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Analyse fonctionnelle, spectrale et probabiliste sur domaines d'extension de Sobolev

Gabriel Claret

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CentraleSupélec

université
PARIS-SACLAY

Thèse de Master

Analyse fonctionnelle, spectrale
et probabiliste sur domaines
d'extension de Sobolev

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Notations

General notations

We will use the following notations:

- $a := b$ when the object a is defined as equal to the object b ;
- n is an integer non-inferior to 2 (unless stated otherwise) referring to the dimension of the ambient space;
- a domain of \mathbb{R}^n is a nonempty open connected set of \mathbb{R}^n ; all domains considered coincide with the interior of their closure;
- $\lambda^{(n)}$ is the n -dimensional Lebesgue measure on \mathbb{R}^n ;
- σ denotes the $(n - 1)$ -dimensional Lebesgue measure on the boundary of a Lipschitz domain of \mathbb{R}^n (regardless of that domain);
- if $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x)$ denotes the open Euclidean ball of center x and radius r ;
- if A is a Borel subset of \mathbb{R}^n and $k \in \mathbb{N} \cup \{\infty\}$, $C^k(A)$ denotes the set of real-valued k -times continuously differentiable functions on A , and $C_0^k(A)$ those of them which have a compact support in A ;
- compact inclusions are denoted by $\subset\subset$;
- if H is a real Banach space, then $\|\cdot\|_H$ denote the associated norm. $H' := \mathcal{L}(H, \mathbb{R})$ denotes its dual space and $\langle \cdot, \cdot \rangle_{H', H}$ denotes the associated duality pairing; if H is a Hilbert space, $\langle \cdot, \cdot \rangle_H$ denote the associated inner product;
- if K is a subspace of a Hilbert space H , then K^\perp denotes the orthogonal complement of K in H ;
- if $L : V \rightarrow H$ is a bounded linear operator between Hilbert spaces, then $L^* : H' \rightarrow V'$ denotes its adjoint operator;
- if Ω is a domain of \mathbb{R}^n (or $\mathbb{C} \simeq \mathbb{R}^2$), then its boundary is denoted by $\partial\Omega$ and its complement domain by $\Omega^c := \mathbb{R}^n \setminus \overline{\Omega}$.

L^2 spaces will always be identified with their dual spaces, and other Hilbert spaces will always be identified to their bidual according to the Riesz representation theorem.

The term ‘well-posedness’ of a differential equation is understood in the sense of Hadamard: existence and uniqueness of a solution, which depends continuously on the data (such as boundary conditions).

Notations for functional spaces

If U is an open set of \mathbb{R}^n and $X(U)$ denotes a Hilbert space of real-valued functions defined on U , then:

- $X'(U) := (X(U))'$;
- $u \in X_\Delta(U)$ if $u \in X(U)$ and $\Delta u \in L^2(U)$;
- $u \in X_{(0)}(U)$ if $u \in X(U)$ and $\int_U u \, dx = 0$; if $\Omega \subset U$ is open, $u \in X_{(0)\Omega}(U)$ if $u \in X(U)$ and $\int_\Omega u \, dx = 0$;
- $u \in X_{[f]\text{Tr}}(U)$ if $u \in X(U)$ and $[[\text{Tr } u]] = f$ in the sense of Definition 2.14;
- $u \in X_{[g]\partial\nu}(U)$ if $u \in X(U)$ and $[[\frac{\partial u}{\partial \nu}]] = g$ in the sense of Definition 2.14;
- $X_{\mathbb{C}}(U) := X(U) + iX(U)$ where $i^2 = -1$.

The notations defined above can be combined.

Specific function spaces

In addition to the usual Sobolev spaces, spaces relevant to our study are listed hereafter. An equality is said to hold ‘weakly’ if it holds in the distributional sense. By ‘gradient norm’ is meant the norm $\|\nabla \cdot\|_{L^2}$ and sums of norms are understood as the square root of the sum of the norms squared.

If Ω is a one-sided extension domain (Definition 2.5):

- $H_f^1(\Omega) := \{u \in H^1(\Omega) \mid \text{Tr}_i u = f\}$ (Definition 2.8);
- $V_1(\Omega) := \{u \in H^1(\Omega) \mid (-\Delta + 1)u = 0 \text{ weakly on } \Omega\}$ endowed with the H^1 norm on Ω ; $V_1(\Omega^c)$ is defined in the same way replacing Ω with Ω^c ;
- $V_{(0)}(\Omega) := \left\{u \in H_{(0)}^1(\Omega) \mid -\Delta u = 0 \text{ weakly on } \Omega\right\}$ endowed with the gradient norm on Ω ;
- $\mathcal{B}_0(\partial\Omega) := \text{Tr}_i \left(H_{(0)}^1(\Omega) \right)$,
- $\mathcal{B}'_{0,ext}(\partial\Omega) := \{g \in \mathcal{B}'(\partial\Omega) \mid \langle g, 1 \rangle_{\mathcal{B}', \mathcal{B}} = 0\}$;
- if Ω is Lipschitz, $L_0^2(\partial\Omega, \sigma) := \{g \in L^2(\partial\Omega, \sigma) \mid \int_{\partial\Omega} g \, d\sigma = 0\}$.

If Ω is a two-sided extension domain (Definition 2.5):

- $H_{\partial\Omega}^1(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid \nabla u|_{\Omega \cup \Omega^c} \in L^2(\Omega \cup \Omega^c)^n\}$ endowed with the sum of the L^2 norm on \mathbb{R}^n and the gradient norm on $\Omega \cup \Omega^c$;
- $V_{1,\partial\Omega}(\mathbb{R}^n) := \{u \in H_{\partial\Omega}^1(\mathbb{R}^n) \mid -\Delta u + u = 0 \text{ weakly on } \mathbb{R}^n \setminus \partial\Omega\}$ endowed with the sum of the L^2 norm on \mathbb{R}^n and the gradient norm on $\Omega \cup \Omega^c$;

- if $n = 2$, $V_{van}(\mathbb{R}^2) := \{u \in H_{\partial\Omega}^1(\mathbb{R}^2) \mid u \rightarrow 0 \text{ as } |x| \rightarrow +\infty\}$;
- if $n \geq 3$, $W_{\partial\Omega}(\mathbb{R}^n) := \{u \in L_{loc}^2(\mathbb{R}^n) \mid \nabla u|_{\Omega \cup \Omega^c} \in L^2(\Omega \cup \Omega^c)^n, \langle u \rangle_\infty = 0\}$ (defined by Equation (B.12));
- $V_{\partial\Omega}(\mathbb{R}^n) := V_{van}(\mathbb{R}^2)$ if $n = 2$ and $V_{\partial\Omega}(\mathbb{R}^n) := W_{\partial\Omega, \langle 0 \rangle \Omega}(\mathbb{R}^n)$ endowed with the gradient norm on $\Omega \cup \Omega^c$ if $n \geq 3$.

If Ω is a two-sided extension domain and U is a one-sided extension domain with $\Omega \subset\subset U$:

- $H_{0, \partial\Omega}^1(U) := \{u \in L^2(U) \mid \nabla u|_{U \setminus \partial\Omega} \in L^2(U \setminus \partial\Omega)^n, \text{Tr}^{\partial U} u = 0\}$.

If Ω and D are two-sided extension domains with $D \subset\subset \Omega$:

- $H_{\partial D}^1(\Omega) := \{u \in L^2(\Omega) \mid u|_D \in H^1(D), u|_{\Omega \setminus \bar{D}} \in H^1(\Omega \setminus \bar{D})\}$;
- $u \in W_{\partial D, \partial\Omega}(\mathbb{R}^n)$ if $u \in L_{loc}^2(\mathbb{R}^n)$, $\nabla u|_{\mathbb{R}^n \setminus (\partial D \cup \partial\Omega)} \in L^2(\mathbb{R}^n \setminus (\partial D \cup \partial\Omega))^n$ and $\langle u \rangle_\infty = 0$ (defined by Equation (B.12)).

Operators

Operators relevant to our study are listed hereafter. Note that most of them are defined with and without a tilde: the operators connected to problems for $(-\Delta + 1)$ do not have a tilde while those connected to problems for $-\Delta$ do.

- I denotes the identity operator (regardless of the space).

If Ω is a one-sided extension domain:

- $\text{Tr}_i : H^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ is the interior trace operator; it can be seen as $\text{Tr}_i : H_{\langle 0 \rangle}^1(\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$;
- $\frac{\partial_i}{\partial\nu} : H_\Delta^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ is the weak interior normal derivation operator; it can be seen as $\frac{\partial_i}{\partial\nu} : V_{\langle 0 \rangle}(\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$;
- $\mathcal{D}_\Omega : \mathcal{B}(\partial\Omega) \rightarrow H^1(\Omega)$ is the Dirichlet 1-harmonic extension operator and $\tilde{\mathcal{D}}_\Omega : \mathcal{B}_0(\partial\Omega) \rightarrow H_{\langle 0 \rangle}^1(\Omega)$ is the Dirichlet harmonic extension operator (Definitions A.11 and A.12);
- $\mathcal{S}_\Omega : \mathcal{B}'(\partial\Omega) \rightarrow H^1(\Omega)$ is the Neumann 1-harmonic extension operator and $\tilde{\mathcal{S}}_\Omega : \mathcal{B}_0(\partial\Omega) \rightarrow H_{\langle 0 \rangle}^1(\Omega)$ is the Neumann harmonic extension operator (Definition A.7);
- $\mathcal{A} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ and $\tilde{\mathcal{A}} : \mathcal{B}_0(\partial\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$ are the Poincaré-Steklov operator for the problems on Ω associated to $(-\Delta + 1)$ and $-\Delta$ respectively (Definition A.14).

If Ω is a two-sided extension domain:

- $\text{Tr}_e : H^1(\Omega^c) \rightarrow \mathcal{B}(\partial\Omega)$ is the exterior trace operator;
- $\frac{\partial_e}{\partial\nu} : H^1_\Delta(\Omega^c) \rightarrow \mathcal{B}(\partial\Omega)$ is the weak exterior normal derivation operator;
- $[[\text{Tr} \cdot]] := \text{Tr}_i - \text{Tr}_e$ and $[[\frac{\partial \cdot}{\partial\nu}]] := \frac{\partial_i}{\partial\nu} - \frac{\partial_e}{\partial\nu}$ are the jump in trace and in normal derivative across $\partial\Omega$ respectively (Definition 2.14);
- $\mathcal{D}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow H^1_{\partial\Omega}(\Omega)$ and $\tilde{\mathcal{D}}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow V_{\partial\Omega}(\mathbb{R}^n)$ are the double layer potential operators for the transmission problems across $\partial\Omega$ associated to $(-\Delta + 1)$ and $-\Delta$ respectively (Propositions B.3 and B.12);
- $\mathcal{S}_{\partial\Omega} : \mathcal{B}'(\partial\Omega) \rightarrow H^1_{\partial\Omega}(\Omega)$ and $\tilde{\mathcal{S}}_{\partial\Omega} : \mathcal{B}'_0(\partial\Omega) \rightarrow V_{\partial\Omega}(\mathbb{R}^n)$ are the single layer potential operators for the transmission problems across $\partial\Omega$ associated to $(-\Delta + 1)$ and $-\Delta$ respectively (Theorem B.1 and Proposition B.11);
- $\mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ and $\tilde{\mathcal{K}}_{\partial\Omega} : \mathcal{B}_0(\partial\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$ are the Neumann-Poincaré operators for the transmission problems across $\partial\Omega$ associated to $(-\Delta + 1)$ and $-\Delta$ respectively (Definition B.23);
- $\mathcal{V}_{\partial\Omega} := \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ and $\tilde{\mathcal{V}}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$ are the traces of the single layer potential operators for the transmission problems across $\partial\Omega$ associated to $(-\Delta + 1)$ and $-\Delta$ respectively (Equations (B.48) and (B.49));
- $-\mathcal{W}_{\partial\Omega} := \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ and $-\tilde{\mathcal{W}}_{\partial\Omega} : \mathcal{B}'_0(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ are the normal derivatives of the double layer potential operators for the transmission problems across $\partial\Omega$ associated to $(-\Delta + 1)$ and $-\Delta$ respectively (Equations (B.48) and (B.49));
- $\mathcal{A}_{\partial\Omega,i} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ and $\mathcal{A}_{\partial\Omega,e} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ are respectively the interior and exterior Poincaré-Steklov operators for the transmission problems across $\partial\Omega$ associated to $(-\Delta + 1)$ (Theorem B.28).

1 Introduction

Whether it be in medical imaging for the detection of cancer tumors, in the study of metal alloys to detect impurities or in the military field to identify buried objects, imaging problems can arise in a wide range of situations. The idea of such a problem can be described as follows: a domain contains an inclusion characterized by a change in conductivity; we wish to identify the inclusion from data obtained on the boundary of the domain. Naturally, when solving such problems, modelling the boundary under study accounts for an essential step. In the classical theory, all boundaries are assumed to be regular, that is at least Lipschitz, and rather often once continuously differentiable. In that framework, imaging problems have been widely studied: in [61], a representation formula is proved for the solutions to the imaging problem (Problem (4.19) on page 29) and used to identify the inclusion from a single boundary measurement under some hypotheses (for example, when the inclusion is a disk). The book [1] synthesises diverse imaging results from single boundary measurements on Lipschitz boundaries, but also multiple boundary measurements using the Poincaré-Steklov (or Dirichlet-to-Neumann) operator to identify the inclusion. Note that imaging problems are closely connected to Calderón’s problem (conductivity identification from boundary measurements), see for instance [19, 20], as well as inverse scattering problems (reconstruction of obstacles from scattered waves), see for instance [35] for the study of such problems on C^2 domains and [70, 95] on Lipschitz domains.

In the classical case, the imaging problem is solved introducing operators called the single and double layer potential operator [85, 91], which are defined as integrals over the boundary using the fundamental solution to the problem. Those operators link a Neumann or Dirichlet boundary condition respectively to a solution having that condition as jump in trace or in normal derivative respectively. Therefore, the layer potential operators are closely connected to the study of transmission problems (Problem (4.1) on page 21) – at this point, across Lipschitz boundaries; the link between the solution to the problem and the layer potential operators is known as Green’s third identity [92]. Imaging considerations aside, transmission problems themselves have been widely studied in a Lipschitz framework. In [3] is proved the transmission problem on the whole Euclidean space can be approximated by truncated transmission problems, and that the associated solutions converge as the truncation domain grows. It is also possible look into the boundary values of the solution (determined from boundary jumps alone), which leads to the introduction of the Neumann-Poincaré operator [77, 91] as a singular integral over a the boundary. It is well known that operator can be symmetrized [80] and that its (real) spectrum is included in $[-\frac{1}{2}, \frac{1}{2}]$ [44, 62]. The fundamental solution to the problem, used to define the layer potential operators can be understood from a probabilistic perspective [8], relying on the connection between the analytical concept of Dirichlet forms and the probabilistic concept of Feller semi-groups [31].

