic to  $\mathbb{P}_{\mathbb{C}}^1$ .
  - singular fibers are the union of rational curves  $F, F'$  satisfying

$$F^2 = F'^2 = -1, \quad F \cdot F' = 1$$

2. Suppose we have two conic bundles  $(X, \pi), (X', \pi')$ . We call a birational map  $\phi : X \rightarrow X'$  a **birational map of conic bundles** if it sends general fibers of  $\pi$  to general fibers of  $\pi'$  (i.e. it preserves the fiber structure).
3. Suppose we have a conic bundle  $(X, \pi)$ . We call it **minimal** if any birational morphism of conic bundles  $(X, \pi) \rightarrow (X', \pi')$  is an isomorphism.

The classification of minimal  $G$ -surfaces was given by Yu. Manin in the abelian case [Man67] and V.A. Iskovskikh in the general case [Isk79]. We want to apply their result to the examples in the upcoming chapters:

**Theorem 10.3.2.** Suppose  $X$  is a rational surface,  $G \subset \text{Aut}(X)$  is finite,  $(G, X)$  is a minimal pair. Then exactly one of the following holds:

1.  $\text{rk}(\text{Pic}(X)^G) = 1$  and  $X$  is a Del Pezzo surface.
2.  $\text{rk}(\text{Pic}(X)^G) = 2$  and  $X$  is a conic bundle (with the structure invariant by  $G$ ).

In the first case the generator is the canonical divisor and in the second case the generators are the canonical divisor and the divisor class of a fiber.

Some comments:

In the second case, we have a morphism  $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  such that general fibers are rational and irreducible and every singular fiber is the union of two rational curves  $F, F'$  satisfying  $F^2 = F'^2 = -1$ ,  $F \cdot F' = 1$ .  $G$  embeds in the group of automorphisms of the generic fiber of  $\pi$ . This group can be viewed as the group of birational maps of  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  preserving the first projection.

We will not prove this theorem, but we will prove the proposition which is hidden inside it:

**Proposition 10.3.3.** *Suppose  $X$  is a rational surface,  $G \subset \text{Aut}(X)$  is a finite group. If  $\text{rkPic}(X)^G = 1$  the surface  $X$  is a Del Pezzo surface and*

$$\text{Pic}(X)^G = \begin{cases} \frac{1}{3}\mathbb{Z}K_X, & \text{if } X \cong \mathbb{P}_{\mathbb{C}}^2 \\ \frac{1}{2}\mathbb{Z}K_X, & \text{if } X \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\ \mathbb{Z}K_X, & \text{otherwise} \end{cases}$$

where  $\text{Pic}(X)^G$  is the part of the Picard group which is fixed by the group  $G \subset \text{Aut}(X)$ .

*Proof.* 1. The canonical divisor  $K_X$  is preserved by  $G$ . Thus:

$$\text{Pic}(X)^G \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}K_X$$

$X$  is a projective surface. Thus there exists  $H$  an ample divisor on  $X$ . Consider

$$O(H) := \sum_{g \in G} g(H)$$

Note that  $O(H)$  is  $G$  invariant and ample. Thus, there exist  $m \in \mathbb{Q}$  such that  $\tilde{H} := mO(H)$  is an ample generator of  $\text{Pic}(X)^G \otimes_{\mathbb{Z}} \mathbb{Q}$  (since by assumption  $\text{rk}(\text{Pic}(X)^G) = 1$ ). We now have that  $-K_X = a\tilde{H}$  for  $a \in \mathbb{Q}$ ,  $a > 0$  since otherwise the Kodaira dimension  $\kappa(X) \geq 0$  which is not the case for rational surfaces by [Bea96] Chapter VII. So

$$(-K_X)^2 = a^2\tilde{H}^2 >^{\tilde{H}\text{-ample}} 0$$

Now take an irreducible curve  $C$ . We have that:

$$C \cdot (-K_X) = \frac{1}{|G|} \sum_{g \in G} g(C) \cdot (-K_X) \stackrel{(\star)}{=} \frac{1}{|G|} (-bK_X) \cdot (-K_X) = \frac{b}{|G|} (-K_X) \cdot (-K_X) = \frac{b}{|G|} K_X^2 >^{(\star\star)} 0$$

( $\star$ ) is true since the orbit of the curve  $C$  is  $G$ -invariant, ( $\star\star$ ) here  $b \in \mathbb{Q}$ ,  $b > 0$  since otherwise the Kodaira dimension  $\kappa(X) \geq 0$  which is not the case for rational surfaces by [Bea96] Chapter VII. Thus, by proposition 7.1.2  $X$  is a Del Pezzo surface.

2. Note that

$$\text{Pic}(X)^G \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}K_X \Rightarrow \text{Pic}(X)^G = \mathbb{Q}K_X \cap \text{Pic}(X)$$

The picard group has no torsion, since  $X$  is rational.<sup>1</sup>

- Suppose  $X \cong \mathbb{P}_{\mathbb{C}}^2$ . Line is a generator of  $\text{Pic}(\mathbb{P}_{\mathbb{C}}^2)$  and it is given by  $-\frac{1}{3}K_X$ . Thus:

$$\text{Pic}(\mathbb{P}_{\mathbb{C}}^2)^G = \mathbb{Q}K_{\mathbb{P}_{\mathbb{C}}^2} \cap \text{Pic}(\mathbb{P}_{\mathbb{C}}^2) = \frac{1}{3}\mathbb{Z}K_{\mathbb{P}_{\mathbb{C}}^2}$$

<sup>1</sup>This is true for rationally connected varieties over  $\mathbb{C}$  [CT-Sk19] Corollary 12.4.5.

- Suppose  $X \cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ . The generators of  $\text{Pic}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  we will give further in 8.3.1 from which we conclude:

$$\text{Pic}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)^G = \mathbb{Q}K_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1} \cap \text{Pic}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) = \frac{1}{2}\mathbb{Z}K_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1}$$

- Otherwise  $X$  is a blow-up of  $\mathbb{P}_{\mathbb{C}}^2$  at  $k \geq 1$  points  $x_1, \dots, x_k$ . Thus:

$$K_X = -3L + \sum_{i=1}^k E_i$$

here  $L = \pi^*l$ , where  $l$  is the line on  $\mathbb{P}_{\mathbb{C}}^2$ , and  $E_i = \pi^{-1}(x_i)$  for  $i \in \{1, \dots, k\}$ . We see that  $K_X$  is not a multiple of any divisor so we obtain:

$$\text{Pic}(X)^G = \mathbb{Z}K_X$$

□

# Chapter 11

## Conjugacy Classes between Examples

Note that if we consider two minimal pairs  $(G, X)$  and  $(G, X')$  it would be nice to understand if these pairs are birationally conjugate, i.e. if it is possible to find a  $G$ -equivariant birational map  $\phi : X \dashrightarrow X'$ . There exist two main tools for that:

### 11.1 Tool 1: Point-Wise Fixed Curves

Suppose we have an element  $h \in \text{Aut}(X)$  such that it point-wise fixes a smooth irreducible curve  $C$  with  $g(C) > 0$ .  $\phi : X \dashrightarrow X'$  is a birational map (note that  $g(C) > 0 \Rightarrow \phi$  does not contract  $C$ ) and  $h' = \phi \circ h \circ \phi^{-1} \in \text{Aut}(X')$  fixes  $\phi(C) = C'$  point-wise.  $\phi|_C : C \dashrightarrow C'$  is an isomorphism since  $C$  and  $C'$  are smooth curves. The important result from in [Ba-Be00] is:

**Theorem 11.1.1.** *Suppose we have  $G_1, G_2 \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  and  $\text{Ord}(G_1) = \text{Ord}(G_2) = p$  where  $p$  is prime. Then  $G_1, G_2$  are conjugated in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  if and only if one of the following occur:*

1. *There exists an irreducible curve  $C$  point-wise fixed by both  $G_1$  and  $G_2$*
2. *There does not exist an irreducible curve  $C$  with  $\text{genus}(C) > 0$  fixed by  $G_1$  or  $G_2$ . In this case both  $G_1$  and  $G_2$  are conjugate to:*

$$\{[x : y : z] \rightarrow [\alpha x : y : z] \mid \alpha^p = 1\}$$

### 11.2 Tool 2: $G$ -elementary links

We can divide  $G$ -surfaces in two sets ( $X$  is a smooth projective rational surface):

( $\mathbb{D}$ ) Pairs  $(G, X)$ , such that  $G \subset \text{Aut}(X)$  is finite satisfying  $\text{rk}(\text{Pic}(X)^G) = 1$ .

( $\mathbb{C}$ ) Triples  $(G, X, \pi)$ , such that  $(X, \pi)$  is a conic bundle and  $G \subset \text{Aut}(X, \pi)$  satisfying  $\text{rk}(\text{Pic}(X)^G) = 2$ .

Some remarks:

- By theorem 10.3.2 for any minimal pair  $(G, X)$  we have  $(G, X) \in (\mathbb{D})$  or there exists a conic bundle structure  $(X, \pi)$  such that  $(G, X, \pi) \in (\mathbb{C})$ .
- ( $\mathbb{D}$ ) and ( $\mathbb{C}$ ) contain to distinct pairs  $(G, X)$  because of conditions on  $\text{rk}(\text{Pic}(X))$ .
- Suppose  $(G, X) \in (\mathbb{D})$  then  $X$  is a del Pezzo surface by 10.3.3 and the  $(G, X)$  is minimal since it is not possible to contract anything  $G$ -equivariantly (after such action  $\text{rk}(\text{Pic}(X)^G) = 1$  should decrease).



- Suppose  $(G, X, \pi) \in (\mathbb{C})$  then it is possible that  $(G, X)$  is not minimal: the example is  $(G, X) = (\mathbb{F}_1, \pi_1)$  and finite  $G \subset \text{Aut}(\mathbb{F}_1)$ .
- It is possible that  $(G, X, \pi) \in (\mathbb{C})$  is a del Pezzo surface.

Let's define  $G$ -elementary links which are  $G$ -equivariant birational maps

$$\phi : (G, X) \dashrightarrow (G, X')$$

- **G-links of type I:** suppose  $(G, X) \in (\mathbb{D})$ ,  $(G, X', \pi') \in (\mathbb{C})$ ;  $\phi$  is the blow-up of one orbit of points of  $X$  under the action of  $G$  is  $G$ -link of type I.
- **G-links of type II:** we have a minimal resolution:

$$\begin{array}{ccc} & Z & \\ \eta \swarrow & & \searrow \eta' \\ X & \overset{\phi}{\dashrightarrow} & X' \end{array} \quad g = \eta' \circ \eta^{-1}$$

where  $\eta$  and  $\eta'$  are blow-ups of a  $G$ -orbit (on  $X$  and  $X'$  respectively). Moreover, either  $(G, X) \in (\mathbb{D})$ ,  $(G', X') \in (\mathbb{D})$  or  $(G, X, \pi) \in (\mathbb{C})$ ,  $(G', X', \pi') \in (\mathbb{C})$ . For the second case, we ask for  $\alpha \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^1)$  such that  $\pi \circ \eta = \alpha \circ \pi' \circ \eta'$  (so  $\phi$  takes general fibers to general fibers)

- **G-links of type III:** consider  $(G, X, \pi) \in (\mathbb{C})$ ,  $(G', X') \in (\mathbb{D})$  and a birational morphism  $\phi$  is which is the blow-up of one orbit of points of  $x'$  under the action of  $G$ .
- **G-links of type IV:** consider  $(G, X, \pi) \in (\mathbb{C})$ ,  $(G', X', \pi') \in (\mathbb{C})$  and the map  $\phi : X \rightarrow X'$  is an isomorphism which is not compatible with  $\pi$  and  $\pi'$ , i.e. we take another conic bundle structure on the same surface.

The important result from [Isk96] is:

**Proposition 11.2.1.** *Suppose  $(G, X_i)$  is such that either  $(G, X_i) \in (\mathbb{D})$  or  $(G, X_i, \pi_i) \in (\mathbb{C})$  for some conic bundle  $\pi_i$  on  $X_i$  for  $i \in \{1, 2\}$ . Then any birational map  $(G, X_1) \dashrightarrow (G, X_2)$  decomposes into  $G$ -elementary links of type I, II, III, IV.*

## Chapter 12

# Automorphisms of Conic Bundles

In 10.1.2 we proved that we can realize a finite subgroup  $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  as a subgroup of  $\text{Aut}(X)$  where  $X$  is a rational surface. After contracting orbits of disjoint exceptional curves we can assume that  $(X, G)$  is minimal and the study of all  $G$ -surfaces can be reduced to studying only minimal ones. By 10.3.2 we know that if  $X$  is a rational surface,  $G \subset \text{Aut}(X)$  is finite, and  $(G, X)$  is a minimal pair then one of the two options for  $X$  is that  $\text{rk}(\text{Pic}(X)^G) = 2$  and  $X$  is a conic bundle (with the structure invariant by  $G$ ). Thus, we need to study the automorphisms of conic bundles.

### 12.1 Minimal Conic Bundles

**Theorem 12.1.1.** *Suppose we have a conic bundle  $(X, \pi)$ . Then the following are equivalent:*

- (1)  $(X, \pi)$  is a minimal conic bundle  $\Leftrightarrow$  (2)  $\pi$  is a smooth fibration  $\Leftrightarrow$  (3)  $X = \mathbb{F}_n$  for some  $n$ .

*Proof.* (3) $\Rightarrow$  (2) Follows from construction of  $\mathbb{F}_n$ .

(2) $\Rightarrow$  (1) Suppose we have a birational morphism of conic bundles  $\phi : (X, \pi) \rightarrow (X', \pi')$  which is not an isomorphism. In this case  $\phi$  should contract at least one  $(-1)$ -curve  $F$  on  $X$  so that  $\overline{\pi(F)}$  is a point. The fiber structure is preserved thus  $F$  is contained in some singular fiber and (1)  $\Rightarrow$  (2) which is equivalent to (2)  $\Rightarrow$  (1).

(1) $\Rightarrow$  (3) Suppose we have a minimal conic bundle  $(X, \pi)$  then there is no  $(-1)$ -curve contained in a fiber (otherwise we could contract it and get birational morphism which is not an isomorphism to some other conic bundle) we see that  $\text{rk}(\text{Pic}(X)) \geq 2$  so by 9.2.3  $X = \mathbb{F}_n$  (since we exclude  $\mathbb{P}_{\mathbb{C}}^2$  with  $\text{rk}(\text{Pic}(\mathbb{P}_{\mathbb{C}}^2)) = 1$ ).  $\square$

### 12.2 Properties of Conic Bundles

**Proposition 12.2.1.** *Suppose we have a conic bundle  $(X, \pi)$  such that  $X \not\cong \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ :*

1. *We can find a section  $s$  of  $\pi$  such that  $s^2 < 0$ .*
2. *For any section  $s$  such that  $s^2 = -n < 0$ , there exists a birational morphism of conic bundles  $p : X \rightarrow \mathbb{F}_n$  sending  $s$  onto  $E_n$ .*
3. *Suppose that  $s, t$  are the sections of  $\pi$  such that  $s^2 = t^2 = -n < 0$ ; let  $r$  be the number of singular fibers of  $\pi$  and  $k$  be the number of these fibers where  $s, t$  intersect the same component. Then we have*

$$r = 2s \cdot t + 2n + k$$

and there exist birational morphisms of conic bundles  $p_0 : X \rightarrow \mathbb{F}_0$  and  $p_1 : X \rightarrow \mathbb{F}_1$ .

*Proof.* (1) By the previous proposition we have a birational morphism of conic bundles

$$p : (X, \pi) \rightarrow (\mathbb{F}_n, \pi)$$

which can be decomposed as a sequence of blow-ups by 5.4.5.

Suppose  $\mathbf{n} > \mathbf{0}$  then  $\tilde{E}_n^2 < 0$  on  $X$ .

Suppose  $\mathbf{n} = \mathbf{0}$  then  $X \neq \mathbb{F}_0 = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  thus  $p$  contains at least 1 blow-up, so the strict transform  $\tilde{s}$  of the section  $s$  containing this point with  $s^2 = 0$  has  $\tilde{s}^2 \leq s^2 - 1 < 0$  on  $X$ .

Suppose that the number of singular fibers  $\mathbf{r} = \mathbf{0}$ : then  $X = \mathbb{F}_n$ . For (2) we take  $p = id$  and in (3) we have only one section with negative intersection  $E_n^2 = -n$  for which the statement holds.

Suppose  $\mathbf{r} \geq \mathbf{1}$ :

(2) Consider the set  $\{F_i \mid i = 1, \dots, r\}$  where any  $F_i$  is the component of singular fiber verifying  $F_i \cap s = \emptyset$ . Now we blow-down all  $F_i$  and get  $\mathbb{F}_m$  for some  $m$ . Note that since  $F_i \cap s = \emptyset$  then  $p(s)^2 = s^2 = -n \Rightarrow m = n$  and  $p(s) = E_n$ .