Although the Lipschitz model allow to set boundary conditions using definitions in the strong sense, real object often are far more irregular, and it is not uncommon for such shapes to be best modelled using fractals (*i.e.* self-similar or scale invariant geometries) [72, 73]. This consideration gives rise to new questions: how irregular can the domain be? how to define boundary data in that case? The notions of trace operators for irregular domains (see [94] for d -set boundaries (Definition 2.3), [53] for d -upper regular boundaries (Definition 2.4) with $d \in]n - 2, n[$ and [14] for H^1 -extension domains (Definition 2.5), of weak normal derivative [67] and of Sobolev spaces of functions defined on irregular sets [26] allow to consider boundary value problems on a broad class of non-Lipschitz domains, including fractal domains, from a variational perspective. In [25, 26], scattering problem are studied in the case of an obstacle described by a fractal screen (*i.e.*, a relatively open fractal set in \mathbb{R}^{n-1}), and leads to the study of layer potential operators in that case. Though dealing with irregular shapes as well, our framework is different for our domain is open in \mathbb{R}^n , which prevents from using boundary condition in the strong sense. The Dirichlet and Neumann boundary conditions are then understood as elements of an abstract trace space and its dual respectively.

In this thesis, we aim at solving imaging problems set on H^1 -Sobolev extension domains. That class contains smooth domains, Lipschitz domains, fractal domains, but unlike the latter, extension domains are not defined using a specific boundary measure, relying on the framework from [14]. This allows to consider domains with varying boundary dimension, such as multi-fractal domains, or partly smooth and partly fractal domains. In addition, the results obtained do not depend on any boundary measure, including in the case of Lipschitz boundaries.

This manuscript is organised as follows: notations which will be used throughout this manuscript are listed on pages iii–vi. In Section 2, we introduce the class of extension domains on which our problems are set. We introduce the trace operator and the weak normal derivatives in our framework and characterize the trace operator as an isometry (Theorem 2.9). Moreover, we explain the probabilistic approach to solving boundary value problems. In Sections 3 to 5, we give an overview of results regarding boundary value and imaging problems on Lipschitz domains and introduce the results which we generalize to extension domains, while the new results are stated and proved in Appendix A to C. For that matter and although this thesis can be read linearly, we advise the reader to refer to the corresponding subsection in the appendix after having read a subsection in the body of the text (both subsections have the same title apart from Section 5: all of Appendix C refers to Subsection 5.2).

In Section 3 and Appendix A, we study bounded problems set on our class of extension domains with Dirichlet and Neumann boundary conditions for the operators $(-\Delta + 1)$ and $-\Delta$. We introduce Hilbert spaces on which those problems are well-posed and introduce the corresponding (1-)harmonic extension operators. We prove those operators are isometric (Theorems A.8

and A.13). Then, we generalize the notion of Poincaré-Steklov operator in the case of extension domains as a linear and continuous operator, and prove it can be regarded as the Riesz isometry between the trace space and its dual space (Theorem A.17). Our isometry results are condensed in Diagram (3.7).

In Section 4 and Appendix B, we focus on the study of transmission problems across the boundary of an extension domain for $(-\Delta + 1)$ and $-\Delta$. In Subsection 4.1 and Appendix B.1, we identify spaces on which that transmission problem is well-posed and define the associated single and double layer potential operators as linear and continuous operators. Our method relies on a variational approach and is focused on the link those operators make between the boundary conditions of the problem and the solution. We prove those operators can be expressed as convolutions on the boundary with the fundamental solution, which generalize the classical formulae (Definition B.1 and Propositions B.6, B.11 and B.12). Finally, we express the normal derivative of the fundamental solution to the Laplacian, used to define the double layer potential in the classical case, in terms of the harmonic measure (Theorem 4.1). In Subsection 4.2 and Appendix B.2, we study the truncated transmission problem and define associated layer potential operators as linear and continuous operators. We prove the convergence of those operators to the layer potentials for the transmission problem on the whole space as the truncation domain grows (Theorems B.18 and B.22). In Subsection 4.4 and Appendix B.4, we use the layer potential operators in the framework of extension domains to generalize the notion of Poincaré-Steklov operator for transmission problems. We define two operators (one interior and one exterior) and characterize them in terms of solutions to the transmission problem. In Subsection 4.3 and Appendix B.3, we generalize the notion of Neumann-Poincaré operator using the boundary relations of the double layer potential operator. We prove its adjoint is connected to the single layer potential operator in the same way as it is in the Lipschitz case (Proposition B.24). We also define other operators associated to boundary values of the layer potentials and generalize the symmetrization properties of the Neumann-Poincaré operator and the notion of Calderón projectors (Theorem B.27). In Subsection 4.5, we use the boundary values of the layer potentials to define norms on the trace space and its dual, equivalent to the usual ones. We prove contracting properties of the Neumann-Poincaré operator (Theorem B.29) and that its real spectrum is included in $] -\frac{1}{2}, \frac{1}{2}[$ (Theorem B.31). Finally, we estimate the parameters of the equivalence of the norms defined in this subsection. In Subsection 4.6 and Appendix B.6, we generalize the representation formula for the solutions to two-phased transmission problems in the case of extension domains, using the generalized layer potential and Neumann-Poincaré operators (Theorem B.34). We apply it to the identification of inclusions from a single boundary measurement in the monotonous case for extension domains and in the case of disks (inside an extension domain).

In Section 5 and Appendix C, we study the Hilbert transform and its connection to the layer potential operators in the complex plane. We extend

our definitions of layer potential and Neumann-Poincaré operators to complex-valued condition, and express the Hilbert transform in terms of single and double layer potential operators in the Lipschitz case (Theorem 5.2). We give two possible definitions of the Hilbert transform in \mathbb{R}^2 for extension domains. In an attempt to prove those definitions coincide, we prove any domain can be approximated by Lipschitz domains from both the inside and the outside (Theorem C.3), as well as the convergence of Dirichlet harmonic extensions in the context of converging domains (Theorem C.17).

2 Framework

In this section, we introduce the class of domains on which our study will be carried out: the class of one- and two-sided extension domains (Subsection 2.1). We define Dirichlet and Neumann boundary conditions for such domains by means of the trace operators and the weak normal derivative and prove the trace operator defines a partial isometry (Subsection 2.2). In Subsection 2.3, we describe the probabilistic approach to the study of boundary value problems.

2.1 Domain regularity

A general framework for the study of boundary value problems relies on the notion of Lipschitz domains [51, Definition 2.4.5].

Definition 2.1. (*Lipschitz domain*) *A domain Ω of \mathbb{R}^n is said to be a Lipschitz domain if there exist constants $L, a, r > 0$ such that for all $x \in \partial\Omega$, there exist an orthogonal system of coordinates the origin of which is $x = 0$, a cylinder $K = B \times]-a, a[$ where B is an $(n-1)$ -dimensional open ball of radius $r > 0$, and a L -Lipschitz continuous function $\varphi : B \rightarrow]-a, a[$ with $\varphi(0) = 0$ such that:*

- (i) $\partial\Omega \cap K = \{(y, \varphi(y)) \mid y \in B\}$,
- (ii) $\Omega \cap K = \{(y, x_n) \mid y \in B, x_n > \varphi(y)\}$.

In other words, a Lipschitz domain is a domain with a boundary which is locally the graph of a Lipschitz continuous function (condition (i)), and which lies on one side of its boundary (condition (ii)). This class of domains contains smooth domains such as C^1 -domains, but also domains with a boundary which is not differentiable everywhere, as long as the singularity points form a σ -null set. Polygonal shapes and pre-fractal domains are prime examples of Lipschitz domains. However, fractal objects such as the Von Koch snowflake (approximated by Figure 1) are nowhere differentiable, and therefore do not satisfy the Lipschitz property.

The fundamental difference between Lipschitz domains and fractal domains can be understood in terms of Hausdorff dimension of the boundary.

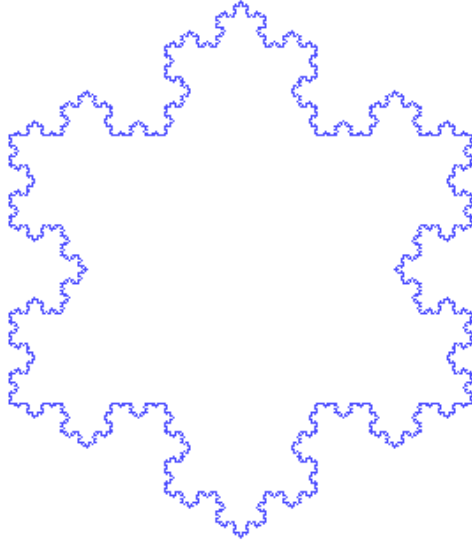


Figure 1: A fifth generation pre-fractal curve of the Von Koch snowflake. (Wikipedia)

Definition 2.2. (*Hausdorff measures and dimension in \mathbb{R}^n*) Let X be a Borel set of \mathbb{R}^n , and let $s > 0$. The s -dimensional Hausdorff measure of X is defined by:

$$H^s(X) := \lim_{r \rightarrow 0} \inf \left\{ \sum_{i \in \mathbb{N}} \text{diam}(A_i)^s \mid \forall i \in \mathbb{N}, \left(A_i \subset \mathbb{R}^n \text{ is a Borel set,} \right. \right. \\ \left. \left. \text{diam}(A_i) < r \right), \text{ and } X \subset \bigcup_{i \in \mathbb{N}} A_i \right\}.$$

The Hausdorff dimension of X is defined by:

$$\dim_H(X) := \inf \{s > 0 \mid H^s(X) = 0\}.$$

While Lipschitz domains have boundaries of Hausdorff dimension $n - 1$, the Hausdorff dimension of fractal domains can be any number in $]n - 2, n[$. In particular, it can take non-integer values, as in the case for the Von Koch snowflake which is of dimension $\frac{\ln 4}{\ln 3} \in]1, 2[$. One idea to widen the class of domains we consider could be to focus on all domains with a boundary of Hausdorff dimension in $]n - 2, n[$.

We may now introduce a class of boundaries which includes fractal shapes such as the Von Koch snowflake: Ahlfors d -regular sets (or d -sets) [57, 58]. In the following definition, we assume $d \in]n - 2, n[$ as it corresponds to the boundaries which are under study; note that the notion of d -set can be defined for all $d \in]0, n[$.

Definition 2.3. (*d -set*) Let $d \in]n - 2, n[$. A Borel set F of \mathbb{R}^n is said to be a d -set if there exists a Borel measure μ supported by F and constants $c_1, c_2 > 0$

such that:

$$\forall x \in F, \forall r \in]0, 1], \quad c_1 r^d \leq \mu(F \cap B_r(x)) \leq c_2 r^d.$$

A measure μ satisfying the condition from Definition 2.3 is called ‘ d -measure’. It was proved in [57, Proposition 1, p. 30] that, on a given d -set, all d -measures are equivalent: Definition 2.3 could be stated using the d -dimensional Hausdorff measure as μ . In particular, the existence of a d -measure supported by the boundary of a domain Ω ensures the Hausdorff dimension of $\partial\Omega$ is d . Naturally, the boundary of any Lipschitz domain is an $(n - 1)$ -set. The Von Koch snowflake is an example of a $\frac{\ln 4}{\ln 3}$ -set.

It goes without saying the class of d -sets is much more general than the class of Lipschitz boundaries for the study of boundary value problems, provided we manage to define boundary condition for such shapes. To go even further, as we explained before, all d -sets have a fixed boundary dimension, and we can be interested in getting rid of that condition. That leads to the introduction of d -upper regular boundaries, studied notably in [53].

Definition 2.4. (*d -upper regular set*) Let $d \in]n - 2, n[$. A Borel set F of \mathbb{R}^n is said to be d -upper regular if there exists a Borel measure μ supported by F and a constant $c > 0$ such that:

$$\forall x \in F, \forall r \in]0, 1], \quad \mu(F \cap B_r(x)) \leq cr^d.$$

In the case of a d -upper regular boundary, the Hausdorff dimension can vary, as long as it is being bounded below by d . For that matter, the class of d -upper regular set greatly expands that of d -sets. Nonetheless, results obtained for domains with such boundaries will depend on the boundary measure μ a priori. It will be pointed out in Subsection 2.2 it is possible to define those boundary conditions for an ever wider class of domains, which is the class of H^1 -extension domains [50].

Definition 2.5. (*H^1 -extension domain*) A domain Ω of \mathbb{R}^n is said to be an H^1 -extension domain if there exists a bounded linear extension operator $E : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$, that is:

- (i) $\forall u \in H^1(\Omega), \quad Eu|_{\Omega} = u,$
- (ii) $\forall u \in H^1(\Omega), \quad \|Eu\|_{H^1(\mathbb{R}^n)} \leq C \|u\|_{H^1(\Omega)},$

where $C > 0$ depends only on Ω (and n).

The class of H^1 -extension domains is very broad: it contains Lipschitz domains [22, 87], d -sets such as the Von Koch snowflake, and even domains with varying boundary dimension (e.g., a Von Koch snowflake on one side, and a Lipschitz boundary on the other side). Also, no boundary measure is specified. A characterization of H^1 -extension domains is given in [50]:

Theorem 2.6. A domain Ω of \mathbb{R}^n is an H^1 -extension domain if and only if it is an n -set (Definition 2.3 for $d = n$), and it holds $H^1(\Omega) = C_2^1(\Omega)$ in the sense of equivalent norms, where $C_2^1(\Omega)$ is the space of sharp maximal functions defined by:

$$C_2^1(\Omega) := \left\{ u \in L^2(\Omega) \mid \left(u_{2,\Omega}^\# : x \mapsto \sup_{r>0} \frac{1}{r} \inf_{c \in \mathbb{R}} \frac{1}{\lambda^{(n)}(B_r(x))} \int_{\Omega \cap B_r(x)} |u - c| \, dy \right) \in L^2(\Omega) \right\},$$

endowed with the norm $\|u\|_{C_2^1(\Omega)} := \|u\|_{L^2(\Omega)} + \|u_{2,\Omega}^\#\|_{L^2(\Omega)}$.