(3) Consider  $p : X \rightarrow \mathbb{F}_n$  from the previous part. In this case we see that

$$\text{Pic}(X) = \mathbb{Z}p^*(E_n) \oplus \mathbb{Z}[f] \oplus \bigoplus_{i=1}^r \mathbb{Z}[F_i]$$

and we can write a section  $t$  as  $t = s + bf - \sum_{i=1}^r a_i F_i$  where  $b \in \mathbb{Z}$ ,  $a_i = t \cdot F_i \in \{0, 1\}$ . Note that  $a_i = 0$  if and only if  $s, t$  intersect the second component of the singular fiber containing  $F_i \Rightarrow k = r - \sum_{i=1}^r a_i$ . We have that:

$$s^2 = t^2 = (s + bf - \sum_{i=1}^r a_i F_i)^2 = s^2 + 2b - \sum_{i=1}^r a_i^2 = s^2 + 2b - (r - k) \Rightarrow 2b = r - k$$

$$s \cdot t = s \cdot (s + bf - \sum_{i=1}^r a_i F_i) = s^2 + b - 0 = b - n \Rightarrow r = 2b + k = 2s \cdot t + 2n + k$$

We observe that  $r \geq 2n$ . We obtain a birational morphism of conic bundles  $p_0 : X \rightarrow \mathbb{F}_0$  such that  $p_0(s) = E_0$  by contracting the set  $\{f \setminus F_i, F_j \mid i = 1, \dots, n; j = n + 1, \dots, r\}$  and we obtain a birational morphism of conic bundles  $p_1 : X \rightarrow \mathbb{F}_1$  such that  $p_1(s) = E_1$  by contracting the set  $\{f \setminus F_i, F_j \mid i = 1, \dots, n - 1; j = n, \dots, r\}$

We call the case when  $r = 2n$  **the exceptional case**.  $\square$

**Definition 12.2.2.** We say that a conic bundle  $(X, \pi)$  is **exceptional** if it has  $2n$  singular fibers and two sections of self-intersection  $-n$ .

## 12.3 De Jonquieres Group

We now want to explore the group of automorphisms of conic bundle. First, consider an example.

**Example 12.3.1.** Suppose we have  $\pi_1 : \mathbb{F}_0 = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  a projection on the first factor. The group of birational maps on  $\mathbb{F}_0$  which preserves the fibration structure is the group of birational maps of the form:

$$(x, y) \mapsto \left( \frac{ax + b}{cx + d}, \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)} \right)$$

$$\text{where } a, b, c, d \in \mathbb{C} : \alpha(x), \beta, \gamma(x), \delta(x) \in \mathbb{C}(x), (ad - bc)(\alpha(x)\delta(x) - \beta(x)\gamma(x)) \neq 0$$

This is **de Jonquieres group** and it is isomorphic to  $\text{PGL}(2, \mathbb{C}(x)) \rtimes \text{PGL}(2, \mathbb{C})$ .

**Definition 12.3.2.** *Suppose we have a conic bundle  $(X, \pi)$ . We denote  $\text{Aut}(X, \pi)$  is the the group of automorphisms of  $X$  which preserve the conic bundle structure.*

*Suppose  $\Lambda$  is a pencil of rational curves a surface  $X$ . Then denote by  $Cr(X, \Lambda)$  the group of birational maps of  $X$  that leaves  $\Lambda$  invariant. In case  $\Lambda$  is induced by a morphism  $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  (i.e. if is base-point-free), we write  $Cr(X, \pi) = Cr(X, \Lambda)$ .*

Suppose that  $G \subset \text{Aut}(X, \pi)$  and we have a birational morphism of conic bundles  $(X, \pi) \rightarrow (X', \pi')$ , where  $(X', \pi') \cong (\mathbb{F}_n, \pi_n)$ , i.e. it is minimal. Note that using elementary links, we can go to  $\mathbb{F}_0$  and see  $G$  as a subgroup of the de Jonquieres group and we can also do the converse. Thus, the study of finite groups of automorphisms of conic bundles is equivalent to the study of finite subgroups of the de Jonquieres group. More precisely:

**Proposition 12.3.3.** • *Suppose  $X$  is a surface,  $\Lambda$  is a pencil of rational curves on  $X$ . Then there exists a birational map*

$$\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$$

*such that  $\phi(\Lambda)$  is the pencil of fibers of  $\pi_1 : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  and  $\phi Cr(X, \Lambda) \phi^{-1} = Cr(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \pi_1)$*

- *Suppose  $G$  is a finite subgroup of  $Cr(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \pi_1)$ . Then there exist a conic bundle  $(X, \pi)$  together with a birational map of conic bundles  $\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  such that  $\phi G \phi^{-1} \subset \text{Aut}(X, \pi)$ .*
- *Suppose  $G$  be a finite subgroup of automorphisms  $\text{Aut}(X, \pi)$  of a conic bundle  $(X, \pi)$ . Then  $G$  is conjugate to a subgroup of  $Cr(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \pi_1)$  by a birational map of conic bundles.*

*Proof.* 1. Consider the pencil of rational curves  $\Lambda$  given in the proposition . We blow-up the base points of  $\Lambda$  such that it becomes base-point free and obtain a surface  $\tilde{X}$  together with the morphism  $\pi : \tilde{X} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ . We made  $\tilde{X}$  in a way such that the general fiber of  $\pi$  are rational. The morphism  $\alpha : \tilde{X} \rightarrow X$  sends fibers of  $\pi$  to  $\Lambda$ . The morphism  $\alpha$  also conjugate  $Cr(X, \Lambda)$  and  $Cr(\tilde{X}, \pi)$ . By Noether-Enriques Theorem([Bea96] Theorem III.4) we obtain a birational map  $\phi : \tilde{X} \dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  such that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{X} & \dashrightarrow^{\phi} & \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\ & \searrow^{\pi} & \swarrow_{\pi_1} \\ & \mathbb{P}_{\mathbb{C}}^1 & \end{array}$$

here  $\pi_1$  is the projection on the first factor.

2. By proposition 10.1.2 we can find a rational surface  $X_1$  together with a birational morphism  $\phi_1 : X_1 \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  such that  $G_1 = \phi_1 \circ G \circ \phi \subset \text{Aut}(X)$ . Fibers of  $\pi_1 : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  are sent to a pencil of rational curves  $\Lambda_1$  on  $X_1$  which is  $G_1$ -invariant.

Now since the base points of  $\Lambda_1$  are  $G_1$ -invariant we can blow them up and get a surface  $X_2$  together with a base point free pencil of rational curves  $\Lambda_2$  on  $X_2$ . Suppose  $\pi' : X_2 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ . We can suppose that any exceptional curve contained in a fiber is not contractible. In this case any reducible fiber is the union of two  $(-1)$ -curves which meet transversely (proposition 12.3.4). Thus,  $(X_2, \pi')$  is a conic bundle.

3. We have that  $G \subset \text{Aut}(X, \pi) \subset Cr(X, \pi)$  which is conjugate by a birational map of conic bundles to  $Cr(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \pi_1)$  by the first part.  $\square$

The first statement of the proposition implies that the group  $Cr(X, \pi)$  is birationally conjugate to the de Jonquieres group.

**Proposition 12.3.4.** *Suppose that  $X$  is a surface,  $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  a morphism such that one of the fibers is isomorphic to  $\mathbb{P}_{\mathbb{C}}^1$ , and  $F = \sum_{i=1}^m a_i F_i$  ( $a_i > 0$ ) is a fiber where  $F_i$ -s are irreducible curves. Then, the possible cases are*

1.  $m = 1, F = F_1 \cong \mathbb{P}_{\mathbb{C}}^1$
2.  $m = 2, F = F_1 \cup F_2$ , with  $F_1 \cong F_2 \cong \mathbb{P}_{\mathbb{C}}^1$ ,  $F_1 \cdot F_2 = 1$  where  $F_1^2 = F_2^2 = -1$
3.  $m > 2$  where:
  - $F_i^2 < 0$ ,  $a_i = 1$  for all  $i$
  - two of the  $F_i$ -s that have self-intersection  $-1$  do not intersect
  - one of the  $F_i$ -s has self-intersection  $\leq -2$

*Proof.* By the Noether-Enriques Theorem ([Bea96] III.4) we can find a birational map  $\phi$  such that the following diagram is commutative:

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\ & \searrow \pi & \swarrow \pi_1 \\ & & \mathbb{P}_{\mathbb{C}}^1 \end{array}$$

here  $\pi_1$  is the projection on the first factor. After that we resolve the indeterminacies of  $\phi$ :

$$\begin{array}{ccc} & \tilde{X} & \\ \eta \swarrow & & \searrow \xi \\ X & \overset{\phi}{\dashrightarrow} & \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \end{array}$$

where  $\eta, \xi$  are the compositions of blow-ups (5.4.6). Note that since the diagram commutes we have that any curve contracted by  $\xi$  is contained in some fiber. So the fibers of  $\pi$  are given by the blow-up of some points of  $\mathbb{P}_{\mathbb{C}}^1$  followed by blow-down the  $(-1)$ -irreducible components. Thus all the fibers are connected and any irreducible component of any fiber is isomorphic to  $\mathbb{P}_{\mathbb{C}}^1$ . Now we can enumerate the irreducible components of any fiber  $F$  in such a way that  $F_i \cdot F_j = 1 \Leftrightarrow (i - j) = \pm 1$ . Now consider three cases:

1.  $m = 1$ : This case is clear.
2.  $m = 2$ : We have:

$$\begin{aligned} 0 = F^2 &= (F_1 + F_2)^2 = F_1^2 + 2F_1 \cdot F_2 + F_2^2 \Rightarrow F_1^2 + F_2^2 = -2 \\ (F_1)^2 &= (F - F_2)^2 = F^2 + 2F \cdot F_2 + (F_2)^2 = 0 + 0 + (F_2)^2 \Rightarrow F_1^2 = F_2^2 = -1 \end{aligned}$$

3.  $m \geq 3$ : Note that  $F_i^2 < 0$  for all  $i$  since:

$$F_i^2 = F_i \cdot (F - \sum_{j \neq i} a_j F_j) < 0, \text{ since } F \cdot F_i = 0 \text{ and } \exists j \text{ s.t. } F_i \cdot F_j > 0 \text{ (} F \text{ is connected)}$$

Suppose that  $F_i \cap F_{i+1} \neq \emptyset$ , with  $F_i^2 = F_{i+1}^2 = -1$ . Note that when we contract  $F_i$  we get that the image of  $F_{i+1}$  has self-intersection 0, so the image of  $F$  becomes a fiber equal to the image of  $F_{i+1}$  and  $m = 2$ . Thus, if we have two components with self-intersection  $(-1)$  and  $m > 2$  then these components are disjoint.

□

## 12.4 Action of a group $G$ on a conic bundle $(X, \pi)$

For a conic bundle  $(X, \pi)$  consider the subgroup  $G \subset \text{Aut}(X, \pi)$  such that it acts birationally on  $(X, \pi)$ . Every element  $g \in G \subset \text{Aut}(X, \pi)$  sends fiber to fiber so we obtain a homomorphism

$$\bar{\pi} : G \rightarrow \text{Aut}(\mathbb{P}_{\mathbb{C}}^1) = \text{PGL}(2, \mathbb{C})$$

satisfying  $\bar{\pi}(g) \circ \pi = \pi \circ g$ : if  $g : f_p \rightarrow f_{p'}$  then  $p \rightarrow p'$  where  $f_p, f_{p'}$  are fibers over  $p, p'$  respectively. The group  $G' = \text{Ker}(\bar{\pi})$  which consists of automorphisms leaving every fiber invariant embeds in the subgroup  $\text{PGL}(2, \mathbb{C}(x))$  of automorphisms of the general fiber  $\mathbb{P}_{\mathbb{C}(x)}^1$ :

$$1 \rightarrow G' \rightarrow G \xrightarrow{\bar{\pi}} \bar{\pi}(G) \rightarrow 1$$

For  $G$  finite we have that  $G'$  and  $\bar{\pi}(G)$  are finite subgroups of  $\text{PGL}(2, \mathbb{C})$  and  $\text{PGL}(2, \mathbb{C}(x))$  so they are either cyclic groups  $C_n$ , dihedral groups  $D_n$  (with  $2n$  elements), the tetrahedral group  $A_4$ , the icosahedral group  $A_5$ , the octahedral group  $S_4$  ([K156]).

## 12.5 Finite subgroups of $\text{Aut}(X, \pi)$ with $(G, X)$ minimal

**Proposition 12.5.1.** *Suppose  $(X, \pi)$  is a conic bundle which contains at least one singular fiber singular fiber,  $G \subset \text{Aut}(X, \pi)$  is a finite group such that a pair  $(G, X)$  is minimal. Then, one of the following occurs:*

1.  $(X, \pi)$  is the exceptional conic bundle;
2.  $G' := \{\text{elements which preserve any fiber}\} \subset G$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  or  $(\mathbb{Z}/2\mathbb{Z})^2$ .

*Proof.* By proposition 12.2.1 we can find a section  $s$  such that  $s^2 = -n < 0$ . By assumption  $(X, \pi)$  is a conic bundle containing at least one singular fiber. Suppose this fiber has components  $F_1, F_2$  and  $s \cap F_1 \neq \emptyset$ . Moreover, we have that the pair  $(G, X)$  is minimal so we can find  $g \in G$  such that  $g(F_1) = F_2$  and  $g(F_2) = F_1$  (note that if there is no such element then we contract the elements in  $\text{Orb}(F_1)$  - contradiction with minimality of  $(G, X)$ ). So we obtain a section  $t$  with the property

$$t = g(s) \neq s \text{ and } t^2 = s^2$$

(1). Suppose we can find an element  $h \in G$  such that  $h \neq id$  but it acts trivially on  $\text{Pic}(X)$  and the basis of the fibration so it preserves  $s$  and  $t$  and fixes them point-wise. Moreover, the set  $\{p \in X \mid h(p) = p\}$  is smooth  $\Rightarrow s \cdot t = 0$  and  $s$  and  $t$  intersect different components of each singular fiber. Thus, in the notation of 12.2.1  $r = 2n$  and  $(X, \pi)$  is the exceptional conic bundle.

(2). Suppose  $G$  does not contain  $h$  as above, so any non-trivial element which acts trivially on the basis fibration gives a non trivial action on the Picard group. Denote the group of such elements and the  $id$  element by:

$$G' = \{g \mid g \text{ acts trivially on the basis of fibration}\}$$

For any  $g \in G'$  we have that  $\text{ord}(g) \in \{1, 2\}$ . To see this we look at the square of each element. Note that  $g \in G'$  preserves the two components of each singular fiber which gives a trivial action on  $\text{Pic}(X)$  (since if we contract  $X$  to the minimal bundle  $\mathbb{F}_n$  we get a trivial action on  $\text{Pic}(\mathbb{F}_n)$ ). By 12.4 we have that  $G' \subset \text{PGL}(2, \mathbb{C}(x))$  thus it can be isomorphic to  $\{1\}$ ,  $\mathbb{Z}/2\mathbb{Z}$  or  $(\mathbb{Z}/2\mathbb{Z})^2$ .

**Suppose  $G' \cong \{1\}$ :** let  $g \in G$  such that  $g(F_1) = F_2$  and  $g(F_2) = F_1$ . Note that  $2 \mid \text{ord}(g)$ . Now by replacing  $g$  we assume that  $\text{ord}(g) = 2^m$ .

$\mathbf{m} = \mathbf{1}$  : Note that  $g$  fixes the point  $p = F_1 \cap F_2$ . Moreover,  $g$  does not act trivially on the tangent directions (since it exchanges  $F_1$  and  $F_2$ ) thus  $p$  is not an isolated fixed point, so we can find a curve  $C \ni p$  such that  $g(C) = C$  so  $g$  acts trivially on the basis and we get  $g \in G' \Rightarrow G' \not\cong \{1\}$ -contradiction.

$\mathbf{m} > \mathbf{1}$  : Consider  $g' = g^{2^{m-1}}$ . Note that  $g'^2 = g^{2^m} = id$ , i.e.  $\text{ord}(g') = 2$ . Moreover,

$$g'(F_1) = g^{2^{m-1}}(F_1) = F_1, \quad g'(F_2) = g^{2^{m-1}}(F_2) = F_2$$

The set  $\{p \in X \mid g'(p) = p\}$  is smooth and  $g'$  does not act trivially on  $F_1$  and  $F_2$ . We conclude that  $g'$  fixes exactly three points on the fiber  $F_1 \cup F_2$ :  $x = F_1 \cap F_2$ ,  $y_1 \in F_1 \setminus F_2$  and  $y_2 = g(y_1) \in F_2 \setminus F_1$ . Note that if we contract  $F_1$  (so that its image is point  $q$ ) we conjugate  $g'$  to an automorphism of order 2 which fixes  $q$ . The action on  $F_1$  is not trivial thus  $q$  is not an isolated fixed point so we can find a curve  $C \ni p$  such that  $g'(C) = C$  so  $g'$  acts trivially on the basis and we get  $g' \in G' \Rightarrow G' \not\cong \{1\}$ -contradiction. We conclude that either  $G' \cong \mathbb{Z}/2\mathbb{Z}$  or  $G' \cong (\mathbb{Z}/2\mathbb{Z})^2$ .  $\square$

## Chapter 13

# Automorphisms of del Pezzo Surfaces

In 10.1.2 we proved that we can realize a finite subgroup  $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  as a subgroup of  $\text{Aut}(X)$  where  $X$  is a rational surface. After contracting orbits of disjoint exceptional curves we can assume that  $(X, G)$  is minimal and the study of all  $G$ -surfaces can be reduced to studying only minimal ones. By 10.3.2 we know that if  $X$  is a rational surface,  $G \subset \text{Aut}(X)$  is finite, and  $(G, X)$  is a minimal pair then one of the two options for  $X$  is that  $\text{rk}(\text{Pic}(X)^G) = 1$  and  $X$  is a Del Pezzo surface. Thus, we need to study the automorphisms of del Pezzo surfaces.

Recall that we call a smooth projective surface  $X$  a **del Pezzo surface** if its anti-canonical divisor  $-K_X$  is ample. In theorem 9.3.1 we proved that a del Pezzo surface  $X$  is one of the following:  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ ,  $\mathbb{P}_{\mathbb{C}}^2$  or  $X$  is the blow-up of  $1 \leq k \leq 8$  points of  $\mathbb{P}_{\mathbb{C}}^2$  such that no 3 are colinear, no 6 belong to the same conic, and no 8 lie on a cubic having a double point at one of them. In order to understand the structure of the Cremona group we need to explore the subgroups of group of automorphisms of these surfaces. The classification of subgroups of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3, \mathbb{C})$  and  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) = (\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})) \rtimes \mathbb{Z}/2\mathbb{Z}$  can be found in [Do-Iz09]. Let's take a closer look at groups of automorphisms of del Pezzo surfaces obtained by non-trivial blow-up of  $\mathbb{P}_{\mathbb{C}}^2$ .

### 13.1 Del Pezzo Surfaces Obtained by Non-Trivial Blow-Up

Suppose  $X$  is the blow-up of  $\mathbb{P}_{\mathbb{C}}^2$  at  $1 \leq k \leq 8$  points in general position (so it has degree  $K_X^2 = (\pi^*K_{\mathbb{P}_{\mathbb{C}}^2} + \sum_{i=1}^k E_i)^2 = 9 - k$ ). Then the group  $\text{Aut}(X)$  acts on the set of  $(-1)$ -curves. We see that the kernel of this action corresponds to the elements of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  which fix the blown-up points. Thus, for  $k \leq 3$  the kernel is infinite and for  $4 \leq k \leq 8$  it is finite. If  $d = 9 - k \geq 3$  we can view  $X_d$  as a subset of  $\mathbb{P}_{\mathbb{C}}^d$  via the embedding given by the linear system  $| -K_{X_d} |$ . In case  $d = 1, 2$   $| -K_{X_d} |$  is not very ample so it does not give an embedding.

**Proposition 13.1.1.** *Suppose  $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^2$  is the blow-up of  $1 \leq k \leq 8$  points  $p_1, \dots, p_k \in \mathbb{P}_{\mathbb{C}}^2$  in general position. Then we can describe all  $(-1)$ -curves of  $X$  as:*

- exceptional curves  $E_i = \pi^{-1}(p_i)$
- strict transforms of the curves given in the following table (here  $k =$  number of blown-up points;  $m =$  degree of the curve):



| $k$      | degree $m$ | multiplicities<br>at the points | number of such curves for |   |   |   |    |    |    |    |
|----------|------------|---------------------------------|---------------------------|---|---|---|----|----|----|----|
|          |            |                                 | 1                         | 2 | 3 | 4 | 5  | 6  | 7  | 8  |
| $\geq 2$ | 1          | (1,1)                           | -                         | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| $\geq 5$ | 2          | (1,1,1,1,1)                     | -                         | - | - | - | 1  | 6  | 21 | 56 |
| $\geq 7$ | 3          | (2,1,1,1,1,1,1)                 | -                         | - | - | - | -  | -  | 7  | 56 |
| 8        | 4          | (2,2,2,1,1,1,1,1)               | -                         | - | - | - | -  | -  | -  | 56 |
| 8        | 5          | (2,2,2,2,2,2,1,1)               | -                         | - | - | - | -  | -  | -  | 28 |
| 8        | 6          | (3,2,2,2,2,2,2,2)               | -                         | - | - | - | -  | -  | -  | 8  |

**Remark.** Note that we just directly compute multiplicities. For example, to compute the first line of the table we say that we need to choose 2 points from  $k$  and get the result

*Proof.* Suppose  $L = \pi^*l$  where  $l \subset \mathbb{P}_{\mathbb{C}}^2$  is a line. Suppose  $E \neq E_i = \pi^{-1}(p_i)$  is a  $(-1)$ -curve on  $X$ . Note that we can write  $E = mL - \sum_{i=1}^k a_i E_i$  for  $m > 0$ ,  $a_i \geq 0$  for  $i \in \{1, \dots, k\}$ . By the genus formula we have:

$$\begin{aligned}
E^2 + E \cdot K_X &= -2 \Rightarrow -1 + E \cdot K_X = -2 \Rightarrow K_X \cdot (mL - \sum_{i=1}^k a_i E_i) = -1 \Rightarrow \\
&\Rightarrow -3m + \sum_{i=1}^k a_i = -1 \Rightarrow \sum_{i=1}^k a_i = 3m - 1 \\
-1 = E^2 &= (mL - \sum_{i=1}^k a_i E_i)^2 = m^2 - \sum_{i=1}^k a_i^2 \Rightarrow \sum_{i=1}^k a_i^2 = m^2 + 1 \\
(\sum_{i=1}^k a_i)^2 &\leq k \sum_{i=1}^k a_i^2 \Rightarrow (3m - 1)^2 \leq k(m^2 + 1) \Rightarrow \\
\Rightarrow P(m) &= (3m - 1)^2 - k(m^2 + 1) = (9 - k)m^2 - 6m + (1 - k) \leq 0
\end{aligned}$$

this is a parabola with branches turned up (since  $(9 - k) > 0$ ) with  $P(0) = 1 - k \leq 0$ ,  $P(7) = 50(8 - k) \geq 0$  and the vertex is at  $m = \frac{3}{9-k}$ . Thus, we only need to check  $m \in \{1, \dots, 7\}$  (the results below are obtained by checking cases). Note that the dimension of the system of curves of degree  $m$  with multiplicities  $a_1, \dots, a_k$  at the points  $p_1, \dots, p_k$  satisfies

$$d(m) = \#|\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(m) \otimes \mathfrak{m}_{p_1}^{a_1} \otimes \dots \otimes \mathfrak{m}_{p_k}^{a_k}| \geq \frac{(m+1)(m+2)}{2} - \sum_i \frac{a_i(a_i+1)}{2}$$

**m = 1 :**  $\sum_{i=1}^k a_i^2 = m^2 + 1 = 2$ ,  $\sum_{i=1}^k a_i = 3m - 1 = 2$  we get the curves of degree 1 (lines) passing through exactly two  $p_i$ -s;  $d(1) \geq 1 \Rightarrow$  there exists such line.

**m = 2 :**  $\sum_{i=1}^k a_i^2 = m^2 + 1 = 5$ ,  $\sum_{i=1}^k a_i = 3m - 1 = 5$  we get the curves of degree 2 (quadratics) passing through exactly five  $p_i$ -s;  $d(2) \geq 1 \Rightarrow$  there exists such curve.

**m = 3 :**  $\sum_{i=1}^k a_i^2 = m^2 + 1 = 10$ ,  $\sum_{i=1}^k a_i = 3m - 1 = 8$  we get the curves of degree 3 passing through seven  $p_i$ -s with multiplicities  $(2, 1, 1, 1, 1, 1, 1)$ ;  $d(3) \geq 1 \Rightarrow$  there exists such curve

**m = 4 :**  $\sum_{i=1}^k a_i^2 = m^2 + 1 = 17$ ,  $\sum_{i=1}^k a_i = 3m - 1 = 11$  we get the curves of degree 4 passing through eight  $p_i$ -s with multiplicities  $(2, 2, 2, 1, 1, 1, 1, 1)$ . ;  $d(4) \geq 1 \Rightarrow$  there exists such curve.

**m = 5 :**  $\sum_{i=1}^k a_i^2 = m^2 + 1 = 26$ ,  $\sum_{i=1}^k a_i = 3m - 1 = 14$  we get the curves of degree 5 passing through eight  $p_i$ -s with multiplicities  $(2, 2, 2, 2, 2, 2, 1, 1)$ ;  $d(5) \geq 1 \Rightarrow$  there exists such curve.

**m = 6 :**  $\sum_{i=1}^k a_i^2 = m^2 + 1 = 37$ ,  $\sum_{i=1}^k a_i = 3m - 1 = 17$  we get the curves of degree 6 passing through eight  $p_i$ -s with multiplicities  $(3, 2, 2, 2, 2, 2, 2, 2)$ ;  $d(6) \geq 1 \Rightarrow$  there exists such curve.

**m = 7 :** no solutions.

Note that the curves of this systems are irreducible since they pass through the points in general position.  $\square$

We see that the overall number of curves is:

|                                              |   |   |   |    |    |    |    |     |
|----------------------------------------------|---|---|---|----|----|----|----|-----|
| k                                            | 1 | 2 | 3 | 4  | 5  | 6  | 7  | 8   |
| Number of $(-1)$ -curves which are not $E_i$ | - | 1 | 3 | 6  | 11 | 21 | 49 | 232 |
| Number of all $(-1)$ -curves                 | 1 | 3 | 6 | 10 | 16 | 27 | 56 | 240 |

**Corollary 13.1.2.** *Suppose  $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^2$  is the blow-up of  $k = 1, 2$ . Then the pair  $(\text{Aut}(X), X)$  is not minimal.*

*Proof.* Note that if we blow-up  $\mathbb{P}_{\mathbb{C}}^2$  at 1 point we get one  $(-1)$ -curve  $E_1$  which is invariant by  $\text{Aut}(X)$  so we can contract it and get that  $(G, X)$  is not minimal for  $G \subset \text{Aut}(X)$ .

If we blow-up  $\mathbb{P}_{\mathbb{C}}^2$  at 2 points we get 3 classes representing  $(-1)$ -curves:  $E_1, E_2$  and  $L - E_1 - E_2$ .  $E_1 \cap E_2 = \emptyset$ ,  $E_1, E_2 \cap L - E_1 - E_2 \neq \emptyset \Rightarrow L - E_1 - E_2$  is invariant by  $\text{Aut}(X)$  so we can contract it and get that the pair  $(G, X)$  is not minimal for  $G \subset \text{Aut}(X)$ .  $\square$

**Proposition 13.1.3.** *Suppose  $X$  is a Del Pezzo surface, which is the blow-up of  $1 \leq k \leq 8$  points  $p_1, \dots, p_k \in \mathbb{P}_{\mathbb{C}}^2$  in general position;  $G \subset \text{Aut}(X)$  is a finite subgroup of automorphisms with  $\text{rk}(\text{Pic}(X)^G) = 1$ . Then  $G$  is not trivial and the size of any orbit of the action of  $G$  on the set of exceptional divisors is divisible by  $\text{deg}(X) = 9 - k$  so  $\text{deg}(X) \mid |\text{Orb}(G)|$ .*

*Proof.* Note that  $\text{rk}(\text{Pic}(X)^G) = 1$  and  $\text{rk}(\text{Pic}(X)) \geq 2 > 1 \Rightarrow G \neq \{1\}$ .

Consider the orbit of  $G$  consisting of  $(-1)$ -curves  $D_1, \dots, D_l$ . Note that  $\sum_{i=1}^l D_i$  is invariant by  $G$  thus we have that  $\sum_{i=1}^l D_i = -aK_X$  where  $a \in \mathbb{Q}$ . From the genus formula for  $D_i$  we deduce  $D_i \cdot K_X = -1$  so:

$$a = D_1 \cdot (-aK_X) = D_1 \cdot (\sum_{i=1}^l D_i) \in \mathbb{Z}$$

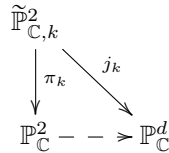
Moreover:

$$a(k - 9) = K_X \cdot (-aK_X) = K_X \cdot (\sum_{i=1}^l D_i) = \sum_{i=1}^l K_X \cdot D_i = -l \Rightarrow (9 - k) \mid l$$

$\square$

Let's also prove the theorem:

**Theorem 13.1.4.** *Suppose we have points  $p_1, \dots, p_k \in \mathbb{P}_{\mathbb{C}}^2$  ( $k \leq 6$ ) in general position. Then the linear system of cubics containing  $p_1, \dots, p_k$  ( $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(3) \otimes \mathfrak{m}_{p_1} \otimes \dots \otimes \mathfrak{m}_{p_k}$ ) gives an embedding:  $j : \tilde{\mathbb{P}}_{\mathbb{C},k}^2 \rightarrow \mathbb{P}_{\mathbb{C}}^d$ :*



We see that  $X_d = j_k(\tilde{\mathbb{P}}_{\mathbb{C},k}^2)$  the surface of degree  $d$  is a **del Pezzo surface**.

*Proof.* To prove that  $j_k$  is an embedding we need to prove that the corresponding linear system separates points and tangent directions. Now consider the case  $\mathbf{k} = \mathbf{6}$ , i.e. we have  $j_6 : \tilde{\mathbb{P}}_{\mathbb{C},6}^2 \rightarrow \mathbb{P}_{\mathbb{C}}^3$ . Suppose that  $\pi_6(x) \neq p_i, p_j$  ( $i < j \leq 6$ ) and consider the linear system of cubics (taking proper powers) defined by  $Q_{i,j}^x$  is the unique conic containing  $\pi_6(x)$ ,  $p_k$ -s (for  $k \neq i, j$ );  $Q_i^x$  is the unique conic containing  $p_k$ -s (for  $k \neq i$ ) so that for strict transforms we have that

$$\tilde{Q}_i \cap \tilde{Q}_j = \emptyset, \quad L_{ij} = \langle p_i, p_j \rangle \text{ for } i \neq j$$

- This system **separates points**: take  $x, y \in \tilde{\mathbb{P}}_{\mathbb{C},6}^2$  such that  $x \neq y$ . Then without loss of generality we have that  $\pi_6(x), \pi_6(y) \neq x_1$  and  $x \notin \tilde{Q}_1$ . In this case we have that  $\tilde{Q}_{1i}^x \cap \tilde{Q}_{1j}^x = \{x\}$  for  $x_j \notin \{p_1, p_i, \pi_6(x)\}$  (since  $Q_{1i} \cap Q_{1j} = \{\pi_6(x), p_k \mid k \neq 1, i, j\}$  - 4 points already: if there were 5 points then the conics would coincide). Thus, there is at most one  $i$  such that  $y \in \tilde{Q}_{1i}^x$ . Then we see that there exists  $j$  such that  $x \in \tilde{Q}_{1j}^x \cup \tilde{L}_{1j}$  but  $y \notin \tilde{Q}_{1j}^x \cup \tilde{L}_{1j}$ . Thus, the system separates points.
- This system **separates tangent directions**. Consider  $x \in \mathbb{P}_{\mathbb{C}}^2 \setminus \{p_1, \dots, p_6\}$  and a cubic  $Q_i \cup \langle x_i, x \rangle$  we have that  $Q_i, Q_j$  for  $i \neq j$  have different tangent directions. Suppose now that  $x \in E_1$ . Then  $Q_{23}^x$  intersect with  $Q_{24}^x$  at  $x$  with multiplicity 2 (since  $x \in E_1$  then "passing trough  $x$ " means being tangent at  $p_1$  in the direction corresponding to  $x$ ). In this case  $x \in \tilde{Q}_{23}^x \cup \tilde{L}_{23}$  and  $x \in \tilde{Q}_{24}^x \cup \tilde{L}_{24}$  have different tangent directions. Thus, this system separates tangent directions.

We conclude that  $j_6$  is an embedding. The result for smaller  $k$  follows. The degree of  $X_{9-k} = X_d$  is given by:

$$\deg(X_d) = \deg(j_k(\tilde{\mathbb{P}}_{\mathbb{C},k}^2)) = d^2 - \sum_i m_i = 9 - k = d$$

□

## 13.2 The del Pezzo Surface of Degree 6

Suppose  $X_6$  is a del Pezzo surface of degree 6. We see that  $X$  is the blow-up of 3 points  $p_1, p_2, p_3$  in general position and up to isomorphism there is only one  $X_6$ . Without loss of generality we may assume that  $p_1 = [1 : 0 : 0]$ ,  $p_2 = [0 : 1 : 0]$ ,  $p_3 = [0 : 0 : 1]$ . Note that  $X_6$  can be viewed as a surface in  $\mathbb{P}_{\mathbb{C}}^6$  given by the embedding corresponding to the linear system of cubics passing through  $p_1, p_2, p_3$ :

$$\begin{array}{ccc} X_6 & & \\ \text{blow-up at } p_1, p_2, p_3 \downarrow & \searrow \text{embedding} & \\ \mathbb{P}_{\mathbb{C}}^2 & \dashrightarrow & \mathbb{P}_{\mathbb{C}}^6 \end{array} \quad [x : y : z] \dashrightarrow [x^2y : x^2z : xy^2 : xyz : xz^2 : y^2z : yz^2]$$

We can also view  $X_6$  as a subset of  $\mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^2$ :

$$X_6 = \{[x : y : z] \times [u : v : w] \mid ux = vy = wz\} \subset \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^2$$

$$\text{proj}_1 : X_6 \rightarrow \mathbb{P}_{\mathbb{C}}^2 \text{ is given by } [x : y : z] \times [u : v : w] \rightarrow [x : y : z]$$

Using this description we can write down explicitly all  $(-1)$ -curves of  $X_6$ :

$$E_1 = \{[1 : 0 : 0] \times [0 : \alpha : \beta] \mid [\alpha : \beta] \in \mathbb{P}_{\mathbb{C}}^1\} = \text{proj}_1^{-1}(p_1)$$

$$E_2 = \{[0 : 1 : 0] \times [\alpha : 0 : \beta] \mid [\alpha : \beta] \in \mathbb{P}_{\mathbb{C}}^1\} = \text{proj}_1^{-1}(p_2)$$

$$E_3 = \{[0 : 0 : 1] \times [\alpha : \beta : 0] \mid [\alpha : \beta] \in \mathbb{P}_{\mathbb{C}}^1\} = \text{proj}_1^{-1}(p_3)$$

$$l_{23} = \{[0 : \alpha : \beta] \times [1 : 0 : 0] \mid [\alpha : \beta] \in \mathbb{P}_{\mathbb{C}}^1\}$$

$$l_{13} = \{[\alpha : 0 : \beta] \times [0 : 1 : 0] \mid [\alpha : \beta] \in \mathbb{P}_{\mathbb{C}}^1\}$$

$$l_{12} = \{[\alpha : \beta : 0] \times [0 : 0 : 1] \mid [\alpha : \beta] \in \mathbb{P}_{\mathbb{C}}^1\}$$

where  $l_{ij}$  are the strict transforms of the lines connecting  $p_i$  and  $p_j$  for  $1 \leq i \neq j \leq 3$ . From this description we see that  $(-1)$ -curves form a hexagon.