The n -set condition ensures extension domains do not possess an outward cusp, *i.e.* they do not become infinitely thin. The domain in Figure 2 is an example of a domain which is not an extension domain. Note that this domain also has an inward cusp, therefore its complement is not an extension domain either.

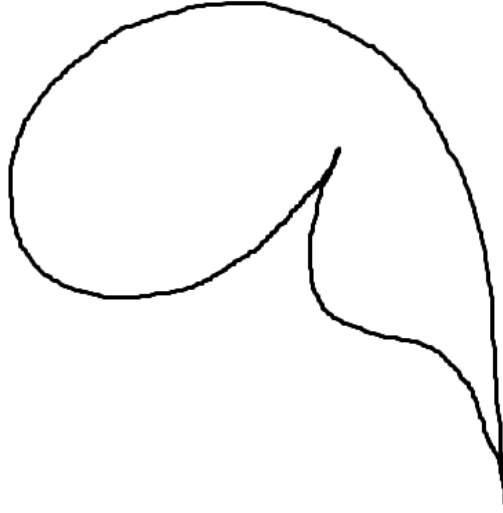


Figure 2: A domain with a cusp.

We introduce the class of domains on which our study will be carried out.

Definition 2.7. (One- and two-sided extension domain) Let Ω be a domain of \mathbb{R}^n . If Ω is bounded and an H^1 -extension domain, then it is a one-sided extension domain.

In addition, if its complement domain $\Omega^c := \mathbb{R}^n \setminus \overline{\Omega}$ is an H^1 -extension domain, and the Hausdorff dimension of $\partial\Omega$ is included in $[n - 1, n[$, then Ω is a two-sided extension domain.

Considering two-sided extension domains will allow to define boundary conditions from both the inside and the outside of the domain, which will be necessary as we consider transmission problems in Subsection 4. The condition

on the dimension of the boundary in Definition 2.7 is to be understood locally, as the dimension can vary along $\partial\Omega$, and ensures Ω is ‘hermetically sealed’ from Ω^c .

2.2 Boundary value problems on extension domains

The study of transmission problems requires the definition of two types of boundary conditions: Dirichlet boundary conditions and Neumann boundary conditions.

The Dirichlet boundary condition is defined using the trace operator. On Lipschitz boundaries, that operator was defined and studied by Marschall [74], shortly after generalized to d -set boundaries by Wallin [94] in 1991, and later on generalized to d -upper regular boundaries [53]. Those approaches rely on the existence of a boundary measure (Lebesgue’s measure in the former case, a d -measure in the latter) to define that operator. In 2009, Biegert [14] uses a different approach based on the H^1 -extension property to define the trace operator on H^1 -extension domains. Here, we rely on the fact that this definition of the trace operator does not depend on the way an element of $H^1(\Omega)$ is extended to $H^1(\mathbb{R}^n)$ [14, Theorem 6.1] to give an equivalent definition, similar to that of Wallin on d -sets [94, Theorem 1] (see also [39, 53]).

Definition 2.8. (Interior trace operator) *Let Ω be a one-sided extension domain of \mathbb{R}^n . The interior trace operator $\text{Tr}_i : H^1(\Omega) \rightarrow \text{Tr}_i(H^1(\Omega))$ is defined quasi-everywhere for $x \in \partial\Omega$ by:*

$$\text{Tr}_i u(x) = \lim_{r \rightarrow 0^+} \frac{1}{\lambda^{(n)}(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) \, dy. \quad (2.1)$$

The ‘quasi-everywhere’ definition is understood in terms of the capacity [14, Definition 5.1]:

$$\text{Cap}(U) := \inf \left\{ \|u\|_{H^1(\Omega)}^2 \mid u \in \overline{C_0^0(\overline{\Omega}) \cap H^1(\Omega)}^{\|\cdot\|_{H^1(\Omega)}} \text{ and } u \geq 1 \text{ on a neighborhood of } U \right\},$$

where U is a subset of $\overline{\Omega}$ and the neighbourhood is in the sense of the trace topology of \mathbb{R}^n on Ω . The use of that capacity allows to circumvent the absence of a specific boundary measure in the definition of H^1 -extension domains (Definition 2.5). The following theorem synthesizes important properties of the interior trace operator.

Theorem 2.9. (Trace theorem) *Let Ω be a one-sided extension domain of \mathbb{R}^n . Then:*

1. *the image of the trace operator $\mathcal{B}(\partial\Omega) := \text{Tr}_i(H^1(\Omega))$ endowed with the norm*

$$\|f\|_{\mathcal{B}(\partial\Omega)} := \min\{\|v\|_{H^1(\Omega)} \mid f = \text{Tr}_i v\}, \quad (2.2)$$

is a Hilbert space.

2. $\text{Ker Tr}_i = H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$ is the closure in $H^1(\Omega)$ of the set of indefinitely differentiable functions with compact support in Ω .
3. the trace operator $\text{Tr}_i : H^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ is a partial isometry with operator norm equal to 1.
4. $\text{Tr}_i : V_1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ defines an isometry, where

$$V_1(\Omega) = \{v \in H^1(\Omega) \mid (-\Delta + 1)v = 0 \text{ weakly on } \Omega\} \quad (2.3)$$

is the space of (weakly) 1-harmonic functions, endowed with the standard norm $\|\cdot\|_{H^1(\Omega)}$. It holds:

$$V_1(\Omega) = \text{Ker}(\text{Tr}_i)^\perp = (H_0^1(\Omega))^\perp. \quad (2.4)$$

Proof. Point 1 follows partially from the definition, see also [52, Theorem 1], [53, Theorem 5.1], [39, Theorem 2], where the norm was defined as:

$$\|f\|_{\text{Tr}_i} = \inf\{\|v\|_{H^1(\Omega)} \mid f = \text{Tr}_i v\}.$$

Here, we specify that the infimum is reached. To prove that, it is sufficient to use the continuity of the trace operator from $H^1(\Omega)$ to its image and Stampacchia's theorem [18, Theorem V.6], ensuring the existence of $u \in H_f^1(\Omega)$, where

$$H_f^1(\Omega) := \{v \in H^1(\Omega) \mid \text{Tr}_i v = f \text{ on } \partial\Omega\}, \quad (2.5)$$

the unique minimizer of $\|\cdot\|_{H^1(\Omega)}$ on $H_f^1(\Omega)$. It is equivalent to the weak well-posedness on $H^1(\Omega)$ of the non-homogeneous Dirichlet boundary value problem for $(-\Delta + 1)$:

$$\begin{cases} -\Delta u + u = 0 & \text{on } \Omega, \\ \text{Tr}_i u = f. \end{cases} \quad (2.6)$$

The weak solution to Problem (2.6) is then understood in the sense of the variational formulation on $H_f^1(\Omega)$:

$$\forall v \in H_f^1(\Omega), \quad \langle u, v \rangle_{H^1(\Omega)} = \|u\|_{H^1(\Omega)}^2, \quad (2.7)$$

which, in particular, implies:

$$\forall v \in H_f^1(\Omega), \quad \left(\|v\|_{H^1(\Omega)} = \|f\|_{\mathcal{B}(\partial\Omega)} \iff u = v \right). \quad (2.8)$$

To prove Point 2, we use the following characterization from [47, Corollary 2.3.1 and Example 2.3.1]:

$$H_0^1(\mathbb{R}^n \setminus \partial\Omega) = \{v \in H^1(\mathbb{R}^n) \mid \tilde{v} = 0 \text{ q.e. on } \partial\Omega\}.$$

$u \in H^1(\Omega)$ can be extended to $H_0^1(\mathbb{R}^n \setminus \partial\Omega)$ if and only if $u \in H_0^1(\Omega)$, and it can be extended to the space on the right hand side of the equality if and only if $\text{Tr}_i u = 0$. Points 3 and 4 are corollaries of the previous points, of Stampacchia's theorem in particular. Let us prove (2.4). For $u \in H^1(\Omega)$, by Green's formula (see Definition 2.12 below), the following equivalences hold:

$$\begin{aligned} u \in V_1(\Omega) &\iff -\Delta u + u = 0 \text{ weakly on } \Omega \\ &\iff \forall v \in \text{Ker}(\text{Tr}_i), \quad \langle u, v \rangle_{H^1(\Omega)} = 0 \\ &\iff u \in \text{Ker}(\text{Tr}_i)^\perp. \end{aligned}$$

Since $\text{Tr}_i : \text{Ker}(\text{Tr}_i)^\perp \rightarrow \mathcal{B}(\partial\Omega)$ is bijection, by (2.8), it is an isometry. Therefore, since $\text{Ker}(\text{Tr}_i)^\perp \subset H^1(\Omega)$ and it is a closed Hilbert space endowed with $\|\cdot\|_{H^1}$ (as an orthogonal complement), the trace operator $\text{Tr}_i : H^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ is a partial isometry of operator norm equal to 1. \square

As it was mentioned before, in the case of a two-sided extension domain, we also define an exterior trace operator in a similar way since Ω^c is an extension domain.

Definition 2.10. (*Exterior trace operator*) *Let Ω be a two-sided extension domain of \mathbb{R}^n . The exterior trace operator $\text{Tr}_e : H^1(\Omega^c) \rightarrow \mathcal{B}(\partial\Omega)$ is defined quasi-everywhere for $x \in \partial\Omega$ by:*

$$\text{Tr}_e u(x) = \lim_{r \rightarrow 0^+} \frac{1}{\lambda^{(n)}(\Omega^c \cap B_r(x))} \int_{\Omega^c \cap B_r(x)} u(y) \, dy. \quad (2.9)$$

The trace operators allow to generalize the notion of 'value at the boundary' and define Dirichlet-type boundary conditions. The idea of the pointwise Definitions 2.8 and 2.10 is to assign to the trace of u at a point $x \in \partial\Omega$ the average of the values taken by u infinitely close to x . A similar result to Theorem 2.9 can be obtained using the exterior trace operator instead of the interior one. In particular, their ranges coincide, in the sense of equivalent norms.

Proposition 2.11. *Let Ω be a two-sided extension domain of \mathbb{R}^n . Then, $\mathcal{B}(\partial\Omega) = \text{Tr}_e(H^1(\Omega^c))$ and the norm defined by:*

$$\|f\|_{\mathcal{B}(\partial\Omega)} := \min\{\|v\|_{H^1(\Omega^c)} \mid f = \text{Tr}_e v\}, \quad (2.10)$$

defines a norm on $\mathcal{B}(\partial\Omega)$ equivalent to $\|\cdot\|_{\mathcal{B}(\partial\Omega)}$.

Proof. We prove the infimum is reached in the same way as in Theorem 2.9. By definition of an H^1 -extension domain, their norms depend on Ω alone, which also defines its complement. Equivalently, an element $u \in H^1(\mathbb{R}^n)$ can be viewed as an extension from either Ω or Ω^c , with coinciding interior and exterior traces on $\partial\Omega$. \square

The Neumann boundary condition is defined using the normal derivation operator, defined in the classical sense by $\nu \cdot \nabla$ at a boundary point, where ν is the outward normal unit vector. Since a Lipschitz boundary is differentiable almost everywhere, that definition of the normal derivative can be used for such boundaries almost everywhere. However, as explained in Subsection 2.1, the normal vector exist for no $x \in \partial\Omega$ *a priori* when working in the general class of extension domains (or even d -sets), since the boundary may be nowhere differentiable (as in the case of the Von Koch snowflake, or any d -set for $d > n - 1$). Therefore, in the case of extension domains, we cannot use the usual definition of the normal derivative as it is done in the regular case. Nonetheless, we may use Green's formula to define a weak normal derivative [67] as a linear continuous functional on the image of the trace, that is as an element of $\mathcal{B}'(\partial\Omega)$.

Definition 2.12. (Weak interior normal derivative) *Let Ω be a one-sided extension domain of \mathbb{R}^n . Let $u \in H_{\Delta}^1(\Omega)$, where that space is defined by (also on pages iii–vi):*

$$H_{\Delta}^1(\Omega) := \{v \in H^1(\Omega) \mid \Delta v \in L^2(\Omega)\}. \quad (2.11)$$

The weak interior normal derivative of u is the element $\psi \in \mathcal{B}'(\partial\Omega)$ such that for all $v \in H^1(\Omega)$, it holds:

$$\langle \psi, \text{Tr}_i v \rangle_{\mathcal{B}', \mathcal{B}} = \int_{\Omega} (\Delta u)v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad (2.12)$$

and we denote $\frac{\partial_i u}{\partial \nu} := \psi$.

The space $H_{\Delta}^1(\Omega)$ is a Hilbert space endowed with the norm [36, Proposition 1.1]:

$$\|\cdot\|_{H_{\Delta}^1(\Omega)} := \sqrt{\|\cdot\|_{H^1(\Omega)}^2 + \|\Delta \cdot\|_{L^2(\Omega)}^2}. \quad (2.13)$$

Note that, in addition to the definition of the weak normal derivative, Definition 2.12 states that Green's formula holds for all $v \in H^1(\Omega)$, *i.e.* that the left-hand side in Equation (2.12) only depends on $\text{Tr}_i v$ rather than v itself [67].

Once again, in the case of a two-sided extension domain, we may also define a weak exterior normal derivative in a similar way.

Definition 2.13. (Weak exterior normal derivative) *Let Ω be a two-sided extension domain of \mathbb{R}^n . Let $u \in H_{\Delta}^1(\Omega^c)$. The weak exterior normal derivative of u is the element $\varphi \in \mathcal{B}'(\partial\Omega)$ such that for all $v \in H^1(\Omega^c)$, it holds:*

$$\langle \varphi, \text{Tr}_e v \rangle_{\mathcal{B}', \mathcal{B}} = - \int_{\Omega^c} (\Delta u)v \, dx - \int_{\Omega^c} \nabla u \cdot \nabla v \, dx, \quad (2.14)$$

and we denote $\frac{\partial_e u}{\partial \nu} := \varphi$.

In the case of a Lipschitz boundary, the weak normal derivatives coincide with those defined in the strong sense. Note the sign convention used in Definition 2.13, which corresponds to using the outward normal vector for the definition of the exterior normal derivative in the regular case. That choice makes the definition of the boundary jump values – which will be used as boundary conditions for transmission problems in Subsection 4 – more intuitive.