**Proposition 13.2.1.** *The action of  $\text{Aut}(X_6)$  on this hexagon gives the exact sequence*

$$1 \rightarrow (\mathbb{C}^*)^2 \rightarrow \text{Aut}(X_6) \xrightarrow{\rho} S_3 \times \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

this sequence splits and we get

$$\boxed{\text{Aut}(X_6) = (\mathbb{C}^*)^2 \rtimes (S_3 \times \mathbb{Z}/2\mathbb{Z})}$$

Here  $(\mathbb{C}^*)^2$  is generated by automorphisms of the form

$$[x : y : z] \times [u : v : w] \rightarrow [x : \alpha y : \beta z] \times [\alpha \beta u : \beta v : \alpha w], \text{ where } \alpha, \beta \in \mathbb{C}^*$$

$(\mathbb{C}^*)^2 \rtimes (S_3 \times \mathbb{Z}/2\mathbb{Z})$  is the lift on  $X_3$  of the group of automorphisms of  $\mathbb{P}_{\mathbb{C}}^2$  that fix the set  $\{p_1, p_2, p_3\}$ .  $\mathbb{Z}/2\mathbb{Z}$  is generated by the automorphism

$$[x : y : z] \times [u : v : w] \rightarrow [u : v : w] \times [x : y : z]$$

which corresponds to the standard quadratic transformation  $[x : y : z] \rightarrow [yz : xz : xy]$  of  $\mathbb{P}_{\mathbb{C}}^2$  which exchanges  $E_i$  and  $l_{jk}$ , for  $\{i, j, k\} = \{1, 2, 3\}$ .  $S_3$  acts on  $(\mathbb{C}^*)^2$  by permuting the coordinates.  $\mathbb{Z}/2\mathbb{Z}$  acts by the inversion.

*Proof.* Note that  $\text{Aut}(X_6)$  preserves the hexagon (since it preserves the  $(-1)$ -curves and the intersection). Note that  $S_3 \times \mathbb{Z}/2\mathbb{Z}$  acts on the hexagon thus we obtain the homomorphism:

$$\rho : \text{Aut}(X_6) \rightarrow S_3 \times \mathbb{Z}/2\mathbb{Z}$$

The kernel of this map consists of those elements which fix all  $(-1)$ -curves thus each element of the kernel corresponds to the element in  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  which fixes points  $p_1, p_2, p_3$ . Note that  $\ker(\rho)$  is generated by automorphisms of the form

$$[x : y : z] \times [u : v : w] \rightarrow [x : \alpha y : \beta z] \times [\alpha \beta u : \beta v : \alpha w], \text{ where } \alpha, \beta \in \mathbb{C}^*$$

So we see that  $\ker(\rho) \cong (\mathbb{C}^*)^2$  and  $(\mathbb{C}^*)^2 \rtimes (S_3 \times \mathbb{Z}/2\mathbb{Z})$  is the lift on  $X_6$  of the group of automorphisms of  $\mathbb{P}_{\mathbb{C}}^2$  that fix the set  $\{p_1, p_2, p_3\}$ .  $S_3$  is generated by:

$$[x : y : z] \times [u : v : w] \rightarrow [y : x : z] \times [v : u : w]$$

$$[x : y : z] \times [u : v : w] \rightarrow [z : y : x] \times [w : v : u]$$

And  $\mathbb{Z}/2\mathbb{Z}$  is generated by:

$$[x : y : z] \times [u : v : w] \rightarrow [u : v : w] \times [x : y : z]$$

So  $\rho$  is in fact surjective and we get the desired splitting.  $\square$

**Proposition 13.2.2.** *Suppose  $X_6$  be the del Pezzo surface of degree 6,  $G \subset \text{Aut}(X_6)$ . Then the pair  $(G, X_6)$  is minimal if and only if the action on the hexagon of  $(-1)$ -curves is transitive.*

*Proof.* Suppose we have a transitive action then no orbit of  $(-1)$ -curves such that it can be contracted. So we see that  $(G, X_6)$  is minimal. Suppose the given action is not transitive. Then there exist two or three invariant disjoint  $(-1)$ -curves of the hexagon. Thus, we can contract them and  $(G, X_6)$  is not minimal.  $\square$

**Proposition 13.2.3.** *There is only one up to conjugation finite cyclic group  $G \subset \text{Aut}(X_6)$  such that  $(G, X_6)$  is minimal. This group is conjugate to a subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  by a birational map  $X_6 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$ .*

*Proof.* Note that by theorem 10.3.2 we have that  $\text{rk}(\text{Pic}(X_6)^G) = 1$ . Thus, we can apply proposition 13.1.3 to get that  $9 - 3 = 6||\text{Orb}(G)|$ . We see that the action of  $G$  on the hexagon is transitive so we can find an element  $g \in G$  which has the form:

$$g : ([x : y : z], [u : v : w]) \rightarrow ([\alpha v : \beta w : u], [\beta y : \alpha z : \alpha \beta x]), \text{ where } \alpha, \beta \in \mathbb{C}^*$$

conjugating it with:

$$h : ([x : y : z], [u : v : w]) \rightarrow ([\beta x : y : \alpha z], [\alpha u : \alpha \beta v : \beta w])$$

we get:

$$\begin{aligned} ([x : y : z], [u : v : w]) &\xrightarrow{h^{-1}} \left( \left[ \frac{1}{\beta} x : y : \frac{1}{\alpha} z \right], \left[ \frac{1}{\alpha} u : \frac{1}{\alpha \beta} v : \frac{1}{\beta} w \right] \right) \xrightarrow{g} \\ &\xrightarrow{g} \left( \left[ \frac{1}{\beta} v : w : \frac{1}{\alpha} u \right], [\beta y : z : \alpha x] \right) \xrightarrow{h} ([x : y : z], [\alpha \beta u : \alpha \beta v : \alpha \beta w]) = ([x : y : z], [u : v : w]) \end{aligned}$$

so we may assume that  $\alpha = \beta = 1$  and we conclude that up to conjugation there is only one finite cyclic group  $G \subset \text{Aut}(X_6)$  such that  $(G, X_6)$  is minimal.

Now let's prove that this group is conjugate to a subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  by a birational map  $X_6 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$ . Consider  $\text{proj}_1 : X_6 \rightarrow \mathbb{P}_{\mathbb{C}}^2$  and define  $\tilde{g} = \text{proj}_1 \circ g \circ \text{proj}_1^{-1} : \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$ :

$$\begin{aligned} \tilde{g} : [x : y : z] &\xrightarrow{\text{proj}_1^{-1}} [x : y : z][u : v : w] \text{ where } ux = vy = wz \xrightarrow{g} \\ &\xrightarrow{g} ([v : w : u], [y : z : x]) \text{ where } ux = vy = wz \xrightarrow{\text{proj}_1} \\ &\xrightarrow{\text{proj}_1} [v : w : u] = [xyzv : xyzw : xyzu] = [xz : xy : yz] \text{ since } ux = vy = wz \end{aligned}$$

Note that  $g$  preserves  $|K_{X_6}|$  (since  $g \in \text{Aut}(X_6)$ ) so  $\tilde{g}$  leaves the linear system of cubics containing  $p_1, p_2, p_3$  invariant. Moreover, it fixes exactly one point in  $\mathbb{P}_{\mathbb{C}}^2$  which is  $p_4 = [1 : 1 : 1]$ . The action of it on the tangent space  $\mathbb{P}(T_{p_4}(\mathbb{P}_{\mathbb{C}}^2))$  at  $p_4$  has order 3. To prove it we consider the action of  $\tilde{g}$  on the affine chart  $y \neq 0$ :

$$\tilde{g} : \left[ \frac{x}{y} : 1 : \frac{z}{y} \right] \rightarrow \left[ \frac{z}{y} : 1 : \frac{x}{y} \right]$$

Denoting  $\tilde{x} = \frac{x}{y}$ ,  $\tilde{z} = \frac{z}{y}$  we get:  $\tilde{g} : (\tilde{x}, \tilde{z}) \rightarrow (\tilde{z}, \frac{\tilde{z}}{\tilde{x}})$ . We can write:

$$f : (\tilde{x}, \tilde{z}) \rightarrow (f_1(\tilde{x}, \tilde{z}), f_3(\tilde{x}, \tilde{z}))$$

$$\partial_{\tilde{x}} f = (\partial_{\tilde{x}} f_1(\tilde{x}, \tilde{z}), \partial_{\tilde{x}} f_3(\tilde{x}, \tilde{z})) = \left( 0, -\frac{\tilde{z}}{\tilde{x}^2} \right), \quad \partial_{\tilde{z}} f = (\partial_{\tilde{z}} f_1(\tilde{x}, \tilde{z}), \partial_{\tilde{z}} f_3(\tilde{x}, \tilde{z})) = \left( 1, \frac{1}{\tilde{x}} \right)$$

Now the action on  $T_{\mathcal{P}} = T_{(1,1)}$  is given by:

$$(t_1, t_3) \rightarrow (t_1, t_3) \begin{pmatrix} \partial_{\tilde{x}} f_1 & \partial_{\tilde{x}} f_3 \\ \partial_{\tilde{z}} f_1 & \partial_{\tilde{z}} f_3 \end{pmatrix} \Big|_{(1,1)} = (t_1, t_3) \begin{pmatrix} 0 & -\tilde{z}/\tilde{x}^2 \\ 1 & 1/\tilde{x} \end{pmatrix} \Big|_{(1,1)} = (t_1, t_3) \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = (t_3, t_3 - t_1)$$

Thus, the order of the action in  $\mathbb{P}(T_{\mathcal{P}})$  is 3:

$$(t_1, t_3) \xrightarrow{\tilde{g}} (t_3, t_3 - t_1) \xrightarrow{\tilde{g}} (t_3 - t_1, -t_1) \xrightarrow{\tilde{g}} (-t_1, -t_3) \sim (t_1, t_3) \text{ in } \mathbb{P}(T_{\mathcal{P}})$$

Exactly two points which correspond to the lines  $(x - y) + \omega^k(z - y) = 0$  ( $\omega = e^{2\pi i/3}$  and  $k = 1, 2$ ) are fixed. To prove this we note that the lines in  $T_{\mathcal{P}}$  are given by the equations  $t_1(x - 1) + t_3(z - 1) = 0$ . When we projectivise  $T_{\mathcal{P}}$  we can look at the action of  $\tilde{g}$  on the chart  $\{t_1 = 1\}$  and get:

$$[1 : t_3] \rightarrow [t_2 : t_3 - 1] = [1 : (t_3 - 1)/t_3]$$

So to find the fixed points we solve the quadratic equation  $t_3^2 - t_3 + 1 = 0$  and get the roots  $e^{\pm 2\pi i/3}$ . We see that the birational map  $\tilde{g}$  preserves the linear system of cubics of  $\mathbb{P}_{\mathbb{C}}^2$  passing through  $p_1, p_2, p_3$ , which have a double point at  $p_4$  and are tangent to the line  $(x - y) + \omega(z - y) = 0$  at  $p_4$ . This linear system induces a birational transformation of  $\mathbb{P}_{\mathbb{C}}^2$  conjugating  $\tilde{g}$  to a linear automorphism.  $\square$

### 13.3 The del Pezzo Surface of Degree 5

Suppose  $X_5$  is a del Pezzo surface of degree 5. We see that  $X_5$  is the blow-up  $\pi$  of 4 points  $p_1, p_2, p_3, p_4$  in general position and up to isomorphism there is only one  $X_5$ . Without loss of generality we may assume that  $p_1 = [1 : 0 : 0]$ ,  $p_2 = [0 : 1 : 0]$ ,  $p_3 = [0 : 0 : 1]$ ,  $p_4 = [1 : 1 : 1]$ . We already described all the  $(-1)$ -curves on  $X_5$ :  $E_i = \pi^{-1}(p_i)$  for  $i \in \{1, 2, 3, 4\}$  and strict pull-backs  $l_{ij}$  of the lines passing through each pair  $p_i, p_j$  where  $i, j \in \{1, 2, 3, 4\}$ . Define the sets:

$$F_1 = \{E_1, l_{23}, l_{24}, l_{34}\}, F_2 = \{E_2, l_{13}, l_{14}, l_{34}\}, F_3 = \{E_3, l_{12}, l_{24}, l_{14}\},$$

$$F_4 = \{E_4, l_{23}, l_{12}, l_{13}\}, F_5 = \{E_1, E_2, E_3, E_4\}$$

**Proposition 13.3.1.** *The action of  $\text{Aut}(X_5)$  on  $F_1, F_2, F_3, F_4, F_5$  gives an isomorphism:*

$$\boxed{\rho : \text{Aut}(X_5) \xrightarrow{\cong} S_5}$$

Moreover, the actions of  $S_n, A_m \subset \text{Aut}(X_5)$  on  $X_5$  given by the usual embedding of these groups into  $S_5$  do not have fixed points if and only if  $n = 3, 4, 5$ , resp.  $m = 4, 5$ .

*Proof.* Note that the elements in  $\ker(\rho)$  leave  $E_1, E_2, E_3, E_4$  invariant. These automorphisms correspond to the automorphisms of  $\mathbb{P}_{\mathbb{C}}^2$  which fix  $p_1, p_2, p_3, p_4$  (denote this set by  $S$ ). So  $\rho$  is injective. To prove that it is surjective we need to prove that we obtain the whole group  $S_5$ . First we see that we have  $S_4$  as the image of the lift of group of automorphisms of  $\mathbb{P}_{\mathbb{C}}^2$  that leave the set  $S$  invariant :  $S_4 = \text{Sym}_{F_1, F_2, F_3, F_4}$ , second, we have the lift of the standard quadratic transformation  $[x : y : z] \dashrightarrow [yz : xz : xy]$  (we also know that it belongs to  $\text{Aut}(X_5)$  since it belongs to  $\text{Aut}(X_6)$  and fixes  $p_4$ ). We see that the image of this lift gives a transposition  $(F_4, F_5)$  and we get the whole  $S_5$ . Let's prove the second statement:

- $\mathbf{S}_2 = \{(), (1, 2)\}$   
 $(1, 2)$  corresponds to the map  $[x : y : z] \rightarrow [y : x : z]$  on  $\mathbb{P}_{\mathbb{C}}^2$  so it fixes all points of the form  $[a : a : b]$  for  $a, b \in \mathbb{C}$  and so we see that  $S_2$  fixes some points.
- $\mathbf{A}_3 = \{(), (1, 2, 3), (1, 3, 2)\}$   
 $(1, 2, 3)$  corresponds to the map  $[x : y : z] \rightarrow [z : x : y]$  on  $\mathbb{P}_{\mathbb{C}}^2$  so it fixes three points of the form  $[a^2 : a : 1]$  for  $a \in \mathbb{C}$  and  $a^3 = 1$ . Moreover,  $(1, 3, 2)$  fix the same set of points and so we see that  $A_3$  fixes some points.
- $\mathbf{S}_3 = \langle (1, 2, 3), (1, 2) \rangle$   
this group fixes only the point  $p_4$  and after we blow-up this point the action becomes fixed point-free.
- $\mathbf{A}_4 = \langle (1, 2, 3), (1, 2)(3, 4) \rangle$   
 $(1, 2, 3)$  corresponds to the map  $[x : y : z] \rightarrow [z : x : y]$  on  $\mathbb{P}_{\mathbb{C}}^2$  so it fixes three points of the form  $[a^2 : a : 1]$  for  $a \in \mathbb{C}$  and  $a^3 = 1$ .  $(1, 2)(3, 4)$  corresponds to the map  $[x : y : z] \rightarrow [z - y : z - x : z]$  on  $\mathbb{P}_{\mathbb{C}}^2$  which does not fix points of this type. Thus,  $A_4$  is fixed point-free.
- For  $\mathbf{S}_4, \mathbf{S}_5, \mathbf{A}_5$  the result follows.