Definition 2.14. (Boundary jump values) Let Ω be a two-sided extension domain of \mathbb{R}^n , and define:

$$H_{\partial\Omega}^1(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid \nabla u|_{\Omega \cup \Omega^c} \in L^2(\Omega \cup \Omega^c)^n\}. \quad (2.15)$$

The jump in trace of $u \in H_{\partial\Omega}^1(\mathbb{R}^n)$ across $\partial\Omega$ is defined by:

$$\llbracket \text{Tr } u \rrbracket := \text{Tr}_i u - \text{Tr}_e u.$$

In addition, if $\Delta u|_{\Omega \cup \Omega^c} \in L^2(\Omega \cup \Omega^c)$, then the jump in normal derivative of u across $\partial\Omega$ is defined by:

$$\left[\left[\frac{\partial u}{\partial \nu} \right] \right] := \frac{\partial_i u}{\partial \nu} - \frac{\partial_e u}{\partial \nu}.$$

Remark 2.15. In Definition 2.14, the gradient ∇u is defined in the weak sense as a distribution on \mathbb{R}^n . The statement ‘ $\nabla u|_{\Omega \cup \Omega^c} \in L^2(\Omega \cup \Omega^c)^n$ ’ is to be understood in the following sense:

$$\exists \psi \in L^2(\Omega \cup \Omega^c), \forall \varphi \in C_0^\infty(\Omega \cup \Omega^c), \quad \langle \nabla u, \varphi \rangle_{(C_0^\infty)', C_0^\infty(\Omega \cup \Omega^c)} = \int_{\Omega \cup \Omega^c} \psi \varphi \, dx,$$

where $C_0^\infty(\Omega \cup \Omega^c)$ is defined on pages iii–vi and $(C_0^\infty(\Omega \cup \Omega^c))'$ the space of distributions on $\Omega \cup \Omega^c$. Since all elements of $C_0^\infty(\Omega \cup \Omega^c)$ vanish at the boundary, that space (and consequently $L^2(\Omega \cup \Omega^c)$) ‘ignores’ the behavior of the distribution ∇u on the boundary.

In what follows, such statements involving weak derivatives and L^2 spaces are always to be understood in that sense.

The space $H_{\partial\Omega}^1(\mathbb{R}^n)$ defined by (2.15) is a Hilbert space endowed with the norm:

$$\| \cdot \|_{H_{\partial\Omega}^1(\mathbb{R}^n)} := \sqrt{\| \cdot \|_{L^2(\mathbb{R}^n)}^2 + \| \nabla \cdot \|_{L^2(\Omega \cup \Omega^c)^n}^2}.$$

It is slightly different from the space $H^1(\Omega \cup \Omega^c)$ which contains distributions only defined on $\Omega \cup \Omega^c$ – i.e. linear continuous forms with domain $C_0^\infty(\Omega \cup \Omega^c)$, the elements of which vanish near the boundary. $H_{\partial\Omega}^1(\mathbb{R}^n)$ on the other hand can be seen as a space of distributions on \mathbb{R}^n , which can be applied to functions in $C_0^\infty(\mathbb{R}^n)$ which contains functions non-null near the boundary. In that sense, setting the transmission problem on $H_{\partial\Omega}^1(\mathbb{R}^n)$ instead of $H^1(\Omega \cup \Omega^c)$ will give more regularity to the solution, for it will be identified on the whole space \mathbb{R}^n , and not only on $\Omega \cup \Omega^c$. We also insist on the fact that the space $H^1(\mathbb{R}^n)$ is not suitable for the study of transmission problems, for being in that space is not compatible with having a non homogeneous jump in trace.

2.3 Probabilistic perspective

2.3.1 Variational-probabilistic correspondence

Although the most intuitive approach to a boundary value problem is an analytical approach, such problems are also widely studied from a probabilistic perspective. The link between those approaches lies in the connections between the concepts of Dirichlet forms and Markov processes. Let us start with reminding the definition of Dirichlet forms [31, Definition 1.1.2, Theorem 1.1.3].

Definition 2.16. (*Dirichlet form*) Let (E, m) be a σ -finite measurable space. A pair $(\mathcal{E}, \text{dom}(\mathcal{E}))$ is said to be a Dirichlet form on $L^2(E, m)$ if

(i) $(\mathcal{E}, \text{dom}(\mathcal{E}))$ is a symmetric form on $L^2(E, m)$: $\text{dom}(\mathcal{E})$ is a dense linear subspace of $L^2(E, m)$ and \mathcal{E} is a non-negative definite symmetric bilinear form defined on $\text{dom}(\mathcal{E}) \times \text{dom}(\mathcal{E})$;

(ii) $(\mathcal{E}, \text{dom}(\mathcal{E}))$ is closed: $\text{dom}(\mathcal{E})$ is complete for the inner product

$$\mathcal{E}_1(u, v) := \langle u, v \rangle_{L^2(E, m)} + \mathcal{E}(u, v), \quad u, v \in \text{dom}(\mathcal{E});$$

(iii) $(\mathcal{E}, \text{dom}(\mathcal{E}))$ is Markovian:

$$\forall u \in \text{dom}(\mathcal{E}), \quad v := (0 \vee u) \wedge 1 \in \text{dom}(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(v, v) \leq \mathcal{E}(u, u).$$

If Ω is a domain of \mathbb{R}^n , a very standard example of Dirichlet form which will be of interest here is the form $(\mathcal{E}^\Delta, H^1(\Omega))$, where \mathcal{E}^Δ is the bilinear form associated to the Laplacian $-\Delta$ on Ω :

$$\mathcal{E}^\Delta : (u, v) \in H^1(\Omega)^2 \longmapsto \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

Let us now recall the definition of Markov processes [31, Definition A.1.8] (see also [16, Chapter I, Definition 1.1] for an alternative definition).

Definition 2.17. (*Markov process*) Let $(E, \mathcal{B}(E))$ be a measurable space (endowed with its Borel σ -field). Consider $E_\partial := E \cup \{\partial\}$, where ∂ is an isolated point, endowed with the σ -field:

$$\mathcal{B}(E_\partial) := \mathcal{B}(E) \cup \{A \cup \{\partial\}, A \in \mathcal{B}(E)\}.$$

$X = (S, \mathcal{F}, (X_t)_{t \in [0, \infty]}, (\mathbb{P}_x)_{x \in E_\partial})$ is said to be a Markov process on $(E, \mathcal{B}(E))$ if:

(i) for all $x \in E_\partial$, $(S, \mathcal{F}, (X_t)_{t \in [0, \infty]}, \mathbb{P}_x)$ is a stochastic process with time parameter set $[0, \infty]$ and state space $(E_\partial, \mathcal{B}(E_\partial))$, such that

$$\forall \omega \in S, \quad X_\infty(\omega) = \partial;$$

(ii) for all $t \geq 0$ and $A \in \mathcal{B}(E_\partial)$, the function $x \in E_\partial \mapsto \mathbb{P}_x(X_t \in A)$ is $\mathcal{B}(E_\partial)$ -measurable;

(iii) (**Markov property**) there exists a filtration $(\mathcal{F}_t)_{t \in [0, \infty]}$ such that:

$$\forall x \in E_\partial, \forall s, t \geq 0, \forall A \in \mathcal{B}(E_\partial), \\ \mathbb{P}_x(X_{s+t} \in A \mid \mathcal{F}_t) = \mathbb{P}_{X_t}(X_s \in A) \quad \mathbb{P}_x\text{-a.s.}$$

(iv) $\forall t \geq 0, \quad \mathbb{P}_\partial(X_t = \partial) = 1;$

(v) (**Normality**) $\forall x \in E_\partial, \quad \mathbb{P}_x(X_0 = x) = 1.$

One particular topic of interest for a Markov process X is its transition function $(P_t)_{t \geq 0}$:

$$\forall t \geq 0, \quad P_t : (x, A) \in E \times \mathcal{B}(E) \longmapsto \mathbb{P}_x(X_t \in A).$$

The transition function $(P_t)_{t \geq 0}$ of a Markov process X is such that [31, Theorem A.1.9]:

1. for all $s, t \geq 0$ and all $f : E \rightarrow \mathbb{R}$ bounded measurable, it holds:

$$P_s P_t f = P_{s+t} f,$$

where

$$P_t f : x \in E \longmapsto \int_E f(y) P_t(x, dy);$$

2. for all $x \in E$, $P_0(x, \cdot)$ is the Dirac measure concentrated at x .

Dirichlet forms are analytical objects while Markov processes (and their transition functions) are probabilistic. However, under some assumptions, a correspondence can be made between both concepts. Assume the transition function $(P_t)_{t \geq 0}$ of a Markov process on a measurable space (E, m) with m σ -finite is such that:

- (i) $(P_t)_{t \geq 0}$ is m -symmetric, *i.e.* for all $t \geq 0$ and all $f, g : E \rightarrow \mathbb{R}_+$ measurable, it holds:

$$\int_E (P_t f) g \, dm = \int_E f (P_t g) \, dm;$$

- (ii) for all $x \in E$ and $f : E \rightarrow \mathbb{R}$ bounded continuous on E , it holds:

$$\lim_{t \rightarrow 0} P_t f(x) = f(x),$$

then [31, Lemma 1.1.14], if $(T_t)_{t \geq 0}$ denotes the family of symmetric linear operators on $L^2(E, m)$ defined by:

$$\forall f : E \rightarrow \mathbb{R} \text{ bounded measurable,} \quad T_t f(x) := \int_E f(y) P_t(x, dy), \quad (2.16)$$

$(T_t)_{t \geq 0}$ is a strongly continuous contraction semi-group on $L^2(E, m)$ (i.e., a Feller semi-group), that is for $s, t \geq 0$ and $f \in L^2(E, m)$:

$$T_s \circ T_t = T_{s+t}, \quad \|T_t f\|_{L^2(E, m)} \leq \|f\|_{L^2(E, m)} \quad \text{and} \quad \lim_{t \rightarrow 0} \|T_t f - f\|_{L^2(E, m)} = 0.$$

That semi-group can be used to define a Dirichlet form $(\mathcal{E}, \text{dom}(\mathcal{E}))$ on $L^2(E, m)$ [31, Theorem 1.1.3]:

$$\text{dom}(\mathcal{E}) := \left\{ f \in L^2(E, m) \mid \lim_{t \rightarrow 0} \frac{1}{t} \langle f - T_t f, f \rangle_{L^2(E, m)} < +\infty \right\},$$

$$\forall f, g \in \text{dom}(\mathcal{E}), \quad \mathcal{E}(f, g) := \lim_{t \rightarrow 0} \frac{1}{t} \langle f - T_t f, g \rangle_{L^2(E, m)}.$$

Conversely, a Dirichlet form $(\mathcal{E}, \text{dom}(\mathcal{E}))$ on $L^2(E, m)$ can be used to define a strongly continuous contraction semi-group [31, Theorem 1.1.3]:

$$T_t : f \in L^2(E, m) \mapsto \lim_{\alpha \rightarrow \infty} e^{-t\alpha} \sum_{k=0}^{+\infty} \frac{(t\alpha)^k}{k!} (\alpha G_\alpha)^k f,$$

where G_α , $\alpha \geq 0$, is defined using the Riesz theorem by:

$$\forall f \in L^2(E, m), \quad G_\alpha f \in \text{dom}(\mathcal{E}) \quad \text{and} \quad (\forall v \in \text{dom}(\mathcal{E}),$$

$$\langle f, v \rangle_{L^2(E, m)} = \langle G_\alpha f, v \rangle_{L^2(E, m)} + \alpha \mathcal{E}(G_\alpha f, v)).$$

If $(T_t)_{t \geq 0}$ is associated to a Markov process on (E, m) in the sense of (2.16), then its transition function is m -symmetric.

2.3.2 Probabilistic incorporation of boundary conditions

In Subsection 2.3.1, we highlighted the fundamental connection between a variational approach and a probabilistic approach to solve elliptic problems. Here, we describe the probabilistic method to incorporate different types of boundary conditions.

Stopped and killed processes. Dirichlet boundary conditions are accounted for by considering processes (described in Subsection 2.3.1) stopped upon reaching the boundary. The most standard example is that of harmonic functions expressed in terms of Brownian diffusions stopped as they exit the domain, see for instance [40, Part 2, Sections III.8, IV.3 and VIII.13] where the link between harmonic functions and martingales through Itô's formula is highlighted. That link is thoroughly used in [8, Chapter II] to establish a number of properties of harmonic functions, including their reconstruction from Dirichlet boundary data using Brownian diffusions:

$$\forall x \in \Omega, \quad u(x) = \mathbb{E}_x(f(B_\tau)),$$

where Ω is a domain of \mathbb{R}^n , u is harmonic (in the strong sense of a C^2 function) on Ω and coincides with $f \in C^0(\partial\Omega)$ on $\partial\Omega$, B is an n -dimensional Brownian motion and τ is its exit time from Ω . This result can be generalized to harmonic functions in the variational sense [6], as it will be pointed out in Subsection 3.1.

Similarly to the variational approach, setting a homogeneous boundary condition can be used to focus on the influence of the source term, which leads to the introduction of killed processes, see for instance [32, Chapter 2]. In that case, we adjoin one cemetery point to the domain Ω , which symbolizes the boundary, and consider the process is trapped at that point when it first exits the domain. In the case of the Brownian motion, if the source term is given by $s \in L^\infty(\Omega)$ and setting $s = 0$ for the cemetery point, then the heat diffusion at time $t \geq 0$ is given by [32, Section 2.1]:

$$\forall x \in \Omega, \quad u(t, x) = \mathbb{E}_x(s(B_t)).$$

One important remark is that the spaces in which the Dirichlet boundary condition and the source term are chosen here are not the intuitive spaces we would use with a variational approach, that is $\mathcal{B}(\partial\Omega)$ and $L^2(\Omega)$ respectively. The probabilistic approach to boundary value problems is not only an alternative to the variational approach, those perspectives are also complementary in that sense. We also refer to [9, 31, 41, 42, 47] for the study of stopped and killed processes.

Reflected processes. Neumann boundary conditions are accounted for by considering processes reflected upon hitting the boundary. The most common approach, see for instance [10] for normally reflected processes on Lipschitz boundaries, is to introduce the semi-group associated to the reflected process, in the spirit of Subsection 2.3.1, and deduce the properties of the diffusion from its study. We also refer to [28, 29, 30, 55, 86] for the study of (normally) reflected processes. In [21] is defined a Brownian motion which is obliquely reflected on the boundary, which accounts for a non-homogeneous Neumann boundary condition. While the papers mentioned here mostly introduce a semi-group which incorporates the boundary condition, it is also possible to define a proper probabilistic normal derivative on non-smooth domains as it is done in [49], which coincides – up to a multiplicative constant – with Definition 2.12.

Absorbed processes. Although it is not the main focus of this study, we say a few words about absorbing boundary conditions of order 1 and 2, that is Robin and Wentzell-type boundary conditions. Robin (or third type or mixed) boundary conditions are expressed in the following way:

$$\left(\frac{\partial u}{\partial \nu} + \alpha \operatorname{Tr} u \right) \Big|_{\partial\Omega} = h,$$

where h is given and α is a complex coefficient. This boundary condition corresponds to a solution which is partly absorbed and partly reflected, depending

on the coefficient α . Problems with such a boundary condition were treated using a probabilistic approach in [82] in 1990 on C^3 boundaries (regularity needed for technical reasons in the proof). That approach relies on the study of the semi-group connected to the solution, similarly to (2.16), and allows to solve the problem with lighter hypotheses on the spectrum of the operator than in the variational approach. Recently, a similar approach allowed to treat problems with a Robin boundary conditions on a d -set boundary in [24].

Another type of absorbing boundary conditions is the Wentzell boundary condition, first introduced in [90] in 1959, which is expressed as:

$$\left(\frac{\partial u}{\partial \nu} + \alpha \operatorname{Tr} u + \beta \Delta_{\partial\Omega} u \right) \Big|_{\partial\Omega} = h,$$

where β is another complex coefficient, and $\Delta_{\partial\Omega}$ denotes the tangential Laplacian along $\partial\Omega$. In addition to the absorption-reflection effect of the Robin boundary condition, the Wentzell boundary condition also contains a term ($\beta \Delta_{\partial\Omega} u$) which refers to an adhesion of the process to the boundary: the process enters the boundary at a certain time and exits at another. A probabilistic study of this problem is carried out in [93], by means of the Dirichlet form associated to the diffusing process, and in [68] on a Von Koch snowflake, using the convergence of the associated Dirichlet forms along the pre-fractal sequence to define a Wentzell boundary condition without defining the tangential Laplacian *per say*. We also refer to [15, 46, 56] for the study of problems with Wentzell boundary conditions.

3 Bounded problems

Let us begin our study of problems set on extension domains by considering bounded problems, *i.e.* problems set on a bounded Ω . In this section, we will consider in turn a Dirichlet and a Neumann boundary condition, that is:

$$\begin{cases} (-\Delta + k)u = 0 & \text{on } \Omega \\ \operatorname{Tr}_i u = f \end{cases} \quad (3.1)$$

on the one hand, and:

$$\begin{cases} (-\Delta + k)u = 0 & \text{on } \Omega \\ \frac{\partial_i u}{\partial \nu} = g \end{cases} \quad (3.2)$$

on the other hand, with $k \in \{0, 1\}$, and f, g given boundary conditions.

In Subsection 3.1 and Appendix A.1, we establish the well-posedness of the problems above when Ω is a one-sided extension domain, and prove the associated (1-) harmonic extension operators can be viewed as isometries. We prove both the Dirichlet and Neumann problem have the same range, and in Subsection 3.2 and Appendix A.2, we highlight that connection through the

definition of the Poincaré-Steklov operator. We prove that operator defines an isometry as well, and explicit the connections between the latter and the (1-)harmonic extension operators.

3.1 Harmonic extension operators

The first question one can ask is whether Problems (3.1) and (3.2) are well-posed for $k = 1$ and $k = 0$, and on what spaces. The case $k = 1$ is simpler since 1 is not an eigenvalue for either the Dirichlet Laplacian or the Neumann Laplacian: both problems are well-posed on $H^1(\Omega)$. Introducing the space of weakly 1-harmonic functions on Ω :

$$V_1(\Omega) := \{u \in H^1(\Omega) \mid (-\Delta + 1)u = 0 \text{ weakly on } \Omega\}, \quad (3.3)$$

we can define a Dirichlet 1-harmonic extension operator associated to Problem (3.1), denoted by $\mathcal{D}_\Omega : \mathcal{B}(\partial\Omega) \rightarrow V_1(\Omega)$, as well as a Neumann 1-harmonic extension operator associated to Problem (3.2), denoted by $\mathcal{S}_\Omega : \mathcal{B}'(\partial\Omega) \rightarrow V_1(\Omega)$. In the same way, we can define on the same spaces exterior 1-harmonic extension operators \mathcal{D}_{Ω^c} and \mathcal{S}_{Ω^c} with range $V_1(\Omega^c)$ for the problems set on the complement domain Ω^c (Definitions A.7 and A.11).

However, 0 is an eigenvalue for the Neumann Laplacian, as any constant is a solution to Problem (3.2) for $k = 0$ and $g = 0$: the problem is ill-posed on $H^1(\Omega)$. We are led to introduce a space similar to $H^1(\Omega)$, but on which constants have been removed:

$$H_{(0)}^1(\Omega) := \left\{ u \in H^1(\Omega) \mid \int_{\Omega} u \, dx = 0 \right\},$$

on which $\|\nabla \cdot\|_{L^2(\Omega)^n}$ defines a norm, equivalent to the usual H^1 norm by Poincaré's inequality. The interior trace operator restricted to $H_{(0)}^1(\Omega)$ is no longer onto; let us denote $\mathcal{B}_0(\partial\Omega) := \text{Tr}_i(H_{(0)}^1(\Omega)) \subset \mathcal{B}(\partial\Omega)$. When the problem is set on $H^1(\Omega)$ (as in the case $k = 1$), all boundary conditions are understood in terms of the range of the trace: Dirichlet boundary conditions in $\mathcal{B}(\partial\Omega)$, Neumann boundary conditions in $\mathcal{B}'(\partial\Omega)$. Now that the problem is set on $H_{(0)}^1(\Omega)$, it seems natural to adapt the space of conditions to the range of the trace on $H_{(0)}^1(\Omega)$: Dirichlet boundary conditions in $\mathcal{B}_0(\partial\Omega)$ and Neumann boundary conditions in its dual space, $\mathcal{B}'_0(\partial\Omega)$. For boundary conditions understood in that sense, Problems (3.1) and (3.2) are well-posed on $H_{(0)}^1(\Omega)$ for $k = 0$. Introducing the space of (weakly) harmonic functions on Ω in which the constants have been removed:

$$V_{(0)}(\Omega) := \{u \in H_{(0)}^1(\Omega) \mid -\Delta u = 0 \text{ weakly on } \Omega\},$$

we can define a Dirichlet harmonic extension operator associated to Problem (3.1), denoted by $\tilde{\mathcal{D}}_\Omega : \mathcal{B}_0(\partial\Omega) \rightarrow V_{(0)}(\Omega)$, as well as a Neumann harmonic extension operator associated to Problem (3.2), denoted by $\tilde{\mathcal{S}}_\Omega : \mathcal{B}'_0(\partial\Omega) \rightarrow V_{(0)}(\Omega)$ (Definitions A.7 and A.12).

In Appendix A.1, we prove the space $\mathcal{B}_0(\partial\Omega)$ is an hyperplane of $\mathcal{B}(\partial\Omega)$ with complement the space of constant functions and equip it with a norm so that $\text{Tr}_i : H_{(0)}^1(\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$ defines an isometry. We define normal derivatives for elements of $H_{(0)}^1(\Omega)$ with Laplacian in $L^2(\Omega)$ as elements of $\mathcal{B}'_0(\partial\Omega)$, which we identify with the subspace of $\mathcal{B}'(\partial\Omega)$ of functions null on constants. We prove that Problems (3.1) and (3.2) are well-posed on the spaces $H^1(\Omega)$ if $k = 1$ and $H_{(0)}^1(\Omega)$ if $k = 0$, and define the associated (1-)harmonic extension operators \mathcal{S}_Ω , \mathcal{D}_Ω and $\tilde{\mathcal{S}}_\Omega$ and $\tilde{\mathcal{D}}_\Omega$ as linear continuous operators on the spaces mentioned above. We prove all four of those operators inherit the isometric properties of the trace operator (on the proper spaces) and define isometries themselves.

The Dirichlet harmonic extension from $\partial\Omega$ to Ω can also be understood from a probabilistic perspective, using the method described in Subsection 2.3, for any bounded Borel set Ω . Let $(B_t)_{t \geq 0}$ be an n -dimensional Brownian motion. Let τ be its exit time from Ω :

$$\tau := \inf\{t \geq 0, B_t \in \partial\Omega\}.$$

If $f \in C^0(\partial\Omega)$, then the function defined on Ω by:

$$u : x \mapsto \mathbb{E}_x(f(B_\tau)) = \int_{\partial\Omega} f(y) \mathbb{P}_x(B_\tau \in dy) \quad (3.4)$$

is the weak solution to the problem [6, Proposition 2.5]:

$$\begin{cases} \Delta u = 0 & \text{on } \Omega, \\ \text{Tr}_i u = f & \text{on } \partial\Omega. \end{cases} \quad (3.5)$$

Note that for all $x \in \Omega$, $\tau < \infty$ a.s. since Ω is bounded. If $n = 2$, it is well known the Brownian trajectories are dense in the plane a.s., therefore $\tau < \infty$ a.s. and by [6, Proposition 2.5] once again, the solution to the exterior Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } \Omega^c, \\ \text{Tr}_e u = f & \text{on } \partial\Omega, \end{cases} \quad (3.6)$$

are given by (3.4), only defined on Ω^c instead of Ω .

While the probabilistic approach allows to define a harmonic extension for any bounded Borel subset of \mathbb{R}^n , that extension can only be applied to bounded continuous functions on the boundary (which ensures formula (3.4) is well-defined). On the other hand, the variational approach allows to define the extension operator $\tilde{\mathcal{D}}_\Omega$ on a more general class of functions, $\mathcal{B}_0(\partial\Omega)$ (and even $\mathcal{B}(\partial\Omega)$), provided however that the domain Ω be an extension domain.

3.2 Poincaré-Steklov operator

As it was pointed out in Subsection 3.1, the spaces of solutions of Problems (3.1) and (3.2) coincide (for a given $k \in \{0, 1\}$). The idea of the

Poincaré-Steklov operator, also referred to as ‘Dirichlet-to-Neumann’ operator, is to highlight the correspondence between those problems, as the mapping $\text{Tr}_i u \mapsto \frac{\partial_i u}{\partial \nu}$ where $u \in H^1(\Omega)$ is a solution to the Dirichlet problem (3.1). Generally speaking, considering the operator $(-\Delta + k)$ with $k \in \mathbb{R}$, we may define that operator when k is outside of the spectrum $\sigma(\Delta_D)$ of the Dirichlet Laplacian (Problem (3.1)). It is known that for an arbitrary domain Ω , all Dirichlet eigenvalues of $-\Delta$ are strictly positive and form an unbounded sequence $(\lambda_j)_{j \in \mathbb{N}}$ with $\lambda_j \rightarrow +\infty$ [45, Paragraph 6.5.1, Theorem 1]: k may only be an eigenvalue of the Dirichlet Laplacian Δ_D if $k < 0$ (*i.e.*, in the case of the Helmholtz operator). If $k \in \mathbb{R} \setminus \sigma(\Delta_D)$, the Dirichlet problem for $(-\Delta + k)$ has a unique solution $u \in H^1(\Omega)$ for each boundary condition $f \in \mathcal{B}(\partial\Omega)$. Hence, we can define the Poincaré-Steklov operator \mathcal{A}_k using that weak solution and establish similar properties regarding continuity and invertibility to those stated in [4] in the case of a d -set boundary ($d \in]n - 2, n[$), and generalized in [84] in the case of a domain with Jonsson boundary measures.

Conversely, when $k \in \mathbb{R} \setminus \sigma(\Delta_D)$ is also outside the spectrum $\sigma(\Delta_N)$ of the Neumann Laplacian (Problem (3.2)), the Neumann problem for $(-\Delta + k)$ has a unique solution $u \in H^1(\Omega)$ for each boundary condition $g \in \mathcal{B}'(\partial\Omega)$. In that case, the Poincaré-Steklov operator is invertible. Otherwise, we need to circumvent the ill-posedness of the Neumann problem by restricting the space of boundary conditions for Problems (3.1) and (3.2) so the Dirichlet and Neumann k -harmonic extension operators (defined as in Subsection 3.1) have the same range: that is why we defined the Dirichlet harmonic extension operator $\tilde{\mathcal{D}}_\Omega$ on $\mathcal{B}_0(\partial\Omega)$ instead of $\mathcal{B}(\partial\Omega)$ in the case $k = 0 \in \sigma(\Delta_N) \setminus \sigma(\Delta_D)$.

In Appendix A.2, we define the Poincaré-Steklov operator associated to the problem for $(-\Delta + k)$ with $k \in \mathbb{R} \setminus \sigma(\Delta_D)$ on a one-sided extension domain Ω (Definition A.14), and prove it is linear, continuous and coincides with its adjoint. Focusing on the cases $k = 1 \notin \sigma(\Delta_N)$ and $k = 0 \in \sigma(\Delta_N)$, we identify spaces on which they are invertible, and rely on the properties of the trace operator Tr_i to prove they define (Riesz) isometries (Theorem A.17). Finally, we connect the Poincaré-Steklov operator with the (1-)harmonic extension operators defined in Subsection 3.1.