□

**Proposition 13.3.2.** *Suppose  $G \subset \text{Aut}(X_5) = S_5$  is cyclic of order 5. Then the pair  $(G, X_5)$  is minimal and  $G$  is conjugate to a subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  by a birational map  $X_5 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$ .*

*Proof.* From the description of  $\text{Aut}(X_5)$  we see that  $(G, X_5)$  is a minimal pair with  $\text{rk}(\text{Pic}(X_5)^G) = 1$ . Note that all subgroups of this type are conjugate in  $\text{Aut}(X_5) = S_5$ , so we can assume that  $G$  is generated by the lift of the birational transformation of  $\mathbb{P}_{\mathbb{C}}^2$ :

$$h : [x : y : z] \dashrightarrow [xy : y(x - z) : x(y - z)]$$

it fixes points  $[\xi + 1 : \xi : 1]$  where  $\xi$  is a root of the polynomial  $x^2 - x - 1 = 0$ . We denote one of these points by  $P$ . The linear system of cubics passing through  $p_1, p_2, p_3, p_4$  which have a double point at  $P$  is invariant by  $h$ . This linear system induces a birational transformation of  $\mathbb{P}_{\mathbb{C}}^2$  conjugating  $h$  to a linear automorphism of  $\mathbb{P}_{\mathbb{C}}^2$ .  $\square$

**Proposition 13.3.3.** *Suppose  $G \subset \text{Aut}(X_5)$ . The pair  $(G, X_5)$  is minimal if and only if the action of  $G$  on the set  $\{F_1, \dots, F_5\}$  is transitive.*

*Proof.* Suppose the action is transitive, then there is an element of order 5 in  $G$  and  $(G, X_5)$  is minimal by the proposition above. Suppose the action is not transitive, then we have that there is an invariant set  $F_i$  for some  $i$  so it can be contracted and we get that  $(G, X_5)$  is not minimal.  $\square$

## 13.4 The del Pezzo Surface of Degree 4

Suppose  $X_4$  is del Pezzo surface of degree 4. We see that  $X_4$  is the blow-up of 5 points  $p_1, \dots, p_5 \in \mathbb{P}_{\mathbb{C}}^2$  in general position. There are infinite (of dimension 1) isomorphic classes of such surfaces. The linear system  $| -K_{X_4} |$  gives an embedding into  $\mathbb{P}_{\mathbb{C}}^4$  and the image is the intersection of two quadrics:

**Proposition 13.4.1.**  *$X_4 \subset \mathbb{P}_{\mathbb{C}}^4$  is a complete intersection of 2 quadrics.*

*Proof.* Consider a smooth curve  $C \in |H|$  (where  $H$  is the hyperplane in  $X_4$ ). Note that we can project  $C$  on a smooth cubic on  $\mathbb{P}_{\mathbb{C}}^2$  so we have  $g(C) = \frac{(3-1)(3-2)}{2} = 1$ . The exact sequence:

$$0 \rightarrow \mathcal{O}_{X_4}(H) \rightarrow \mathcal{O}_{X_4}(2H) \rightarrow \mathcal{O}_C(2H) \rightarrow 0$$

induces a long exact sequence in cohomology from which we get the inequality :

$$h^0(X_4, \mathcal{O}_{X_4}(2)) \leq h^0(X_4, \mathcal{O}_{X_4}(1)) + h^0(C, \mathcal{O}_C(2))$$

Similar the exact sequence:

$$0 \rightarrow \mathcal{O}_{X_4} \rightarrow \mathcal{O}_{X_4}(H) \rightarrow \mathcal{O}_C(H) \rightarrow 0$$

induces a long exact sequence in cohomology from which we get the inequality :

$$h^0(X_4, \mathcal{O}_{X_4}(1)) \leq h^0(X_4, \mathcal{O}_{X_4}) + h^0(C, \mathcal{O}_C(1))$$

We have that  $H \cdot C = H^2 = 4$ . We also have:

$$h^0(C, \mathcal{O}_C(C)) - h^1(C, \mathcal{O}_C(C)) = \chi(C, \mathcal{O}_C(C)) = \deg(C) + g(C) - 1 = 4 + 1 - 1 = 4$$

and

$$h^1(C, \mathcal{O}_C(C)) = h^0(C, \mathcal{O}_C(K_C - C)) \stackrel{(\star)}{=} h^0(C, \mathcal{O}_C(-C)) = 0 \Rightarrow h^0(C, \mathcal{O}_C(C)) = 4$$

( $\star$ ) is true since the canonical divisor on the elliptic curve is trivial. Similarly:

$$h^0(C, \mathcal{O}_C(2C)) - h^1(C, \mathcal{O}_C(2C)) = \chi(C, \mathcal{O}_C(2C)) = \deg(2C) + g(C) - 1 = 8 + 1 - 1 = 8$$

and

$$h^1(C, \mathcal{O}_C(2C)) = h^0(C, \mathcal{O}_C(K_C - 2C)) = h^0(C, \mathcal{O}_C(-2C)) = 0 \Rightarrow h^0(C, \mathcal{O}_C(2C)) = 8$$

so we obtain:

$$\begin{aligned} h^0(X_4, \mathcal{O}_{X_4}(2)) &\leq h^0(X_4, \mathcal{O}_{X_4}(1)) + h^0(C, \mathcal{O}_C(2)) \leq \\ &\leq h^0(X_4, \mathcal{O}_{X_4}) + h^0(C, \mathcal{O}_C(1)) + h^0(C, \mathcal{O}_C(2)) = 1 + 4 + 8 = 13 \Rightarrow h^0(X_4, \mathcal{O}_{X_4}(2)) \leq 13 \end{aligned}$$

We know that  $h^0(\mathbb{P}_{\mathbb{C}}^4, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^4}(2)) = \binom{4+2}{2} = 15$ . From the short exact sequence<sup>1</sup>:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^4}(2) \otimes \mathcal{I}_{X_4} \rightarrow \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^4}(2) \rightarrow \mathcal{O}_{X_4}(2) \rightarrow 0$$

and the induced long exact sequence in cohomology we get that

$$h^0(\mathbb{P}_{\mathbb{C}}^4, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^4}(2) \otimes \mathcal{I}_{X_4}) \geq h^0(\mathbb{P}_{\mathbb{C}}^4, \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^4}(2)) - h^0(X_4, \mathcal{O}_{X_4}(2)) \geq 2$$

so there are at least two irreducible quadrics  $Q_1, Q_2$  containing  $X_4$ , i.e.  $X_4 \subset Q_1 \cap Q_2$ . Note that  $\deg(X_4) = 4$  thus by means of degree of  $X_4 \subset \mathbb{P}_{\mathbb{C}}^4$  is the intersection of these quadrics.  $\square$

Using linear change of coordinates we can write the equation of this surface as:

$$\sum_{i=0}^5 x_i^2 = \sum_{i=0}^5 \lambda_i x_i^2$$

The group  $\text{Aut}(X_4)$  is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \text{Aut}(X_4, \pi, 5)$ , where  $\text{Aut}(X_4, \pi, 5)$  is the lift of the group of automorphisms of  $\mathbb{P}_{\mathbb{C}}^2$  that leave the set  $\{p_1, \dots, p_5\}$  invariant. More precisely:

**Proposition 13.4.2.** *Suppose  $X_4$  is a del Pezzo surface of degree 4,  $E_i = \pi^{-1}(p_i)$ ,  $L$  is a pull-back of a line of  $\mathbb{P}_{\mathbb{C}}^2$ . Then we have:*

1. There are 10 conic bundle structures on  $X_4$  with fibers:  $L - E_i, -K_{X_4} - (L - E_i)$  for  $i \in \{1, \dots, 5\}$
2. The action of  $\text{Aut}(X_4)$  on the five pairs of divisors  $\{L - E_i, -K_{X_4} - (L - E_i)\}$  gives rise to a split exact sequence

$$0 \rightarrow (\mathbb{Z}/2\mathbb{Z})^4 \rightarrow \text{Aut}(X_4) \xrightarrow{\rho} S_5,$$

where  $(\mathbb{Z}/2\mathbb{Z})^4 \cong \{(a_1, \dots, a_5) \in (\mathbb{Z}/2\mathbb{Z})^5 \mid \sum a_i = 0\}$  and the automorphism  $(a_1, \dots, a_5)$  permutes the pair  $\{L - E_i, -K_{X_4} - (L - E_i)\}$  if and only if  $a_i = 1$ .

3. We have

$$\boxed{\text{Aut}(X_4) = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \text{Aut}(X_4, \pi, 5)}$$

where  $\text{Aut}(X_4, \pi, 5)$  acts on  $(\mathbb{Z}/2\mathbb{Z})^4$  by permutation of the  $a_i$ -s.

*Proof.* 1). Suppose  $F = mL - \sum_{i=1}^5 a_i E_i$  is a fiber obtained from some conic bundle structure on  $X_4$ . We have that  $F^2 = 0$  and from genus formula we have  $F \cdot K_{X_4} = -2$ . This gives us equalities:

$$\sum_{i=1}^5 a_i = m^2 \text{ and } \sum_{i=1}^5 a_i = 3m - 2$$

We also have that

$$\left(\sum a_i\right)^2 \leq 5 \sum a_i^2 \Rightarrow (3m - 2)^2 \leq 5m^2 \Rightarrow 4(m^2 - 3m + 1) \leq 0 \Rightarrow m \in \{1, 2\}$$

**m = 1 :**  $\sum_{i=1}^5 a_i = 1$  and  $\sum_{i=1}^5 a_i = 1$  so we obtain  $F = L - E_i$  for  $i \in \{1, \dots, 5\}$ .

**m = 2 :**  $\sum_{i=1}^5 a_i = 4$  and  $\sum_{i=1}^5 a_i = 4$  so we obtain:

$$F = 2L - \sum_{i \neq j} E_i = -\left(-3L + \sum_{k=1}^5 E_k\right) - (-L - E_j) = -K_{X_4} - (L - E_j), \text{ for } j \in \{1, \dots, 5\}$$

<sup>1</sup>here  $\mathcal{I}_{X_4}$  is the ideal sheaf of  $X_4$



2). We see that  $\text{Aut}(X_4)$  acts on the set  $\mathcal{S} = \bigcup_{i=1}^5 \{L - E_i, -K_{X_4} - (L - E_i)\}$  and since  $K_{X_4}$  is fixed there is an induced action on the set of pairs  $\{L - E_i, -K_{X_4} - (L - E_i)\}$ . So we obtain the homomorphism  $\rho : \text{Aut}(X_4) \rightarrow S_5$ . Moreover, by the description of the structure of the action on conic bundle we have embedding  $\ker(\rho) \rightarrow (\mathbb{Z}/2\mathbb{Z})^5$ .

We can assume that  $p_1 = [1 : 0 : 0]$ ,  $p_2 = [0 : 1 : 0]$ ,  $p_3 = [0 : 0 : 1]$ ,  $p_4 = [1 : 1 : 1]$  and  $p_5 = [a : b : c]$ , where  $a, b, c \in \mathbb{C}^*$ . Consider the rational map:

$$\tau : [x : y : z] \dashrightarrow [ayz : bxz : cxy]$$

We can lift it to  $\pi^{-1} \circ \tau \circ \pi \in \text{Aut}(X_4)$  whose induced action on  $\text{Pic}(X_4)$  in the basis  $(E_1, E_2, E_3, E_4, E_5, L)$  is described by:

$$\begin{pmatrix} 0 & -1 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Note that  $\pi^{-1} \circ \tau \circ \pi \in \ker(\rho)$ . For  $i = 1, 2, 3$  it leaves the pair  $(L - E_i, -K_{X_4} - (L - E_i))$  exactly the same, for  $i = 4, 5$  it exchanges the elements of such pair but preserves the pair itself. We see that for  $\rho$  the action on  $\mathcal{S}$  is equivalent to  $(a_1, a_2, a_3, a_4, a_5) = (0, 0, 0, 1, 1) \in (\mathbb{Z}/2\mathbb{Z})^5$ . Permuting points in set  $\{p_1, p_2, p_3, p_4, p_5\}$  we get the elements  $(a_1, a_2, a_3, a_4, a_5) \in (\mathbb{Z}/2\mathbb{Z})^5$  such that exactly two  $a_i$ -s equal 1 and others are 0-s. These elements generate:

$$\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \mathbb{Z}/2\mathbb{Z}, \sum a_i = 0\} \cong (\mathbb{Z}/2\mathbb{Z})^4$$

and we see that  $(\mathbb{Z}/2\mathbb{Z})^4 \subset \ker(\rho)$ . Now we need to prove the equality. Note that it is enough to prove that  $(1, 1, 1, 1, 1) \notin \ker(\rho)$ . Note that  $K_{X_4} = -3L + \sum_{i=1}^5 E_i \Rightarrow L = 1/2(K_{X_4} + \sum_{i=1}^5 (L - E_i))$ . Then  $(1, 1, 1, 1, 1)$  sends  $L$  to the element:

$$\begin{aligned} 1/2 \left( K_{X_4} + \sum_{i=1}^5 (L - E_i) \right) &\xrightarrow{(1,1,1,1,1)} 1/2 \left( K_{X_4} + \sum_{i=1}^5 (-K_{X_4} - L + E_i) \right) = \\ &= 1/2(-2L - 3K_{X_4}) \notin \text{Pic}(X_4) \Rightarrow (1, 1, 1, 1, 1) \notin \ker(\rho) \end{aligned}$$

Thus,  $\ker(\rho) = (\mathbb{Z}/2\mathbb{Z})^4$ .

3). Suppose we have some permutation  $\sigma \in S_5$  lying in the image of  $\rho$ . In this case we can find  $g \in \text{Aut}(X_4)$  such that  $\rho(g) = \sigma$ . Consider the element  $\alpha \in \text{Aut}(\text{Pic}(X_4))$  sending  $E_i \rightarrow E_{\sigma(i)}$  and fixing  $L$ . Note that  $\text{Aut}(X_4)$  can be viewed as a subgroup of  $\text{Aut}(\text{Pic}(X_4))$  and  $g \circ \alpha^{-1} \in \text{Aut}(\text{Pic}(X_4))$  fixes our pairs of conic bundles. We can find  $h \in (\mathbb{Z}/2\mathbb{Z})^4 \subset \text{Aut}(X_4)$  such that  $h \circ g \circ \alpha^{-1}$  either fixes the divisor of every conic bundle from part 1) or permutes the divisors of conic bundles in each pair. We already showed that second case is not possible; thus,  $h \circ g \circ \alpha^{-1}$  fixes  $L - E_1, \dots, L - E_5$  and  $K_{X_4}$  so  $h \circ g \circ \alpha^{-1}$  acts trivially on  $\text{Pic}(X_4)$  and  $\alpha = h \circ g \in \text{Aut}(X_4)$ .

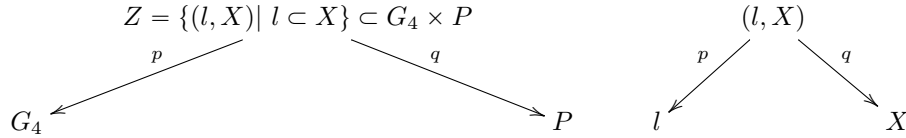
We see that  $\alpha$  is a lift of an automorphism of  $\mathbb{P}_{\mathbb{C}}^2$  the action of which on  $\{p_1, \dots, p_5\}$  coincides with the action of  $\sigma$  on  $\{1, \dots, 5\}$ . Vice versa, we see that any  $\phi \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  leaving the set  $\{p_1, \dots, p_5\}$  invariant can be lifted to  $\pi^{-1} \circ \phi \circ \pi \in \text{Aut}(X_4)$  whose action on the pairs of conic bundles coincides with the action of  $\phi$  on  $\{p_1, \dots, p_5\}$ .  $\square$

### 13.5 The del Pezzo Surface of Degree 3

Suppose  $X_3$  is del Pezzo surface of degree 3. We see that it is the blow-up of 6 points  $p_1, \dots, p_6 \in \mathbb{P}_{\mathbb{C}}^2$  in general position. The linear system  $| -K_{X_3} |$  gives an embedding into  $\mathbb{P}_{\mathbb{C}}^3$  and the image is a smooth cubic surface by theorem 13.1.4. Moreover, this way we can obtain all smooth cubic surfaces. To prove this we first prove two propositions:

**Proposition 13.5.1.** *A smooth cubic surface  $X \subset \mathbb{P}_{\mathbb{C}}^3$  contains a line.*

*Proof.* Consider the space of cubics  $P = |\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^3}(3)|$  in  $\mathbb{P}_{\mathbb{C}}^3$ , the space of lines in  $\mathbb{P}_{\mathbb{C}}^3$  is given by  $G_4 = G(2, 4)$  where  $\dim G_4 = 4$  (note that we can view it as a smooth hypersurface of  $\mathbb{P}_{\mathbb{C}}^5$  since we get the single quadratic equation in  $\mathbb{P}_{\mathbb{C}}^5$  using Plücker coordinates.) Consider  $Z \subset G_4 \times P$  together with projections  $p$  and  $q$ :



Suppose  $X$  contains a line  $l$  defined by  $(Z = 0, T = 0)$  where  $X, Y, Z, T$  are the coordinates on  $\mathbb{P}_{\mathbb{C}}^3$ . Then in the equation which defines  $X$  we should have 0 coefficients before the terms which do not contain  $Z$  or  $T$ , i.e.  $X^3, X^2Y, XY^2, Y^3$ . Thus,  $\dim(\text{fiber of } p) = \dim(P) - 4$  so  $\dim(Z) = \dim(\text{fiber of } p \times G_4) = \dim(P)$ .

Suppose that  $X$  does not contain lines. In this case we see that  $\text{codim}(q(Z)) \geq 1$ . Note that  $q^{-1}(X)$  is either empty or has positive dimension. We see that  $q$  is surjective. Thus, a smooth cubic surface  $X \subset \mathbb{P}_{\mathbb{C}}^3$  contains a line.  $\square$

**Proposition 13.5.2.** *Suppose we have a smooth cubic surface  $S \subset \mathbb{P}_{\mathbb{C}}^3$  and a line  $l \subset S$ . Then we have 5 disjoint pairs of concurrent lines intersecting  $l$ . Hence, we can find 2 disjoint lines on  $S$ .*

*Proof.* Suppose  $\{P_{\lambda}\}_{\lambda \in \mathbb{P}_{\mathbb{C}}^1}$  is the pencil of planes in  $\mathbb{P}_{\mathbb{C}}^3$  such that for all  $\lambda \in \mathbb{P}_{\mathbb{C}}^1$  we have  $l \in P_{\lambda}$ . Then we have that  $X \cap P_{\lambda} = l \cup C_{\lambda}$ , where  $C_{\lambda}$  is a conic.