Condensing all isometry results for the trace operator Tr_i , the 1-harmonic extension operators \mathcal{S}_Ω and \mathcal{D}_Ω , and the Poincaré-Steklov operator for $(-\Delta + 1)$ $\mathcal{A} = \mathcal{A}_1$, we obtain Diagram (3.7), where ϕ denotes the isometry from the Riesz representation theorem from a Hilbert space to its dual (regardless of that space):

$$\begin{array}{ccc}
V_1(\Omega) = H_0^1(\Omega)^\perp & \begin{array}{c} \xrightarrow{\text{Tr}_i = \mathcal{D}_\Omega^{-1}} \\ \xleftarrow{\text{Tr}_i^{-1} = \mathcal{D}_\Omega} \end{array} & \text{Tr}_i(H^1(\Omega)) = \mathcal{B}(\partial\Omega) \\
\begin{array}{c} \curvearrowright \\ \phi^{-1} \\ \phi \\ \curvearrowleft \end{array} & \begin{array}{c} \searrow \\ \mathcal{S}_\Omega^{-1} = \frac{\partial_i}{\partial \nu} \\ \mathcal{S}_\Omega \end{array} & \begin{array}{c} \curvearrowleft \\ \mathcal{A}^{-1} = \phi^{-1} \\ \mathcal{A} = \phi \\ \curvearrowright \end{array} \\
V_1'(\Omega) = (H_0^1(\Omega)^\perp)' & \begin{array}{c} \xrightarrow{(\text{Tr}_i^*)^{-1}} \\ \xleftarrow{\text{Tr}_i^*} \end{array} & (\text{Tr}_i(H^1(\Omega)))' = \mathcal{B}'(\partial\Omega)
\end{array} \tag{3.7}$$

An analogous diagram can be drawn with the operators associated with the Laplacian instead, by removing the constants in all the spaces. Doing so, the Poincaré-Steklov operator for $-\Delta$, $\tilde{\mathcal{A}} = \mathcal{A}_0$, becomes invertible. An analogous diagram of isomorphisms can also be drawn for the exterior problem associated to $(-\Delta + 1)$. The operators mentioned will not be isometric given our definition of $\|\cdot\|_{\mathcal{B}(\partial\Omega)}$ in Theorem 2.9 which relies on the interior trace operator. However, as it was discussed in Proposition 2.11, we could define an equivalent norm using the operator Tr_e instead. Considering that norm for $\mathcal{B}(\partial\Omega)$ (and modifying the norm on $\mathcal{B}'(\partial\Omega)$ accordingly), the operators in the diagram for the exterior problem would be isomorphisms.

4 Transmission problems

In this section, we consider the following transmission problem:

$$\begin{cases} (-\Delta + k)u = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega, \\ \llbracket \text{Tr } u \rrbracket = -f, \\ \llbracket \frac{\partial u}{\partial \nu} \rrbracket = g, \end{cases} \tag{4.1}$$

where $k \in \{0, 1\}$ and f, g are given boundary condition. The spirit of such a problem is to identify the solution based on the influence of the boundary (*i.e.* its jumps across the boundary), rather than its boundary values themselves as in Section 3. The choice of the values of k is due to the fact that 1 is not an eigenvalue of the transmission Laplacian, while 0 is. Naturally, the results described here can be adapted to other values of k , bearing that distinction in mind.

In Subsection 4.1, we introduce the layer potential operators in the Lipschitz case, which are used to find the solution the transmission problem. Those operators are defined using the fundamental solution to the operator under study, and we generalize the link between that function and the harmonic measure in the case of extension domains. The notions of layer potential operators are generalized in Appendix B.1 as linear continuous operators. In Subsection 4.2 and Appendix B.2, we consider the truncated transmission problem

(*i.e.*, the transmission problem set on a large truncation domain instead of \mathbb{R}^n) and prove convergence results of the solutions to those of the original transmission problem in the case of extension domains. In Subsection 4.4, we aim towards generalizing the notion of Poincaré-Steklov operators for transmission problems by giving two possible definitions; we prove those definitions coincide in Appendix B.4. In Subsection 4.3, we describe the operators relevant to the study of boundary values for the layer potential operators in the Lipschitz case, notably the Neumann-Poincaré operator. We generalize those notions in the framework of two-sided extension domains and prove symmetrization formulae for the Neumann-Poincaré operator in Appendix B.3. In Subsection 4.5 and Appendix B.5, we recall the link between contracting operators and Neumann series. We introduce norms on $\mathcal{B}(\partial\Omega)$ and $\mathcal{B}'(\partial\Omega)$ using the operators from the previous section and prove contracting properties for the Neumann-Poincaré operator. We prove the real spectrum of that operator is included in $] -\frac{1}{2}, \frac{1}{2}[$. Finally, in Subsection 4.6, we state results proved in the Lipschitz framework regarding the resolution of imaging problems: the representation formula for the solution and the identification of subdomains in the monotonous case and in the case of disks. We generalize those results in the framework of extension domains in Appendix B.6.

4.1 Layer potential operators

In the classical case – *i.e.*, when the boundary is Lipschitz –, solving the transmission problem (4.1) for $k = 0$ leads to the introduction of the so-called *layer potential operators* (see for instance [1, 61, 75]), defined by:

$$\tilde{\mathcal{S}}_{\partial\Omega} g(x) = \int_{\partial\Omega} \tilde{G}(x-y)g(y) \sigma(dy), \quad x \in \mathbb{R}^n,$$

for the single layer potential operator, and

$$\tilde{\mathcal{D}}_{\partial\Omega} f(x) = \int_{\partial\Omega} \frac{\partial \tilde{G}}{\partial \nu_y}(x-y)g(y) \sigma(dy), \quad x \in \mathbb{R}^n \setminus \partial\Omega,$$

for the double layer potential operator, where \tilde{G} is the fundamental solution to $-\Delta$ on \mathbb{R}^n :

$$\tilde{G} : x \in \mathbb{R}^n \longmapsto \begin{cases} \frac{1}{2\pi} \ln |x| & \text{if } n = 2, \\ \frac{1}{(2-n)\omega_n} |x|^{2-n} & \text{if } n \geq 3, \end{cases}$$

where ω_n is the area of the unit sphere of \mathbb{R}^n . Here, the elements f and g are in $L^2(\partial\Omega, \sigma)$, and $\int_{\partial\Omega} g d\sigma = 0$. For such f and g , the layer potentials $\tilde{\mathcal{S}}_{\partial\Omega} g$ and $\tilde{\mathcal{D}}_{\partial\Omega} f$ are harmonic on $\mathbb{R}^n \setminus \partial\Omega$, and the solution to the transmission problem (4.1) for $k = 0$ can be expressed in terms of the layer potentials as [85, Proposition 3.5.1]:

$$u = \tilde{\mathcal{S}}_{\partial\Omega} g - \tilde{\mathcal{D}}_{\partial\Omega} f. \tag{4.2}$$

This formula is known as Green's third identity.

In the case of a two-sided extension domain Ω , the classical definitions of the layer potentials can no longer be used: Ω has no specified boundary measure, and even then, the normal derivative $\frac{\partial}{\partial \nu} \tilde{G}$ is understood as an element of $\mathcal{B}'(\partial\Omega)$ and might not be integrable; same goes for the boundary condition $g \in \mathcal{B}'_0(\partial\Omega)$. That is why, in order to generalize the notion of layer potential operators in the case of two-sided extension domains, we focus on the connection those operators have with the solution to the transmission problem (Formula (4.2)).

In Appendix B.1, we introduce spaces on which the transmission problems (4.1) for $k = 1$ and $k = 0$ are well-posed. We use a variational approach to define the layer potential operators for the transmission problems (4.1) for $k = 1$ and $k = 0$ and prove those operators can be written in terms of convolutions with the fundamental solution to the associated problem, which goes to show our new definitions generalize the classical definitions of those operators. Such an approach was used in [7], only in a Lipschitz framework. We prove the newly defined operators are linear and continuous, and generalize Green's third identity (4.2) in that framework.

We generalize the connection between the normal derivative of the fundamental solution, used to define the double layer potential operator, and the harmonic measure, proved in [8, Proposition 3.11] in the case of strongly harmonic functions on regular domains. Let Ω be a one-sided extension domain of \mathbb{R}^n . Let Γ be the Green function for the Laplacian on Ω with homogeneous Dirichlet boundary condition:

$$\forall x \in \Omega, \quad \begin{cases} \Delta \Gamma(x, \cdot) = \delta_x & \text{on } \Omega, \\ \text{Tr } \Gamma(x, \cdot) = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.3)$$

If \tilde{G} denotes the fundamental solution to the Laplacian on \mathbb{R}^n , then for $x \in \Omega$, it holds:

$$\Gamma(x, \cdot) = \tilde{G}(x - \cdot) - h_x \quad \text{on } \Omega, \quad (4.4)$$

where h_x is harmonic on Ω and has the same trace as $\tilde{G}(x - \cdot)$ on $\partial\Omega$. Denoting $\Gamma_x := \Gamma(x, \cdot)$, it follows that:

$$\forall x \in \Omega, \quad \Gamma_x \in H_0^1(\Omega). \quad (4.5)$$

The harmonic correction can either be expressed using a variational approach as $h_x = \tilde{\mathcal{D}}(\text{Tr } \tilde{G}(x - \cdot))$, where $\tilde{\mathcal{D}}$ is the harmonic extension defined in Subsection 3.1, or using the probabilistic formula (3.4).

Theorem 4.1. *Let Ω be a one-sided extension domain of \mathbb{R}^n . Then, for all $x \in \Omega$ and $f \in \mathcal{B}(\partial\Omega)$ bounded, it holds:*

$$\left\langle \frac{\partial \Gamma_x}{\partial \nu}, f \right\rangle_{\mathcal{B}', \mathcal{B}(\partial\Omega)} = \frac{1}{\omega_n} \int_{\partial\Omega} f(y) \mathbb{P}_x(B_\tau \in dy), \quad (4.6)$$

where ω_n is the measure of the unit sphere in \mathbb{R}^n .

Proof. We adapt the proof of [8, Proposition 3.11]. Let $x \in \Omega$. Let $\varepsilon > 0$ small enough so that $\overline{B_\varepsilon(x)} \subset \Omega$, where $B_\varepsilon(x)$ is the ball of \mathbb{R}^n of center x and radius ε . Then, by Green's formula on $\Omega \setminus \overline{B_\varepsilon(x)}$, it holds (normal derivatives in duality pairings are in $\mathcal{B}'(\partial\Omega)$ on $\partial\Omega$; those in integrals on in $L^2(\partial B_\varepsilon(x), \sigma)$ on $\partial B_\varepsilon(x)$):

$$\begin{cases} \left\langle \frac{\partial u}{\partial \nu}, \text{Tr } \Gamma_x \right\rangle_{\mathcal{B}', \mathcal{B}(\partial\Omega)} - \int_{\partial B_\varepsilon(x)} \frac{\partial u}{\partial \nu} \Gamma_x \, d\sigma = \int_{\Omega \setminus \overline{B_\varepsilon(x)}} \nabla u \cdot \nabla \Gamma_x \, dy, \\ \left\langle \frac{\partial \Gamma_x}{\partial \nu}, \text{Tr } u \right\rangle_{\mathcal{B}', \mathcal{B}(\partial\Omega)} - \int_{\partial B_\varepsilon(x)} \frac{\partial \Gamma_x}{\partial \nu} u \, d\sigma = \int_{\Omega \setminus \overline{B_\varepsilon(x)}} \nabla u \cdot \nabla \Gamma_x \, dy, \end{cases} \quad (4.7)$$

where σ denotes the $(n-1)$ -dimensional Lebesgue measure.

Since u is harmonic on $B_\varepsilon(x)$, it holds:

$$\int_{\partial B_\varepsilon(x)} \frac{\partial u}{\partial \nu} \Gamma_x \, d\sigma = \int_{B_\varepsilon(x)} \nabla u \cdot \nabla \Gamma_x \, dy \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (4.8)$$

using the regularity of Γ_x from (4.5). In the same way, it holds:

$$\int_{\partial B_\varepsilon(x)} \frac{\partial h}{\partial \nu} u \, d\sigma \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where h is defined by (4.4). Moreover, since $B_\varepsilon(x)$ is a ball, it holds [1, Lemma 2.2]:

$$\int_{\partial B_\varepsilon(x)} \frac{\partial G(x, \cdot)}{\partial \nu} u \, d\sigma = \frac{\varepsilon^{1-n}}{\omega_n} \int_{\partial B_\varepsilon(x)} u \, d\sigma \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\omega_n} u(x).$$

The continuity of u by Weyl's lemma [96, Lemma 2] guarantees the convergence. Hence:

$$\int_{\partial B_\varepsilon(x)} \frac{\partial \Gamma_x}{\partial \nu} u \, d\sigma \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\omega_n} u(x) = \frac{1}{\omega_n} \int_{\partial\Omega} f(y) \mathbb{P}_x(B_\tau \in dy), \quad (4.9)$$

using the representation (3.4) of the harmonic function u in terms of the harmonic measure on $\partial\Omega$.

Rewriting the Green formulae (4.7) with the boundary conditions (4.3) and (3.5), the limits (4.8) and (4.9), and calculating the difference yields (4.6). \square

Corollary 4.2. *Let Ω be a one-sided extension domain of \mathbb{R}^n . Let $x \in \Omega$ and assume there exists a d -upper regular measure μ_x on $\partial\Omega$ for some $d \in]n-2, n[$ (i.e., a measure as in Definition 2.4) such that $\frac{\partial}{\partial \nu} \Gamma_x \in L^1(\partial\Omega, \mu_x)$. Then, the normal derivative of the Green function Γ_x is expressed in terms of the harmonic measure $\mathbb{P}_x(B_\tau \in dy)$ and the boundary measure μ_x as:*

$$\frac{\partial \Gamma_x}{\partial \nu}(y) = \frac{1}{\omega_n} \frac{\mathbb{P}_x(B_\tau \in dy)}{\mu_x(dy)}.$$

4.2 Approximation by truncated problems

A usual approximation [3, 4] of the transmission problem (4.1) consists in considering that the exterior domain Ω^c has been truncated. In other words, we consider a two-sided extension domain Ω of \mathbb{R}^n , and a bounded one-sided extension domain U such that $\Omega \subset\subset U$. We wish to approximate the transmission problem on the whole space (4.1) with $k \in \{0, 1\}$, by a problem set on U :

$$\begin{cases} -\Delta u + ku = 0 & \text{on } U \setminus \partial\Omega, \\ \llbracket \text{Tr } u \rrbracket_{\partial\Omega} = -f, \quad \left[\left[\frac{\partial u}{\partial \nu} \right] \right]_{\partial\Omega} = g, \\ \text{Tr}_i^{\partial U} = 0, \end{cases} \quad (4.10)$$

with $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'(\partial\Omega)$ if $k = 1$ and $g \in \mathcal{B}'_0(\partial\Omega)$ if $k = 0$, where $\llbracket \cdot \rrbracket_{\partial\Omega}$ denotes the boundary jump value across $\partial\Omega$ and $\text{Tr}_i^{\partial U}$ denotes the interior trace operator on ∂U (similar notations will be used for trace operators on $\partial\Omega$).

Note that, in this subsection, only the interior trace operator on ∂U will be considered, as well as normal derivatives on $\partial\Omega$; we will simply denote them by $\text{Tr}^{\partial U}$ and $\frac{\partial_i}{\partial \nu}, \frac{\partial_e}{\partial \nu}$ respectively.

Naturally, our main interest regarding the truncated problem is its convergence towards the transmission problem set on the whole space, when the truncation domain U grows to \mathbb{R}^n . In Appendix B.2, we define the single and double layer potential operators for the truncated problem (4.10) for $k = 1$ and $k = 0$ in the case of extension domains, and prove the space of solutions to the truncated problem can be regarded as the space of solutions to the problem on the whole space (4.1) (using the Dirichlet boundary condition on U). In that space, we establish the convergence of the layer potential operators for the truncated problems to those on the whole space when the truncating domains grows at least as fast as a growing sequence of balls, generalizing the result from [3] in the Lipschitz case and [4] on d -sets.

4.3 Neumann-Poincaré operator and boundary values

The layer potential operators allow to determine the solution to the transmission problem (4.1) from its jumps in trace and normal derivative. One thing we can be interested in knowing about that solution is what its boundary values actually are, since all that was fixed when setting the problem where the jumps. From Proposition B.7, it is known any solution to the transmission problem can be split into a single layer potential part and a double layer potential part, and it seems natural to keep that decomposition in mind when determining the boundary values. The trace value of a single layer potential (denoted by $\tilde{\mathcal{V}}_{\partial\Omega}$ if $k = 0$) as well as the normal derivative value of a double layer potential (denoted by $-\tilde{\mathcal{W}}_{\partial\Omega}$ if $k = 0$) are easily defined [63, 89, 91], due to the absence of jumps in the quantities in question.

From their definition, double layer potentials have a jump in trace. Knowing that jump, it is enough to have access to the average of the traces to recover its trace values. In the case of the transmission problem (4.1) for $k = 0$ across a Lipschitz boundary, that average is given by the Neumann-Poincaré operator [1, 64, 80], defined as the following singular integral:

$$\tilde{\mathcal{K}}_{\partial\Omega}f(x) = \frac{1}{\omega_n} \text{p.v.} \int_{\partial\Omega} \frac{\langle y-x, \nu_y \rangle}{|x-y|^n} f(y) \sigma(dy),$$

with p.v. the Cauchy principal value, ω_n the area of the unit sphere of \mathbb{R}^n and ν the outward normal vector field to $\partial\Omega$. It holds:

$$\tilde{\mathcal{K}}_{\partial\Omega} = \frac{1}{2} (\text{Tr}_i + \text{Tr}_e) \circ \tilde{\mathcal{D}}_{\partial\Omega}. \quad (4.11)$$

In the classical case, that operator is understood as $\tilde{\mathcal{K}}_{\partial\Omega} : L^2(\partial\Omega, \sigma) \rightarrow L^2(\partial\Omega, \sigma)$. One important property of the Neumann-Poincaré operator is that its adjoint, expressed as:

$$\tilde{\mathcal{K}}_{\partial\Omega}^*f(x) = \frac{1}{\omega_n} \text{p.v.} \int_{\partial\Omega} \frac{\langle x-y, \nu_x \rangle}{|x-y|^n} f(y) \sigma(dy),$$

satisfies (see for instance [66, Proposition 12]):

$$\tilde{\mathcal{K}}_{\partial\Omega}^* = \frac{1}{2} \left(\frac{\partial_i}{\partial\nu} + \frac{\partial_e}{\partial\nu} \right) \circ \tilde{\mathcal{S}}_{\partial\Omega}. \quad (4.12)$$

The four boundary operators introduced above are used to defined the Claderón projectors [80, Theorem 3.1.3]:

$$\tilde{\mathcal{C}}_i := \frac{1}{2}I + \tilde{\mathcal{M}} \quad \text{and} \quad \tilde{\mathcal{C}}_e := \frac{1}{2}I - \tilde{\mathcal{M}},$$

where

$$\tilde{\mathcal{M}} := \begin{pmatrix} -\tilde{\mathcal{K}}_{\partial\Omega} & \tilde{\mathcal{V}}_{\partial\Omega} \\ \tilde{\mathcal{W}}_{\partial\Omega} & \tilde{\mathcal{K}}_{\partial\Omega}^* \end{pmatrix} : H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega),$$

the study of which allows to prove the Neumann-Poincaré operators and boundary value operators can be symmetrized:

$$\begin{cases} \tilde{\mathcal{K}}_{\partial\Omega} \tilde{\mathcal{V}}_{\partial\Omega} = \tilde{\mathcal{V}}_{\partial\Omega} \tilde{\mathcal{K}}_{\partial\Omega}^*, \\ \tilde{\mathcal{W}}_{\partial\Omega} \tilde{\mathcal{K}}_{\partial\Omega} = \tilde{\mathcal{K}}_{\partial\Omega}^* \tilde{\mathcal{W}}_{\partial\Omega}, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\mathcal{K}}_{\partial\Omega}^2 + \tilde{\mathcal{V}}_{\partial\Omega} \tilde{\mathcal{W}}_{\partial\Omega} = \frac{1}{4}I, \\ (\tilde{\mathcal{K}}_{\partial\Omega}^*)^2 + \tilde{\mathcal{W}}_{\partial\Omega} \tilde{\mathcal{V}}_{\partial\Omega} = \frac{1}{4}I. \end{cases} \quad (4.13)$$

In Appendix B.3, we define the trace of the single layer potential operator and the normal derivative of the double layer potential operator in the case of two-sided extension domains, as linear and continuous on the trace spaces and their duals. We generalize the notion of Neumann-Poincaré operator for

the transmission problem (4.1) for $k = 1$ and $k = 0$, using Relation (4.11) as our definition, and prove the connection (4.12) between its adjoint and our generalized single layer potential operator still holds. We prove the Neumann-Poincaré operator and its adjoint are continuous. Finally, we generalize the notions of Calderón projectors and prove the symmetrization formulae (4.13) in the framework of two-sided extension domains.

4.4 Poincaré-Steklov operators for transmission problems

We wish to define Poincaré-Steklov operators for the transmission problem associated to $(-\Delta + 1)$ on \mathbb{R}^n . In the case of the bounded problem set on Ω , the interior Poincaré-Steklov operator $\mathcal{A} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ is defined by (see Subsection 3.2):

$$\mathcal{A}(\text{Tr}_i u) = \frac{\partial_i u}{\partial\nu},$$

for every $u \in V_1(\Omega)$ (defined by (2.3)). Although one could be tempted to proceed in the same way, *i.e.* map a Dirichlet boundary condition to a Neumann boundary condition, the solutions to the transmission problem are such that the jump in trace and the jump in normal derivative can be chosen independently. If the space of solutions to the transmission problem is denoted by:

$$V_{1,\partial\Omega}(\mathbb{R}^n) := \{u \in H_{\partial\Omega}^1(\mathbb{R}^n) \mid -\Delta u + u = 0 \text{ weakly on } \mathbb{R}^n \setminus \partial\Omega\},$$

then it holds (Theorem B.7):

$$V_{1,\partial\Omega}(\mathbb{R}^n) = \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega)) \oplus \mathcal{D}_{\partial\Omega}(\mathcal{B}(\partial\Omega)), \quad (4.14)$$

where the direct sum is orthogonal with respect to the norm on $H_{\partial\Omega}^1(\mathbb{R}^n)$ by Green's formula: for $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'(\partial\Omega)$,

$$\begin{aligned} \langle \mathcal{S}_{\partial\Omega} g, \mathcal{D}_{\partial\Omega} f \rangle_{H_{\partial\Omega}^1} &= \int_{\mathbb{R}^n} (\mathcal{S}_{\partial\Omega} g)(\mathcal{D}_{\partial\Omega} f) \, dx + \int_{\Omega \cup \Omega^c} \nabla(\mathcal{S}_{\partial\Omega} g) \cdot \nabla(\mathcal{D}_{\partial\Omega} f) \, dx \\ &= \left\langle \left[\left[\frac{\partial}{\partial\nu}(\mathcal{D}_{\partial\Omega} f) \right] \right], \left[\text{Tr}(\mathcal{S}_{\partial\Omega} g) \right] \right\rangle_{\mathcal{B}', \mathcal{B}} = 0 \end{aligned}$$

The latter spaces are described by:

$$\begin{aligned} \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega)) &= \{u \in V_{1,\partial\Omega}(\mathbb{R}^n) \mid \left[\text{Tr} u \right] = 0\}, \\ \mathcal{D}_{\partial\Omega}(\mathcal{B}(\partial\Omega)) &= \left\{ u \in V_{1,\partial\Omega}(\mathbb{R}^n) \mid \left[\left[\frac{\partial u}{\partial\nu} \right] \right] = 0 \right\}. \end{aligned}$$

Instead of considering any solution to the transmission problem, it seems natural from Decomposition (4.14) to consider solutions with either a homogeneous jump in trace ($u = \mathcal{S}_{\partial\Omega} \left[\left[\frac{\partial u}{\partial\nu} \right] \right]$) or a homogeneous jump in normal derivative ($u = -\mathcal{D}_{\partial\Omega} \left[\text{Tr} u \right]$). In this way, the Poincaré-Steklov operators we wish to

define must link the jump in trace of a certain solution $u = -\mathcal{D}_{\partial\Omega}[\text{Tr } u]$ to the jump in normal derivative of *another* solution $v = \mathcal{S}_{\partial\Omega}[\frac{\partial v}{\partial\nu}]$, where u and v are connected somehow. From [89], in the Lipschitz case, the (interior) Poincaré-Steklov operator for the transmission problem can be defined as follows:

$$\begin{aligned} \mathcal{V}_{\partial\Omega}^{-1} \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega} \right) : \mathcal{B}(\partial\Omega) &\longrightarrow \mathcal{B}'(\partial\Omega) \\ -\llbracket \text{Tr } u \rrbracket &\xrightarrow{(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega})} \text{Tr}_i u = \text{Tr } v \xrightarrow{\mathcal{V}_{\partial\Omega}^{-1}} \left\llbracket \frac{\partial v}{\partial\nu} \right\rrbracket \end{aligned} \quad (4.15)$$

where $u = -\mathcal{D}_{\partial\Omega}[\text{Tr } u]$ and $v = \mathcal{S}_{\partial\Omega}[\frac{\partial v}{\partial\nu}]$.

As explained before, this operator links two solutions to the transmission problem: a solution u with no jump in normal derivative, and a solution v with no jump in trace. The connection between those solutions is done using the interior trace of u . However, one could choose to use the normal derivative of u as the connection, and define the interior Poincaré-Steklov operator as follows:

$$\begin{aligned} - \left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^* \right)^{-1} \mathcal{W}_{\partial\Omega} : \mathcal{B}(\partial\Omega) &\longrightarrow \mathcal{B}'(\partial\Omega) \\ -\llbracket \text{Tr } u \rrbracket &\xrightarrow{-\mathcal{W}_{\partial\Omega}} \frac{\partial u}{\partial\nu} = \frac{\partial_i v}{\partial\nu} \xrightarrow{(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^*)^{-1}} \left\llbracket \frac{\partial v}{\partial\nu} \right\rrbracket \end{aligned} \quad (4.16)$$

where $u = -\mathcal{D}_{\partial\Omega}[\text{Tr } u]$ and $v = \mathcal{S}_{\partial\Omega}[\frac{\partial v}{\partial\nu}]$.

In Appendix B.4, we prove the operators defined by (4.15) and (4.16) coincide, and generalize the notion of interior Poincaré-Steklov operator for the transmission problem (4.1) for $k = 1$ in the framework of two-sided extension domains, and define the exterior Poincaré-Steklov operator in a similar way. Moreover, we give an interpretation of those operators in terms of solutions to the transmission problem.

4.5 Norm estimates and Neumann series

We are interested in the spectral properties of the Neumann-Poincaré operators defined in Subsection 4.3. When defined on $L^2(\partial\Omega, \sigma)$ for Lipschitz boundaries, the spectrum of the operator $\tilde{\mathcal{K}}_{\partial\Omega}^*$ (hence, that of its adjoint $\tilde{\mathcal{K}}_{\partial\Omega}$) lies in $[-\frac{1}{2}, \frac{1}{2}]$ [1, 44, 62]. That property is used to give an explicit formula to two-phased transmission problems (Equation (4.19)), see Subsection 4.6.

It is well-known and easy to verify that the resolvent of an operator L can be expressed in terms of the associated Neumann series:

$$\sum_{k=0}^{+\infty} L^k = (I - L)^{-1}, \quad (4.17)$$

provided the series converges. In particular, that equality is valid anytime $\|L\| < 1$. In the Lipschitz case [89, Theorem 3.2], it is proved the operators

$(-\frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega})$ is (strictly) contractive for the norm on $L^2(\partial\Omega, \sigma)$ defined by:

$$\|\cdot\|_{\tilde{\mathcal{V}}_{\partial\Omega}^{-1}} := \sqrt{\langle \tilde{\mathcal{V}}_{\partial\Omega}^{-1} \cdot, \cdot \rangle_{L^2(\partial\Omega, \sigma)}}, \quad (4.18)$$

where $\tilde{\mathcal{V}}_{\partial\Omega} : L^2(\partial\Omega, \sigma) \rightarrow L^2(\partial\Omega, \sigma)$ is the trace of the single layer potential operator for the Laplacian, defined in Subsection 4.3. In particular, the corresponding Neumann series (4.17) converges.

In Appendix B.5, we define norms in the spirit of Equation (4.18) associated to $\mathcal{V}_{\partial\Omega}$, $\mathcal{V}_{\partial\Omega}^{-1}$ and $\mathcal{W}_{\partial\Omega}$, as well as $\tilde{\mathcal{V}}_{\partial\Omega}$, $\tilde{\mathcal{V}}_{\partial\Omega}^{-1}$ and $\tilde{\mathcal{W}}_{\partial\Omega}$ in the framework of two-sided extension domains: those norms are defined on the trace space and its dual space. In our case, those operators are defined using the trace spaces $\mathcal{B}(\partial\Omega)$ and $\mathcal{B}_0(\partial\Omega)$ and their dual $\mathcal{B}'(\partial\Omega)$ and $\mathcal{B}'_0(\partial\Omega)$. We use the continuity of the operators mentioned above to prove those norms are equivalent to the usual norms on $\mathcal{B}(\partial\Omega)$, $\mathcal{B}'(\partial\Omega)$ and $\mathcal{B}_0(\partial\Omega)$, $\mathcal{B}'_0(\partial\Omega)$, and later on express the equivalence constants using the two-sided extension properties of the domain under study. We generalize the contraction property of the operator $(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega})$ from [89] using the generalized definition of the operator from Appendix B.3, prove a similar property for the operator $(-\frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega})$, and use results from perturbation theory to show the real spectra of the operators $\mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ and $\mathcal{K}_{\partial\Omega}^* : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ are included in $] -\frac{1}{2}, \frac{1}{2}[$, which generalizes the Lipschitz case [27, 83]. We prove similar results for the operators $\tilde{\mathcal{K}}_{\partial\Omega} : \mathcal{B}_0(\partial\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$ and $\tilde{\mathcal{K}}_{\partial\Omega}^* : \mathcal{B}'_0(\partial\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$.