We want to explore the cases when  $C_{\lambda}$  contains a line (which means that actually  $C_{\lambda}$  is the union of two lines). Now suppose  $C_{\lambda}$  is contained in a plane  $P = 0$  (linear form),  $C_{\lambda}$  is a double line defined by the equation  $L_2 = 0$  (linear form),  $l$  is defined by the equation  $L_1 = 0$  (linear form). In this case  $S$  is defined by the equation:

$$PQ + L_1L_2^2 = 0, \text{ where } Q \text{ is a quadratic form}$$

Note that in this case the following system has solution:

$$\begin{cases} PQ + L_1L_2^2 = 0 \\ P'Q + PQ' + L_1'L_2^2 + 2L_2L_2'L_1 = 0 \end{cases} \Rightarrow \begin{cases} Q = 0 \\ L_2 = 0 \\ P = 0 \end{cases} \text{ has a solution}$$

Which gives us singular points-contradiction.

Suppose  $C_{\lambda}$  is the union of two concurrent lines.  $(X, Y, Z, T)$  are the coordinates on  $\mathbb{P}_{\mathbb{C}}^3$  and  $l$  is given by  $Z = T = 0$ . Then  $S$  is given by:

$$S : A_1(Z, T)X^2 + 2B_1(Z, T)XY + C_1(Z, T)Y^2 + 2D_2(Z, T)X + 2E_2(Z, T)Y + F_3(Z, T) = 0$$

where  $A_i(Z, T), B_i(Z, T), C_i(Z, T), D_i(Z, T), E_i(Z, T), F_i(Z, T)$  are homogeneous polynomial in  $Z, T$  of degree for some  $i$ . (Note that we do not have terms  $X^3, X^2Y, XY^2, Y^3$  since they should vanish identically when we plug in  $Z = T = 0$ .) Suppose  $Z = \lambda T$  then  $S$  is given by:

$$S : A_1(\lambda, 1)TX^2 + 2B_1(\lambda, 1)TXY + C_1(\lambda, 1)TY^2 + 2D_2(\lambda, 1)T^2X + 2E_2(\lambda, 1)T^2Y + F_3(\lambda, 1)T^3 = 0$$

Dividing by  $T$  we get the equation of  $C_\lambda$ :

$$C_\lambda : A_1(\lambda, 1)X^2 + 2B_1(\lambda, 1)XY + C_1(\lambda, 1)Y^2 + 2D_2(\lambda, 1)TX + 2E_2(\lambda, 1)TY + F_3(\lambda, 1)T^2 = 0$$

We want  $C_t$  to be singular such that it is not a double line. Thus, we want  $\Delta(Z, T)$  the determinant of the equation above to be 0 and not to have a double root:

$$\Delta(Z, T) = \begin{vmatrix} A_1(\lambda, 1) & B_1(\lambda, 1) & D_2(\lambda, 1)T \\ B_1(\lambda, 1) & C_1(\lambda, 1) & E_2(\lambda, 1)T \\ D_2(\lambda, 1)T & E_2(\lambda, 1)T & F_3(\lambda, 1)T^2 \end{vmatrix} \text{ a homogeneous polynomial of degree 5 in } Z, T$$

Changing the coordinates we can suppose that  $(Z = 0)$  is a root with corresponding conic  $C_0$ ,  $\alpha \in C_0$  is a singular point on  $C_0$  (i.e. the intersection of lines). We need to explore two cases:  $\alpha \in l$  and  $\alpha \notin l$ :

**1). triangle** ( $\alpha \notin l$ ). By changing coordinates we assume that  $C_0 : XY = 0 \Rightarrow$  we see that all coefficients of  $C_\lambda$  except  $B_1(\lambda, 1)$  are divisible by  $\lambda$ . Thus, expanding  $\Delta(Z, T)$  we see that all the terms except  $(B_1(\lambda, 1))^2 F_3(\lambda, 1)T^2$  are divisible by  $\lambda^2$ . We know that  $\lambda \nmid B_1(\lambda, 1)$ . Let's prove that  $\lambda^2 \nmid F_3(\lambda, 1)$ . Suppose not, i.e.  $\lambda^2 \mid F_3(\lambda, 1)$ . Then the equation of  $S$  is given by:

$$S : A_1(Z, T)X^2 + 2B_1(Z, T)XY + C_1(Z, T)Y^2 + 2D_1(Z, T)ZX + 2E_1(Z, T)ZY + F_1(Z, T)Z^2 = 0$$

where  $A_1(Z, T)$ ,  $B_1(Z, T)$ ,  $C_1(Z, T)$ ,  $D_1(Z, T)$ ,  $E_1(Z, T)$ ,  $F_1(Z, T)$  are homogeneous polynomial in  $Z, T$  of degree 1. Substitute  $T = 1$ :

$$S : A_1(Z, 1)X^2 + 2B_1(Z, 1)XY + C_1(Z, 1)Y^2 + 2D_1(Z, 1)ZX + 2E_1(Z, 1)ZY + F_1(Z, 1)Z^2 = 0$$

So we see that  $S$  is singular at  $[X : Y : Z : T] = [0 : 0 : 0 : 1]$  - contradiction with  $S$  being smooth. Thus,  $\lambda$  is a simple root.

**2). asterisk** ( $\alpha \notin l$ ). As in previous case by changing coordinates we can assume that  $\lambda = 0$  and  $C_0$  is given by  $X^2 - T^2 = 0$ . We see that  $A_1(\lambda, 1) = -F_3(\lambda, 1)$  and all coefficients of  $C_\lambda$  except  $A_1(\lambda, 1), F_3(\lambda, 1)$  are divisible by  $\lambda$ . Thus, expanding  $\Delta(Z, T)$  we see that all the terms except  $A_1(\lambda, 1)C_1(\lambda, 1)F_3(\lambda, 1)T^2$  are divisible by  $\lambda^2$ .  $A_1(\lambda, 1)C_1(\lambda, 1)F_3(\lambda, 1)T^2$  is not divisible by  $\lambda^2$  since  $A_1(\lambda, 1), F_3(\lambda, 1)$  are not divisible by  $\lambda$  and  $C_1(\lambda, 1)$  is a linear polynomial. Thus,  $\lambda$  is a simple root.

Thus, there are exactly suitable 5 distinct singular conics (the ones meeting  $l$ ) where each of them is the unions of 2 lines. Finally, we can take 2 conics defined above  $C_0 = l_0 \cup l'_0$  and  $C_1 = l_1 \cup l'_1$ . Note that from the construction we have that  $l, l_0, l_1$  are not coplanar. Thus,  $l_0 \cap l_1 = \emptyset$ .  $\square$

**Theorem 13.5.3.** *Suppose  $X \subset \mathbb{P}_\mathbb{C}^3$  is smooth cubic surface. Then  $X \cong X_3$ .*

*Proof.* From the proposition above we know that there are some disjoint lines  $l, l' \subset X$ . Now define rational maps  $\phi$  and  $\psi$  that are the inverses of each other:

$$\phi : l \times l' \dashrightarrow X, \text{ where } \langle x, x' \rangle \rightarrow x'' = \langle x, x' \rangle \cap X$$

where  $(x, x') \in l \times l'$  and a line  $\langle x, x' \rangle$  meets  $X$  at  $x''$ .

$$\psi : X \dashrightarrow l \times l', \text{ where } y \rightarrow \langle x, x' \rangle = \langle l \cap \langle y, l' \rangle, l' \cap \langle y, l \rangle \rangle$$

it is defined for  $y \in X \setminus (l \times l')$ . Note that we can view  $\psi$  as a morphism: for  $y \in l$  or  $y \in l'$  we can just take tangent plane to  $X$  at  $y$ . In this case  $\psi$  is a birational morphism so thus it can be seen as the composition of blow-ups. The curve is contracted by  $\psi$  if it intersects both  $l$  and  $l'$ .

By the proposition 13.5.2 we know that there are 5 pairs of lines which meet  $l$ , i.e.  $d_i, d'_i$  for  $i \in \{1, \dots, 5\}$  and  $d_i, d'_i$  lie in the same plane  $P_i$ . Note that  $P_i \cap l'$  is a point which belongs exactly either to  $d_i$  or

$d'_i \deg(X \cap P_1) = 3$  so we have  $X \cap P_i = l \cup d_i \cup d'_i$ , moreover  $l \cap l' = \emptyset$  thus  $l'$  intersect either  $d_i$  or  $d'_i$ ; it does not intersect both of them since  $d_i, d'_i, l'$  are not coplaner). We obtain exactly 5 lines which are contracted by  $\psi$ , i.e. we can view  $X$  as the blow-up of  $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$  at 5 points of the form  $(x_i, x'_i)$  and  $x_i \neq x_j, x'_i \neq x'_j$  for  $i \neq j$  which is isomorphic to the blow-up of  $\mathbb{P}^2_{\mathbb{C}}$  at 6 points. Moreover,  $j : X \rightarrow \mathbb{P}^3_{\mathbb{C}}$  is the anticanonical embedding thus  $X \cong X_3$ .  $\square$

In general a cubic  $X_3$  has a trivial group of automorphisms ([MaKo88]). However, there are many of examples of cubic surfaces  $X$  for which  $\text{Aut}(X) \neq \{1\}$ . The classification of group of automorphisms of cubic surfaces can be found in [Dol10] § 9.5.

**Example 13.5.4.** Consider the Fermat cubic surface, which is a Del Pezzo surface of degree 3

$$S_F = \{[w : x : y : z] \in \mathbb{P}^3_{\mathbb{C}} \mid w^3 + x^3 + y^3 + z^3 = 0\}$$

$S_F$  admits non-trivial automorphisms. For example:

$$\rho_1 : [w : x : y : z] \rightarrow [w : e^{2i\pi/3}y : z : x]$$

$$\rho_2 : [w : x : y : z] \rightarrow [w : e^{4i\pi/3}y : z : x]$$

## 13.6 The del Pezzo Surface of Degree 1 and 2

Recall that in case  $d = 1, 2$  we have that  $|-K_{X_d}|$  is not very ample so it does not give an embedding.

### 13.6.1 The del Pezzo surface of degree 2:

Suppose  $X_2$  is a del Pezzo surface of degree 2. We see that it is the blow-up of 7 points  $p_1, \dots, p_7 \in \mathbb{P}^2_{\mathbb{C}}$  in general position. The system  $|-K_{X_2}|$  gives a morphism  $j_2 : X_2 \rightarrow \mathbb{P}^2_{\mathbb{C}}$ :

$$\begin{array}{ccc} \tilde{\mathbb{P}}^2_{\mathbb{C},7} = X_2 & & \\ \text{blow-up at } p_1, \dots, p_7 \downarrow & \searrow j_2 & \\ \mathbb{P}^2_{\mathbb{C}} & \dashrightarrow & \mathbb{P}^2_{\mathbb{C}} \end{array}$$

is surjective but not an isomorphism. Since we have  $K^2_{X_2} = (\pi^*K_{\mathbb{P}^2_{\mathbb{C}}} + E_1 + \dots + E_7)^2 = 9 - 7 = 2$  then it is a double covering (since  $-K_{X_2} = \pi^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$  thus  $2 = K^2_{X_2} = \pi^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1) \cdot \pi^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1) = \deg(j_2)\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1) \cdot \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1) \Rightarrow \deg(j_2) = 2$ ). Note that for a general line on  $\mathbb{P}^2_{\mathbb{C}}$  we have that its preimage corresponds to a smooth cubic through the 7 points so it has genus  $\frac{(3-1)(3-2)}{2} = 1$ . By Riemann-Hurwitz, the restriction of the double covering to the curve has four ramification points. This shows that the ramification curve of  $j_2$  is a smooth (since  $X_2$  is smooth) quartic. So one can see  $X_2$  as a surface which is given by the equation

$$w^2 = F_4(x, y, z), \text{ where } F_4 \text{ is a form of degree 4}$$

in a weighted projective space  $\mathbb{P}(2, 1, 1, 1) = \{(w, x, y, z) \in \mathbb{C}^4 \setminus \{(0, 0, 0, 0)\} / \sim \text{ where } (w, x, y, z) \sim (\lambda^2 w, \lambda x, \lambda y, \lambda z) \text{ for } \lambda \in \mathbb{C}^*\}$ . Moreover, all smooth quartic curves are obtained this way ([Kol96]). Now the 56  $(-1)$ -curves of  $X_2$  correspond to the 28 bitangents of the smooth quartic. The involution  $\iota_G$  of  $X_2$  obtained from the double covering is called *Geiser involution* and there exists an exact sequence:

$$1 \rightarrow \langle \iota_G \rangle \rightarrow \text{Aut}(X_2) \rightarrow \text{Aut}(\Gamma) \rightarrow 1$$

where  $\text{Aut}(\Gamma)$  is the group of automorphisms of the curves. The structure of the group of automorphisms of del Pezzo surfaces of degree 2 can be deduced from [Do-Iz09], §6.6.

### 13.6.2 The del Pezzo surface of degree 1:

Suppose  $X_1$  is a del Pezzo surface of degree 1. We see that it is the blow-up of 8 points  $p_1, \dots, p_8 \in \mathbb{P}_{\mathbb{C}}^2$  in general position. The system  $| -K_{X_1} |$  gives a rational map  $j_1 : X_1 \dashrightarrow \mathbb{P}_{\mathbb{C}}^1$ :

$$\begin{array}{ccc}
 P \in \tilde{\mathbb{P}}_{\mathbb{C},8}^2 = X_1 & & \\
 \text{blow-up at } p_1, \dots, p_8 \downarrow & \searrow j_1 & \\
 \mathbb{P}_{\mathbb{C}}^2 & \dashrightarrow & \mathbb{P}_{\mathbb{C}}^1
 \end{array}$$

The fibers of  $j_1$  correspond to the cubics on  $\mathbb{P}_{\mathbb{C}}^2$  containing the points  $p_1, \dots, p_8$ . Two cubics intersect in 9 points thus this pencil has the ninth point  $P$ . This point is fixed by any automorphism, and it is a unique base-point of  $j_1$ .

The linear system  $| -2K_{X_1} |$  induces a morphism  $X_1 \rightarrow \mathbb{P}_{\mathbb{C}}^3$ , which lies in a quadric cone  $Q$ :

$$\begin{array}{ccc}
 P \in \tilde{\mathbb{P}}_{\mathbb{C},8}^2 = X_1 & & \\
 \text{blow-up at } p_1, \dots, p_8 \downarrow & \searrow | -2K_{X_1} | & \\
 \mathbb{P}_{\mathbb{C}}^2 & \dashrightarrow & \mathbb{P}_{\mathbb{C}}^3
 \end{array}$$

The restriction gives a double covering of  $Q$ , ramified over the vertex  $v$  of  $Q$  and a smooth curve  $C$  of genus 4 such that  $C$  is the intersection of  $Q$  with a cubic surface ([Ba-Be00], [dFe04], [Do-Iz09]). Note that  $Q$  is isomorphic to  $\mathbb{P}(1, 1, 2)^2$  and  $C$  has the equation of degree 6 there. Using the change of coordinates  $X_1$  is given in  $\mathbb{P}(3, 1, 1, 2)$  by ([Kol96]):

$$w^2 = z^3 + F_4(x, y)z + F_6(x, y), \text{ where } F_4, F_6 \text{ are forms of degrees 4, 6 respectively}$$

the vertex of the quadric is  $v = [1 : 0 : 0 : 1] = [(-1)^3 \cdot 1 : (-1) \cdot 0 : (-1) \cdot 0 : (-1)^2 \cdot 1] = [-1 : 0 : 0 : 1]$ .

The involution associated to the double covering given by:  $\sigma([w : x : y : z]) = [w : x : y : z]$  is called *the Bertini involution*.

A family of del Pezzo surfaces of degree 1 has a complicated group of automorphisms described in [Do-Iz09], §6.7.

<sup>2</sup>Consider  $\pi : \mathbb{A}_{x,y,z} \setminus \{0\} \rightarrow \mathbb{P}_{[X:Y:Z:T]}^3$ , where  $(x, y, z) \mapsto [x^2 : xy : y^2 : z]$ . Note that we have  $\pi((x, y, z)) = \pi((x', y', z')) \Leftrightarrow \exists \lambda \in \mathbb{C}^*$  such that  $x' = \lambda x, y' = \lambda y, z' = \lambda^2 z$ . Thus, the image of  $\mathbb{P}(1, 1, 2)$  is given by the quadratic cone  $XY = Z^2$  in  $\mathbb{P}_{\mathbb{C}}^3$ .

# Chapter 14

## Examples

In this chapter we will take a closer look at some examples of the conjugacy classes of the finite subgroups in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .

### 14.1 Conjugacy Invariants

Suppose  $G$  is a finite abelian subgroup of  $\text{Aut}(X)$  where  $X$  is the surface. Before studying the examples it is important to mention some birational invariants which help us to prove that subgroups of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  have different conjugacy classes.

#### 14.1.1 Invariant Pencil of Rational Curves

*The existence of a pencil of rational curves of the surface  $X$  invariant by the group  $G$  is the conjugacy invariant.* The fact that it is invariant is equivalent to the existence of a birational map  $\phi : X \dashrightarrow X'$  where  $(X', \pi)$  is a conic bundle such that  $\phi G \phi^{-1}$  acts biregularly on the conic bundle. Note that we obtained such  $\phi$  in the proof of the Proposition 12.3.3.

#### 14.1.2 Pairs of Invariant Pencils

*The existence of two  $G$ -invariant pencils of rational curves on the surface  $X$ , with some given free intersection is the conjugacy invariant.* This is equivalent to the existence of a  $G$ -equivariant rational map  $\phi : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  which such that  $G$  acts on the target as a subgroup of  $\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})$ .

#### 14.1.3 The Existence of Fixed Points

If there exist some points on the surface  $X$  fixed by  $G \subset \text{Aut}(X)$  is a finite abelian group.  $\phi : X \dashrightarrow X'$  is a birational map of surfaces,  $G' \subset \text{Aut}(X')$  a finite abelian group such that  $\phi$  conjugate  $G$  and  $G'$ . Then, by Proposition A.2 [Ko-Sz99], there is at least one point on  $X'$  fixed by  $G'$ .

## 14.2 Finite Abelian subgroups of $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$

In this section we will classify the conjugacy classes of Finite Abelian subgroups of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .

**Definition 14.2.1.** We call a automorphism  $[x : y : z] \rightarrow [ax : by : cz]$  of  $\mathbb{P}_{\mathbb{C}}^2$  **the diagonal automorphism** and denote it  $[a : b : c]$ . We also denote  $\zeta_m = e^{2\pi i/m}$ .

**Proposition 14.2.2.** *Every finite Abelian subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3, \mathbb{C})$  is conjugate, in the Cremona group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ , to one and only one of the following:*

1. a diagonal group, isomorphic to  $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ , where  $n|m$ , generated by

$$[1 : \zeta_n : 1] \text{ and } [\zeta_m : 1 : 1]$$

When  $n = 1$  we obtain the cyclic groups

2.  $V_9 \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ , generated by

$$[1 : \zeta_3 : \zeta_3^2] \text{ and } [x : y : z] \rightarrow [y : z : x]$$

Thus, two isomorphic finite Abelian subgroups different from  $V_9$  of  $\text{PGL}(3, \mathbb{C})$  are conjugate in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .

*Proof.* From the Theorems 4.7, 4.8 of [Do-Iz09] we deduce that every finite Abelian subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3, \mathbb{C})$  is conjugated in  $\text{PGL}(3, \mathbb{C})$  to

- A cyclic group which is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ , generated by

$$[1 : \zeta_n^a : \zeta_n^b] \text{ where } \gcd(n, a, b) = 1$$

- A diagonal group, isomorphic to  $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ , such that  $m|n$  generated by

$$[1 : 1 : \zeta_m] \text{ and } [1 : \zeta_n^a : \zeta_n^b] \text{ where } \gcd(n, a, b) = 1$$

- The special group  $V_9$ , isomorphic to  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ , generated by

$$[1 : \zeta_3 : \zeta_3^2] \text{ and } [x : y : z] \rightarrow [y : z : x]$$

$V_9$  is not diagonalizable in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  since it does not have fixed points, but a diagonal group fixes  $[1 : 0 : 0]$ ,  $[0 : 1 : 0]$ ,  $[0 : 0 : 1]$  (by 14.1.3).

Linear automorphisms of  $\mathbb{P}^2$  of the same finite order are conjugate in the Cremona group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  ([Be-B104], Proposition 2.1.). Thus cyclic groups of  $\text{PGL}(3, \mathbb{C})$  of order  $n$  are conjugate in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  to a cyclic group isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ , generated by  $[\zeta_n : 1 : 1]$ .

Suppose  $G$  is a finite abelian subgroup of  $\text{PGL}(3, \mathbb{C})$  such that it is not cyclic and not conjugate to  $V_9$ . In this case it is diagonal and is isomorphic to  $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ , such that  $m|n$ . The group  $\text{GL}(2, \mathbb{Z}) \subset \text{Cr}(\mathbb{P}_{\mathbb{C}}^2)$  of birational maps of the form

$$g : (x, y) \mapsto (x^a y^b, x^c y^d)$$

normalizes the torus  $\mathcal{T}$  of diagonal automorphisms ( $gTg^{-1} = T$  for  $T \in \mathcal{T}$ ). We can conjugate the group  $G$  by an element of  $\text{GL}(2, \mathbb{Z})$  [Be-B104], Proposition 2.1.) in such a way that it contains  $[\zeta_n : 1 : 1]$ . Since  $g$  normalizes the torus,  $G$  stays diagonal and contains a subgroup isomorphic to  $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ . This subgroup contains all elements of  $\mathcal{T}$  of order  $m$  including  $[1 : \zeta_m : 1]$ , so we get a diagonal group, isomorphic to  $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ , where  $m|n$ , generated by  $[1 : \zeta_m : 1]$  and  $[\zeta_n : 1 : 1]$ .  $\square$

### 14.3 Finite Abelian subgroups of $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$

In this section we will classify the conjugacy classes of finite abelian subgroups of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . First consider an example of a subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) = (\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})) \rtimes \mathbb{Z}/2\mathbb{Z}$ :

**Example 14.3.1.** Consider a subgroup  $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$  where the action of  $(\mathbb{C}^*)^2$  is given by:

$$(x, y) \rightarrow (\alpha x, \beta y)$$

and the action of  $\mathbb{Z}/2\mathbb{Z}$  is given by

$$(x, y) \rightarrow (y, x)$$

Consider a birational map:

$$h : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2, \text{ where } (x, y) \dashrightarrow [x : y : 1]$$

$h$  conjugates  $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z} \subset \text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  to the subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  generated by:

$$[x : y : z] \rightarrow [\alpha x : \beta y : z] \text{ and } [x : y : z] \rightarrow [y : x : z], \text{ where } \alpha, \beta \in \mathbb{C}^*$$

**Proposition 14.3.2.** Every finite Abelian subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  is conjugate, in the Cremona group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ , to one of the following groups:

- (i)  $G \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ ,  $(x, y) \rightarrow (\zeta_n x, y)$  and  $(x, y) \rightarrow (x, \zeta_m y)$
- (ii)  $G \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/n\mathbb{Z}$ ,  $(x, y) \rightarrow (\pm x^{\pm 1}, y)$  and  $(x, y) \rightarrow (x, \zeta_n y)$
- (iii)  $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$ ,  $(x, y) \rightarrow (x^{-1}, y)$  and  $(x, y) \rightarrow (-x, \zeta_{2n} y)$
- (iv)  $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ ,  $(x, y) \rightarrow (\pm x^{\pm 1}, y)$
- (v)  $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ ,  $(x, y) \rightarrow (-x, -y)$  and  $(x, y) \rightarrow (x^{-1}, y)$
- (vi)  $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ ,  $(x, y) \rightarrow (-x, -y)$  and  $(x, y) \rightarrow (x^{-1}, y^{-1})$
- (vii)  $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ ,  $(x, y) \rightarrow (\pm x, \pm y)$  and  $(x, y) \rightarrow (x^{-1}, y)$
- (viii)  $G \cong (\mathbb{Z}/2\mathbb{Z})^4$ ,  $(x, y) \rightarrow (\pm x^{\pm 1}, \pm y^{\pm 1})$
- (ix)  $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ ,  $(x, y) \rightarrow (x^{-1}, y^{-1})$  and  $(x, y) \rightarrow (-y, x)$
- (x)  $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ ,  $(x, y) \rightarrow (-y, -x)$ ,  $(x, y) \rightarrow (x^{-1}, y^{-1})$  and  $(x, y) \rightarrow (y, x)$

Where  $m, n \in \mathbb{Z}_{>0}$  and  $n|m$ . Moreover, the groups of type (1) are conjugate to subgroups of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$ .

*Proof.* 1. Suppose  $G \subset (\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$  then by the example 14.3.1 we can conjugate  $G$  with the subgroup in  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . We see that  $G$  is conjugate to the group in case (i).

2. Suppose  $G \subset \text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})$  and  $\pi_i : \text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C}) \rightarrow \text{PGL}(2, \mathbb{C})$  are the projections on  $i$ -th factor. Note that  $\pi_i(G)$  is a finite abelian subgroup for  $i = 1, 2$ . Let's explore the cases of the structure of  $\pi_i(G)$ :

- $\pi_i(G)$  is cyclic for  $i = 1, 2$ : in this case we have that  $G$  is conjugate to the subgroup of  $(\mathbb{C}^*)^2$  which consists of automorphisms of the form:

$$(x, y) \rightarrow (\alpha x, \beta y), \text{ where } \alpha, \beta \in \mathbb{C}^*$$

Thus, we are in the case (i).

- **exactly one of  $\pi_i(G)$  is cyclic:** after conjugation in  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  we can assume that  $\pi_1(G)$  is a non-cyclic group which consists of the maps of the form  $x \dashrightarrow \pm x^{\pm 1}$  and  $\pi_2(G)$  is a cyclic group generated by  $y \rightarrow \zeta_m y$ .<sup>1</sup> Consider the exact sequence:

$$1 \rightarrow G \cap \ker(\pi_2) \rightarrow G \rightarrow \pi_2(G) \rightarrow 1$$

<sup>1</sup>finite subgroups of  $\text{PGL}(2, \mathbb{C})$  are up to isomorphism exactly the following: cyclic groups  $C_n$ , dihedral groups  $D_n$  (with  $2n$  elements), the tetrahedral group  $A_4$ , the icosahedral group  $A_5$ , the octahedral group  $S_4$ . Thus, the only abelian groups are cyclic groups  $C_n$  and  $D_1 \cong \mathbb{Z}/2\mathbb{Z}$  (cyclic),  $D_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$ .



We can view  $G$  as a subgroup of  $\pi_1(G) \times \pi_2(G) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/m\mathbb{Z}$ . By assumption  $\pi_2(G)$  is cyclic, but  $G$  is not cyclic, thus  $G \cap \ker(\pi_2) \not\cong \{1\}$ . So we have that  $G \cap \ker(\pi_2)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  or  $(\mathbb{Z}/2\mathbb{Z})^2$ .

If  $G \cap \ker(\pi_2) \cong (\mathbb{Z}/2\mathbb{Z})^2$  we have that  $G \cong \pi_1(G) \times \pi_2(G) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/m\mathbb{Z}$  and we get (ii) for  $m > 1$  and (iv) for  $m = 1$ .

In case  $G \cap \ker(\pi_2) \cong \mathbb{Z}/2\mathbb{Z}$  we have that  $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$  or  $G \cong \mathbb{Z}/2m\mathbb{Z}$ . By assumption  $G$  is not cyclic thus  $m$  is even. Using conjugation we can assume that  $G \cap \ker(\pi_2)$  is generated by  $x \rightarrow x^{-1}$  and  $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$  and we get (iii) with  $n = m/2$  for  $m > 2$  and (v) for  $m = 2$ .

- $\pi_1(G) \cong \pi_2(G) \cong (\mathbb{Z}/2\mathbb{Z})^2$ : after conjugation in  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  we can assume that  $\pi_1(G)$  consists of the maps of the form  $x \mapsto \pm x^{\pm 1}$  and  $\pi_2(G)$  consists of the maps of the form  $y \mapsto \pm y^{\pm 1}$ . Consider the exact sequence:

$$1 \rightarrow G \cap \ker(\pi_2) \rightarrow G \rightarrow \pi_2(G) \rightarrow 1$$

We can view  $G$  as a subgroup of  $\pi_1(G) \times \pi_2(G) \cong (\mathbb{Z}/2\mathbb{Z})^4$ . Then  $G$  is one of the following:

$$\begin{cases} G \cap \ker(\pi_2) \cong \{1\} \Rightarrow \text{case (vi)} \\ G \cap \ker(\pi_2) \cong \mathbb{Z}/2\mathbb{Z} \Rightarrow \text{case (vii)} \\ G \cap \ker(\pi_2) \cong (\mathbb{Z}/2\mathbb{Z})^2 \Rightarrow \text{case (viii)} \end{cases}$$

3. Suppose  $G \not\subset \text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})$  and  $\phi \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) \setminus \text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})$ . Then we can write  $\phi$  as:

$$\phi : (x, y) \rightarrow (\alpha(y), \beta(x)), \text{ where } \alpha, \beta \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^1)$$

Now we conjugate  $\phi$  by  $\psi : (x, y) \rightarrow (\beta(x), y)$ :

$$\psi \circ \phi \circ \psi^{-1} : (x, y) \xrightarrow{\psi^{-1}} (\beta^{-1}(x), y) \xrightarrow{\phi} (\alpha(y), x) \xrightarrow{\psi} (\beta(\alpha(y)), x)$$

Thus, up to conjugation  $\phi$  is of the form:

$$(x, y) \rightarrow (\alpha(y), x), \text{ where } \alpha \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^1)$$

Note that if  $\phi$  is of finite order then  $\alpha$  is of finite order, so up to conjugation  $\alpha$  is of the form  $y \rightarrow \lambda y$  ( $\lambda \in \mathbb{C}^*$ ) and  $\phi$  is of the form  $(x, y) \rightarrow (\lambda y, x)$ .

We see that if  $G \subset \text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$ ,  $G \not\subset \text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C})$  is a cyclic group, then we get case (i). If  $G$  is not a cyclic group, then it is generated by the group

$$H = G \cap (\text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C}))$$

together with the element of finite order (thus diagonalizable) which is conjugate to  $\phi_\lambda : (x, y) \rightarrow (\lambda y, x)$  ( $\lambda \in \mathbb{C}^*$ ).  $G$  is an abelian group so for any element  $h \in H$  given by  $h : (x, y) \rightarrow (h_1(x), h_2(y))$  we get:

$$\phi_\lambda \circ h : (x, y) \xrightarrow{h} (h_1(x), h_2(y)) \xrightarrow{\phi_\lambda} (\lambda h_2(y), h_1(x))$$

$$h \circ \phi_\lambda : (x, y) \xrightarrow{\phi_\lambda} (\lambda y, x) \xrightarrow{h} (h_1(\lambda y), h_2(x))$$

$$\phi_\lambda \circ h = h \circ \phi_\lambda \Rightarrow (\lambda h_2(y), h_1(x)) = (h_1(\lambda y), h_2(x)) \Rightarrow h_1(x) = h_2(x) \text{ and } \lambda h_1(y) = h_1(\lambda y)$$

Thus, any element of  $H$  has the form

$$(x, y) \rightarrow (\alpha(x), \alpha(y)), \text{ where } \alpha(\lambda x) = \lambda \alpha(x)$$

Let's look at the elements of  $\text{PGL}(2, \mathbb{C})$  which commute with  $\alpha$ . The possible cases depend on  $\lambda$ :

- $\lambda = 1$  :

We suppose that  $H$  is either diagonal or equal to the group generated by  $(x, y) \rightarrow (-x, -y)$  and  $(x, y) \rightarrow (x^{-1}, y^{-1})$  after conjugating it by some element  $(x, y) \rightarrow (\gamma(x), \gamma(y))$ . If  $H$  is diagonal then  $G \subset (\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$  and we are in case (i). The second option gives case (x).

- $\lambda = -1$  :

$H$  contains the square of  $(x, y) \rightarrow (-y, x)$ , which is  $\alpha : (x, y) \rightarrow (-x, -y)$ . The group  $H$  is either cyclic or generated by  $(x, y) \rightarrow (-x, -y)$  and  $(x, y) \rightarrow (x^{-1}, y^{-1})$ . If  $H$  is cyclic, it is diagonal, so  $G \subset (\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$  and we are in case (i). The second option gives (ix).

- $\lambda \neq \pm 1$  :

The group  $H$  is diagonal and then  $G$  is contained in  $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ .

□

Let's now prove that some of these groups are conjugate in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ :

**Proposition 14.3.3.** *Every finite Abelian subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  is conjugate, in the Cremona group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ , to one and only one of the following groups:*

- |                                                                                                     |                                                                                                                                                                                                                                     |
|-----------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) $G \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ ,                                | $(x, y) \rightarrow (\zeta_n x, y)$ and $(x, y) \rightarrow (x, \zeta_m y)$                                                                                                                                                         |
| (2) $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$ ,                               | $(x, y) \rightarrow (x^{-1}, y)$ and $(x, y) \rightarrow (-x, \zeta_{2n} y)$                                                                                                                                                        |
| (3) $G \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/2n\mathbb{Z}$ ,                           | $(x, y) \rightarrow (\pm x^{\pm 1}, y)$ and $(x, y) \rightarrow (x, \zeta_{2n} y)$                                                                                                                                                  |
| (4) $G \cong (\mathbb{Z}/2\mathbb{Z})^2$                                                            | $(x, y) \rightarrow (\pm x^{\pm 1}, y)$                                                                                                                                                                                             |
| (5) $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ ,                                                          | $(x, y) \rightarrow (\pm x, \pm y)$ and $(x, y) \rightarrow (x^{-1}, y)$                                                                                                                                                            |
| (6) $G \cong (\mathbb{Z}/2\mathbb{Z})^4$ ,                                                          | $(x, y) \rightarrow (\pm x^{\pm 1}, \pm y^{\pm 1})$                                                                                                                                                                                 |
| (7) $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ ,<br>conjugate to the group g.b. | $(x, y) \rightarrow (x^{-1}, y^{-1})$ and $(x, y) \rightarrow (-y, x)$<br>$(x, y) \rightarrow (x^{-1}, y^{-1})$ and $(x, y) \rightarrow (-x^{-1}, -x^{-1}y)$                                                                        |
| (8) $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ ,<br>conjugate to the group g.b.                           | $(x, y) \rightarrow (-y, -x)$ , $(x, y) \rightarrow (x^{-1}, y^{-1})$ and $(x, y) \rightarrow (y, x)$<br>$(x, y) \rightarrow (x^{-1}, -x^{-1}y)$ , $(x, y) \rightarrow (x^{-1}, y^{-1})$ and $(x, y) \rightarrow (x^{-1}, x^{-1}y)$ |

Where  $m, n \in \mathbb{Z}_{>0}$  and  $n|m$ . Moreover, the groups of type (1) are conjugate to subgroups of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$ .

*Proof.* We now use the results from the previous proposition to determine conjugacy classes.

1. The case (i) corresponds to (1).

2. Suppose we have the group of type (ii)  $G \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/m\mathbb{Z}$  with generators  $(x, y) \rightarrow (\pm x^{\pm 1}, y)$  and  $(x, y) \rightarrow (x, \zeta_m y)$ .

- for  $m = 2n$  we get (3).
- for  $m = 2n + 1$  consider the map:

$$\phi : (x, y) \mapsto \left( x, \frac{y}{x + x^{-1}} \right)$$

it conjugates  $f : (x, y) \rightarrow (-x, y)$  to  $(x, y) \rightarrow (-x, -y)$ :

$$\phi \circ f \circ \phi^{-1} : (x, y) \xrightarrow{\phi^{-1}} (x, y(x+x^{-1})) \xrightarrow{f} (-x, y(x+x^{-1})) \xrightarrow{\phi} \left( x, \frac{y(x+x^{-1})}{-x-x^{-1}} \right) = (-x, -y)$$

Moreover  $\phi$  commutes with  $g : (x, y) \rightarrow (x^{-1}, y)$  and  $h : (x, y) \rightarrow (x, \zeta_m y)$ :

$$\left. \begin{aligned} g \circ \phi : (x, y) &\xrightarrow{\phi} (x, y(x+x^{-1})) \xrightarrow{g} (x^{-1}, y(x+x^{-1})) \\ \phi \circ g : (x, y) &\xrightarrow{g} (x^{-1}, y) \xrightarrow{\phi} (x^{-1}, y(x^{-1}+x)) \end{aligned} \right\} \Rightarrow g \circ \phi = \phi \circ g$$

$$\left. \begin{aligned} h \circ \phi : (x, y) &\xrightarrow{\phi} (x, y(x+x^{-1})) \xrightarrow{h} (x, \zeta_m y(x+x^{-1})) \\ \phi \circ h : (x, y) &\xrightarrow{h} (x, \zeta_m y) \xrightarrow{\phi} (x, \zeta_m y(x+x^{-1})) \end{aligned} \right\} \Rightarrow h \circ \phi = \phi \circ h$$

We get the group of type (2) with  $m = n$ .

3. The case (iii) corresponds to (2).

4. Suppose we have a group of type (iv)  $G \cong (\mathbb{Z}/2\mathbb{Z})^2$  with generators  $(x, y) \rightarrow (\pm x^{\pm 1}, y)$ .

- Consider a birational map:

$$\phi : (x, y) \dashrightarrow \left( x, \frac{y+x^2}{y+x^{-2}} \right)$$

Suppose  $g : (x, y) \rightarrow (x^{-1}, y)$ ,  $h : (x, y) \rightarrow (-x, y)$ . Now we have that  $g$  is conjugate to  $(x, y) \rightarrow (x^{-1}, y^{-1})$ :

$$g_1 : (x, y) \xrightarrow{\phi^{-1}} \left( x, \frac{x^2 - x^{-2}y}{y-1} \right) \xrightarrow{g} \left( -x, \frac{x^2 - x^{-2}y}{y-1} \right) \xrightarrow{\phi} \left( x^{-1}, \frac{\frac{x^2 - x^{-2}y}{y-1} + x^{-2}}{\frac{x^2 - x^{-2}y}{y-1} + x^2} \right) = (x^{-1}, y^{-1})$$

$$\left. \begin{aligned} h \circ \phi : (x, y) &\xrightarrow{\phi} \left( x, \frac{y+x^2}{y+x^{-2}} \right) \xrightarrow{h} \left( -x, \frac{y+x^2}{y+x^{-2}} \right) \\ \phi \circ h : (x, y) &\xrightarrow{h} (-x, y) \xrightarrow{\phi} \left( -x, \frac{y+(-x)^2}{y+(-x)^{-2}} \right) \end{aligned} \right\} \Rightarrow \phi \circ h = h \circ \phi$$

We can conjugate this group to the group of type (v) by the element in  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$ :

$$\psi : (x, y) \rightarrow \left( \frac{x+1}{x-1}, \frac{y+1}{y-1} \right)$$

Then we have:

$$\psi \circ g_1 \circ \psi^{-1} : (x, y) \xrightarrow{\psi^{-1}} \left( \frac{x+1}{x-1}, \frac{y+1}{y-1} \right) \xrightarrow{g_1} \left( \frac{x-1}{x+1}, \frac{y-1}{y+1} \right) \xrightarrow{\psi} \left( \frac{\frac{x-1}{x+1} + 1}{\frac{x-1}{x+1} - 1}, \frac{\frac{y-1}{y+1} + 1}{\frac{y-1}{y+1} - 1} \right) = (-x, -y)$$

$$\psi \circ h \circ \psi^{-1} : (x, y) \xrightarrow{\psi^{-1}} \left( \frac{x+1}{x-1}, \frac{y+1}{y-1} \right) \xrightarrow{h} \left( -\frac{x+1}{x-1}, \frac{y+1}{y-1} \right) \xrightarrow{\psi} \left( \frac{-\frac{x+1}{x-1} + 1}{-\frac{x+1}{x-1} - 1}, \frac{\frac{y+1}{y-1} + 1}{\frac{y+1}{y-1} - 1} \right) = (x^{-1}, y)$$

- Consider a birational map:

$$\phi : (x, y) \dashrightarrow \left( x, x \frac{y+x^2}{y+x^{-2}} \right)$$

Suppose  $g : (x, y) \rightarrow (-x, y)$ ,  $h : (x, y) \rightarrow (x^{-1}, y)$ . Now we have that  $g$  is conjugate to  $(x, y) \rightarrow (-x, -y)$

$$(x, y) \xrightarrow{\phi^{-1}} \left( x, \frac{x^3 - yx^{-2}}{y-x} \right) \xrightarrow{g} \left( -x, \frac{x^3 - yx^{-2}}{y-x} \right) \xrightarrow{\phi} \left( -x, x \frac{\frac{x^3 - yx^{-2}}{y-x} + x^2}{\frac{x^3 - yx^{-2}}{y-x} + x^{-2}} \right) = (-x, -y)$$

$h$  is conjugate to  $(x, y) \rightarrow (x^{-1}, y^{-1})$

$$(x, y) \xrightarrow{\phi^{-1}} \left( x, \frac{x^3 - yx^{-2}}{y-x} \right) \xrightarrow{h} \left( x^{-1}, \frac{x^3 - yx^{-2}}{y-x} \right) \xrightarrow{\phi} \left( x^{-1}, x^{-1} \frac{\frac{x^3 - yx^{-2}}{y-x} + x^{-2}}{\frac{x^3 - yx^{-2}}{y-x} + x^2} \right) = (x^{-1}, y^{-1})$$

Thus, (iv) and (vi) are birationally conjugate so (iv), (v), (vi) are birationally conjugate to (4).

5. Cases (vii) and (viii) correspond to (5) and (6) respectively.

6. Suppose we have the group of type (ix), i.e.  $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  generated by  $g : (x, y) \rightarrow (x^{-1}, y^{-1})$  and  $h : (x, y) \rightarrow (-y, x)$  so we are in case (7). Let's prove that it is conjugate to the group with the generators given in the proposition. Consider the birational map

$$\phi : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2, \text{ where } (x, y) \dashrightarrow [x : y : 1]$$

Then:

$$\begin{aligned} g' &= \phi \circ g \circ \phi^{-1} : [x : y : z] \xrightarrow{\phi^{-1}} \left( \frac{x}{z}, \frac{y}{z} \right) \xrightarrow{g} \left( \frac{z}{x}, \frac{z}{y} \right) \xrightarrow{\phi} [zy : zx : xy] \\ h' &= \phi \circ h \circ \phi^{-1} : [x : y : z] \xrightarrow{\phi^{-1}} \left( \frac{x}{z}, \frac{y}{z} \right) \xrightarrow{h} \left( -\frac{y}{z}, \frac{x}{z} \right) \xrightarrow{\phi} [-y : x : z] \end{aligned}$$

So we get the group of birational maps of  $\mathbb{P}_{\mathbb{C}}^2$  generated by  $g' : [x : y : z] \dashrightarrow [zy : zx : xy]$  and  $h' : [x : y : z] \dashrightarrow [-y : x : z]$  up to conjugation. This group leaves invariant the pencil of lines through point  $[0 : 0 : 1]$  which is  $\{ax + by \mid [a : b] \in \mathbb{P}_{\mathbb{C}}^1\}$ . Now consider the birational map

$$\psi : \mathbb{P}_{\mathbb{C}}^2 \rightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1, \text{ where } [x : y : z] \dashrightarrow [x : y] \times [x : z]$$

Then

$$\begin{aligned} \psi \circ g' \circ \psi^{-1} : (\alpha, \beta) &= [x : y] \times [x' : z'] = [x : y] \times [x : z] \xrightarrow{\psi^{-1}} [x : y : z] \xrightarrow{g'} \\ &\xrightarrow{g'} [yz : xz : xy] \xrightarrow{\psi} [yz : xz] \times [yz : xy] = [y : x] \times [z : x] = (\alpha^{-1}, \beta^{-1}) \\ \psi \circ h' \circ \psi^{-1} : (\alpha, \beta) &= [x : y] \times [x' : z'] = [x : y] \times [x : z] \xrightarrow{\psi^{-1}} [x : y : z] \xrightarrow{h'} \\ &\xrightarrow{h'} [-y : x : z] \xrightarrow{\psi} [-y : x] \times [-y : z] = (-\alpha^{-1}, -\alpha^{-1}\beta) \end{aligned}$$

So our group is conjugate to the group generated by  $(x, y) \dashrightarrow (x^{-1}, y^{-1})$  and  $(x, y) \dashrightarrow (-x^{-1}, -x^{-1}y)$

7. Suppose we have a group of type (x):  $G \cong (\mathbb{Z}/2\mathbb{Z})^3$  generated by  $f : (x, y) \rightarrow (-y, -x)$ ,  $g : (x, y) \rightarrow (x^{-1}, y^{-1})$  and  $h : (x, y) \rightarrow (y, x)$  so we are in case (8). Let's prove that it is conjugate to the group with the generators given in the proposition. Consider the birational map

$$\phi : (x, y) \dashrightarrow (x/y, x)$$

Now we have:

$$\begin{aligned} \phi \circ f \circ \phi^{-1} : (x, y) &\xrightarrow{\phi^{-1}} (y, x^{-1}y) \xrightarrow{f} (-x^{-1}y, -y) \xrightarrow{\phi} (x^{-1}, -x^{-1}y) \\ \phi \circ g \circ \phi^{-1} : (x, y) &\xrightarrow{\phi^{-1}} (y, x^{-1}y) \xrightarrow{g} (y^{-1}, xy^{-1}) \xrightarrow{\phi} (x^{-1}, y^{-1}) \\ \phi \circ h \circ \phi^{-1} : (x, y) &\xrightarrow{\phi^{-1}} (y, x^{-1}y) \xrightarrow{h} (x^{-1}y, y) \xrightarrow{\phi} (x^{-1}, x^{-1}y) \end{aligned}$$

and the first projection is invariant.

Let's now prove that different cases give different conjugacy classes in  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ .

8. The groups of type (1) fix at least one point and others don't so they are different from other cases.

9. Note that the only candidates for being conjugate are:

- (5) and (8) with  $G \cong (\mathbb{Z}/2\mathbb{Z})^3$
- (2) for  $n = 2$  and (7) with  $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

Note that (2) and (5) leave invariant two pencils of curves given by the projections  $\pi_i : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  on  $i$ -th coordinate and which intersect freely in exactly one point. Let's prove that it is not true for (7) and (8).

Consider  $G$  as a subgroup of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  such that  $G$  permutes fibrations. By proposition 10.3.3 we have that:

$$\text{Pic}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)^G = -1/2K_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1}\mathbb{Z} =: d\mathbb{Z}$$

Suppose that we can find two  $G$ -invariant pencils of rational curves  $\Lambda_1 = n_1d$  and  $\Lambda_2 = n_2d$  ( $n_1, n_2 \in \mathbb{Z}_{>0}$ ). Consider the intersection :

$$\Lambda_1 \cdot \Lambda_2 = \left(-\frac{1}{2}K_{\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1}\right)^2 n_1 n_2 = (E_0 + f)^2 n_1 n_2 = 2n_1 n_2 - \text{even}$$

Moreover, the fixed part of the intersection is even since  $\text{Ord}(G) = 8$  (in both cases) and  $G$  does not have fixed points on  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ . Thus, the free part of the intersection is also even thus they cannot intersect freely in exactly one point.

Thus, the members of pairs (5)+(8) and (2) + (7) are not birationally conjugate to each other.  $\square$

# Chapter 15

## Conclusion

We proved in 9.3.3 that the minimal rational surfaces are given by  $X$  is a smooth projective rational surface  $X$  where  $X \cong \mathbb{P}_{\mathbb{C}}^2$  or  $X \cong \mathbb{F}_n$  for  $n \neq 1$ . Then we have that the groups of automorphisms of minimal rational surfaces are given by:

$$\begin{cases} \text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3, \mathbb{C}) \\ \text{Aut}(\mathbb{F}_n) = \mathbb{C}^{n+1} \rtimes \text{GL}(2, \mathbb{C}) / \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \mid \mu^n = 1 \right\} \end{cases}$$

We proved in 12.1.1 that the minimal conic bundles are given by  $X = \mathbb{F}_n$  for some  $n$ . Moreover, 12.5.1 gives us the result on the finite subgroups of groups of automorphisms of conic bundles containing singular fibers:

**Theorem.** *Suppose  $(X, \pi)$  is a conic bundle which contains a singular fiber (at least 1),  $G \subset \text{Aut}(X, \pi)$  is a finite group such that a pair  $(G, X)$  is minimal. Then, one of the following occurs:*

1.  $(X, \pi)$  is the exceptional conic bundle
2.  $G' = \{\text{elements which preserve any fiber}\} \subset G$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$  or  $(\mathbb{Z}/2\mathbb{Z})^2$ .

Suppose we have a finite subgroup  $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ . By proposition 10.1.2 we can find a rational surface  $X$  together with a birational morphism  $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^2$  such that  $\pi^{-1}G\pi \subset \text{Aut}(X)$ , i.e. we can realize the subgroup of  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  as a subgroup of  $\text{Aut}(X)$  where  $X$  is a rational surface. After contracting orbits of disjoint  $(-1)$ -curves we can assume that  $(X, G)$  is minimal. Then we study minimal pairs  $(X, G)$  using theorem 10.3.2:

**Theorem.** *Suppose  $X$  is a smooth projective rational surface,  $G \subset \text{Aut}(X)$  is finite. Suppose we have a minimal pair  $(G, X)$ . Then exactly one of the following holds:*

1.  $\text{rk}(\text{Pic}(X)^G) = 1$ . In this case,  $X$  is a Del Pezzo surface.
2. The surface  $X$  has a conic bundle structure invariant by  $G$ , and  $\text{rk}(\text{Pic}(X)^G) = 2$ .

In the first case  $X$  is a del Pezzo surface by proposition 10.3.3. For the second case recall proposition 12.3.3 which proves that the study of finite groups of automorphisms of conic bundles is equivalent to the study of finite subgroups of the group of birational maps of  $\mathbb{F}_0$  which preserve the conic bundle structure. This group is called de Jonquieres group and it is isomorphic to  $\text{PGL}(2, \mathbb{C}(x)) \rtimes \text{PGL}(2, \mathbb{C})$ . Combined with the results from chapters 12 and 13 it gives us the following result:

**Theorem.** *Let  $G \subset \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  be a finite group. Then  $G$  is conjugated to the subgroup of one of the following groups:*

1. *De Jonquieres group which is isomorphic to  $\text{PGL}(2, \mathbb{C}(x)) \rtimes \text{PGL}(2, \mathbb{C})$ .*
2.  *$\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) \cong \text{PGL}(3, \mathbb{C})$ .*
3.  *$\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) = (\text{PGL}(2, \mathbb{C}))^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ .*
4.  *$\text{Aut}(X_6) \cong (\mathbb{C}^*)^2 \rtimes (S_3 \times \mathbb{Z}/2\mathbb{Z})$ , where  $X_6$  is the del Pezzo surface of degree 6.*
5.  *$\text{Aut}(X_5) \cong S_5$  where  $X_5$  is the del Pezzo surface of degree 5.*
6.  *$\text{Aut}(X_4) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \text{Aut}(X_4, \pi, 5)$ , where  $\text{Aut}(X_4, \pi, 5)$  is the lift of the group of automorphisms of  $\mathbb{P}_{\mathbb{C}}^2$  that leave the set  $\{p_1, \dots, p_5\}$  invariant.*
7.  *$\text{Aut}(X_k)$  where  $X_k$  is a del Pezzo surface of degree  $k$  for  $k = 1, 2, 3$ .*

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