4.6 Imaging

Consider a bounded domain Ω inside which is an inclusion $D \subset\subset \Omega$, characterized by its conductivity $k \in]0, 1[\cup]1, +\infty[$ (Figure 3). We try to identify the inclusion D based only on boundary values on $\partial\Omega$ of the solution to the following problem:

$$\begin{cases} \nabla \cdot \left((1 + (k-1)\mathbf{1}_D) \nabla u \right) = 0 & \text{on } \Omega, \\ \frac{\partial_i u}{\partial \nu} \Big|_{\partial\Omega} = g, \\ \int_{\Omega} u \, dx = 0, \end{cases} \quad (4.19)$$

where g is a given Neumann boundary condition and $\mathbf{1}_D$ denotes the indicator function of the subset D .

In the Lipschitz case [1, 61], the Neumann boundary condition is understood as $g \in L_0^2(\partial\Omega, \sigma)$, where:

$$L_0^2(\partial\Omega, \sigma) = \left\{ g \in L^2(\partial\Omega, \sigma) \mid \int_{\partial\Omega} g \, d\sigma = 0 \right\}.$$

That space is consistent with being the normal derivative of a harmonic function, by Green's formula. Using the classical definition of layer potentials on

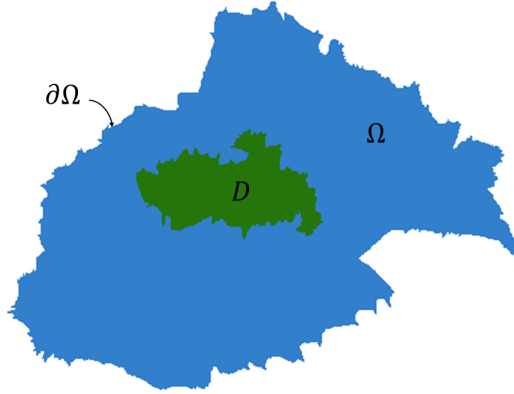


Figure 3: Configuration of an imaging problem in the case of extension domains.

$L^2(\partial\Omega, \sigma)$ given in Subsection 4.1, the solution u to the imaging problem (4.19) can be expressed as [61, Theorem 2.1]:

$$u = H - \tilde{\mathcal{S}}_{\partial D} \varphi \quad \text{where} \quad H = \tilde{\mathcal{S}}_{\partial\Omega} g - \tilde{\mathcal{D}}_{\partial\Omega} f, \quad (4.20)$$

with

$$\left(\frac{k+1}{2(k-1)} I + \tilde{\mathcal{K}}_{\partial D}^* \right) \varphi = \frac{\partial H}{\partial \nu} \Big|_{\partial D},$$

defined using the adjoint of the Neumann-Poincaré operator in the classical sense, see Subsection 4.3, and $f = \text{Tr}_i u$ on $\partial\Omega$. That representation formula is the key ingredient when it comes to subdomain identification; we are specifically interested in two identification results [1] said ‘with one measurement’ (only one value of g is needed): the identification in the monotonous case which allows to prove two possible inclusions are the same under the assumption that one contains the other, and the identification in the case of disks in \mathbb{R}^2 , which identifies inclusions of a fixed shape.

In Appendix B.6, we generalize the representation formula (4.20) in the case of a two-sided extension domain Ω . The Neumann boundary condition is understood in the sense of $\mathcal{B}'_0(\partial\Omega)$ and the operators involved are understood in the generalized variational sense. Then, we generalize the identification results in the monotonous case and in the case of disks of \mathbb{R}^2 . Note that, although the latter result only applies to disks in \mathbb{R}^2 , which are smooth, the larger domain Ω is still a two-sided extension domain: our approach relies on the abstract definition of the layer potential operators and on the general notions of Dirichlet and Neumann boundary conditions, in the sense of the trace operators and the weak normal derivatives.

5 Hilbert transform and Cauchy integral

In this section, we study the Hilbert transform and the Cauchy integrals as operators connected to the layer potential operators. The most fundamental definition of the Hilbert transform can be seen as a convolution with the kernel $y \in \mathbb{R} \mapsto \frac{1}{\pi y}$. That kernel being singular at point 0, that convolution is actually defined as a singular integral [8, Chapter IV, Section 2]:

$$\mathfrak{H}f : x \mapsto \lim_{\substack{\varepsilon \rightarrow 0 \\ N \rightarrow \infty}} \frac{1}{\pi} \int_{\varepsilon < |y| < N} \frac{f(x-y)}{y} dy,$$

for $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the limit exists. In other words, the Hilbert transform is defined as the principal value of the convolution described above [88, Prologue]. Understanding the action of this Hilbert transform can be done by determining its frequency response [8, Chapter IV, Section 2, Proposition 2.3]:

Proposition 5.1. *Let $f \in C_0^1(\mathbb{R})$. Then, the Fourier transform of the Hilbert transform of f is given by:*

$$\widehat{\mathfrak{H}f}(\xi) = -i \operatorname{sgn}(\xi) \hat{f}(\xi) \quad a.e.$$

Therefore, the Hilbert transform acts as a rotation of $\pm \frac{\pi}{2}$ of the frequency components, depending on their sign.

The Hilbert transform is also a topic of interest from a probabilistic perspective, and many of its properties are proved using probabilistic tools, such as martingale transforms and projections. Let $(B_t)_{t \geq 0}$ be an n -dimensional Brownian motion and $(\mathcal{F}_t)_{t \in [0, \infty]}$ be the filtration generated by the process. If $X \in L^2(\mathcal{F}_\infty)$, then there exists a process $(M_t)_{t \geq 0}$ $(\mathcal{F}_t)_{t \geq 0}$ -adapted and satisfying:

$$\mathbb{E} \left(\int_0^{+\infty} |M_s|^2 ds \right) < +\infty,$$

such that [69, Section 5.3, Theorem 5.7]:

$$X = \mathbb{E}(X) + \int_0^{+\infty} H_s \cdot dB_s.$$

Then, if A is an $n \times n$ real matrix, the martingale transform of X by A is defined by [11, Section 1]:

$$A * X := \int_0^{+\infty} AH_s \cdot dB_s \in L^2(\mathcal{F}_\infty).$$

Considering projections of those martingale transforms (*i.e.* considering conditional expectations of those processes) allows to get insightful inequalities, which can be used to prove the Hilbert transform is of strong type (p, p) (see [97, Definition 9.3]) for all $p \in]1, +\infty[$, see [11, 12].

In the complex plane \mathbb{C} , alternative definitions of the Hilbert transform [63, 79, 81] use the boundary of a Lipschitz domain $\partial\Omega$ as singularity points, rather than 0:

$$\mathcal{H}_{\partial\Omega}f(z) := \frac{1}{\pi} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $f : \partial\Omega \rightarrow \mathbb{C}$ is such that the integral is defined and $z \in \mathbb{C}$. In that case, the Hilbert transform is directly connected to the Cauchy integral [81]:

$$\Phi_{\partial\Omega}f = \frac{1}{2i} \mathcal{H}_{\partial\Omega}f.$$

This integral has been widely studied for Lipschitz curves [23, 34, 33], but also on more general curves, while still assuming the existence of a regular enough boundary measure [76]. We wish to use the generalized definitions of the layer potential operators given in Subsection 4.1 to define the Hilbert transform and Cauchy integral over the boundary of any two-sided extension domain.

In Subsection 5.1, we highlight the connection between Hilbert transform and layer potential operators in the complex plane in the Lipschitz case. In Subsection 5.2, we introduce two possible definitions of the Hilbert transform in the case of a two-sided extension domain and describe a method to prove those definitions coincide. In Appendix C, we prove results following that method: the dyadic approximation of general domains and the convergence of the Dirichlet harmonic extension operators.

5.1 Lipschitz case: connection with the layer potential operators

First, let us highlight the connection between the layer potential operators and the Hilbert transform (or the Cauchy integral) in the complex plane. Let Ω be a Lipschitz domain of \mathbb{C} (*i.e.* the image of a Lipschitz domain of \mathbb{R}^2 by the canonical bijection $\mathbb{R}^2 \rightarrow \mathbb{C}$). In a complex framework, we can extend the definitions of the layer potential operators and of the Neumann-Poincaré operator for the transmission problem for the Laplacian across a Lipschitz boundary, given in [1] for instance:

$$\begin{aligned} \tilde{\mathcal{D}}_{\partial\Omega} : H_{\mathbb{C}}^{\frac{1}{2}}(\partial\Omega) &\longrightarrow H_{\mathbb{C}}^1(\mathbb{C} \setminus \partial\Omega) \\ f_R + if_I &\longmapsto \tilde{\mathcal{D}}_{\partial\Omega}f_R + i\tilde{\mathcal{D}}_{\partial\Omega}f_I, \end{aligned}$$

for the double layer potential,

$$\begin{aligned} \tilde{\mathcal{S}}_{\partial\Omega} : H_{\mathbb{C}}^{-\frac{1}{2}}(\partial\Omega) &\longrightarrow H_{\mathbb{C}}^1(\mathbb{C} \setminus \partial\Omega) \\ g_R + ig_I &\longmapsto \tilde{\mathcal{S}}_{\partial\Omega}g_R + i\tilde{\mathcal{S}}_{\partial\Omega}g_I, \end{aligned}$$

for the single layer potential, and

$$\begin{aligned}\tilde{\mathcal{K}}_{\partial\Omega} &: H_{\mathbb{C}}^{\frac{1}{2}}(\partial\Omega) \longrightarrow H_{\mathbb{C}}^{\frac{1}{2}}(\partial\Omega) \\ f_R + if_I &\longmapsto \tilde{\mathcal{K}}_{\partial\Omega}f_R + i\tilde{\mathcal{K}}_{\partial\Omega}f_I,\end{aligned}$$

for the Neumann-Poincaré operator. The definitions of those complex-valued operators display their algebraic decompositions.

Let $f \in H_{\mathbb{C}}^{\frac{1}{2}}(\partial\Omega)$. The Cauchy integral of f over $\partial\Omega$ counter-clockwise is defined by:

$$\Phi_{\partial\Omega}f(z) := \frac{1}{2i\pi} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C}.$$

It is holomorphic on $\mathbb{C} \setminus \partial\Omega$ [79, Section 15]. Moreover, it is connected to the Neumann-Poincaré operator in the following way [63, Section 6]:

$$\tilde{\mathcal{K}}_{\partial\Omega}f(z) = \int_{\partial\Omega} f(\zeta) \operatorname{Re} \left(\frac{d\zeta}{2i\pi(\zeta - z)} \right), \quad z \in \partial\Omega. \quad (5.1)$$

If $f_R \in H^{\frac{1}{2}}(\partial\Omega)$ (\mathbb{R} -valued), then (5.1) yields:

$$\tilde{\mathcal{K}}_{\partial\Omega}f_R(z) = \operatorname{Re}(\Phi_{\partial\Omega}f_R(z)), \quad z \in \partial\Omega. \quad (5.2)$$

Since $\Phi_{\partial\Omega}f_R$ is holomorphic on $\mathbb{C} \setminus \partial\Omega$, its real part is harmonic and by (5.2), $\Phi_{\partial\Omega}f_R$ can be written as:

$$\Phi_{\partial\Omega}f_R(z) = -\tilde{\mathcal{D}}_{\partial\Omega}f_R(z) + ih_R(z), \quad z \in \mathbb{C}, \quad (5.3)$$

where h_R is the harmonic conjugate of $-\tilde{\mathcal{D}}_{\partial\Omega}f_R$. By the Plemelj formulae [79, Section 17], it holds $[[\operatorname{Tr} \Phi_{\partial\Omega}f_R]] = f_R$, which implies h_R (which is harmonic) can be expressed as a single layer potential.

If $f_I \in iH^{\frac{1}{2}}(\partial\Omega)$ ($i\mathbb{R}$ -valued), then (5.1) yields:

$$\tilde{\mathcal{K}}_{\partial\Omega}(if_I)(z) = \operatorname{Im}(\Phi_{\partial\Omega}(if_I)(z)), \quad z \in \partial\Omega. \quad (5.4)$$

In the same way as before, $\Phi_{\partial\Omega}(if_I)$ can be written as:

$$\Phi_{\partial\Omega}(if_I)(z) = \tilde{\mathcal{S}}_{\partial\Omega}g_I(z) - i\tilde{\mathcal{D}}_{\partial\Omega}(if_I)(z), \quad z \in \mathbb{C},$$

where $\tilde{\mathcal{S}}_{\partial\Omega}g_I$ is the anti-harmonic conjugate of $-\tilde{\mathcal{D}}_{\partial\Omega}(if_I)$.

Those considerations yield the following theorem.

Theorem 5.2. *Let Ω be a Lipschitz domain of \mathbb{C} . Let $f \in H_{\mathbb{C}}^{\frac{1}{2}}(\partial\Omega)$.*

1. *If f is real-valued, then $\operatorname{Re}(\Phi_{\partial\Omega}f)$ is a double layer potential and $\operatorname{Im}(\Phi_{\partial\Omega}f)$ is a single layer potential;*
2. *If f is purely imaginary-valued, then $\operatorname{Re}(\Phi_{\partial\Omega}f)$ is a single layer potential and $\operatorname{Im}(\Phi_{\partial\Omega}f)$ is a double layer potential.*

5.2 Non-Lipschitz case

Generalizing a singular integral operator such as the Hilbert transform to two-sided extension domains can be rather tricky, due for instance to the lack of a specified boundary measure. It is often more fruitful to rely on properties of the operator in question to define it from another angle, as it was done in Subsection 4.1. In the case of the Hilbert transform, we wish to use its connection with the Neumann-Poincaré operator given by equation (5.1), and the consequent Theorem 5.2 to give a definition on irregular domains. However, the Plemelj formulae [79, Section 17], which allow to identify the harmonic conjugate in equation (5.3) as a single layer potential in the Lipschitz case, are proved on Lipschitz boundaries, relying on geometrical considerations. Therefore, at this point, we can give two definitions of the Cauchy integral, one as a holomorphic function using the harmonic conjugate:

Definition 5.3. (*Cauchy integral on two-sided extension domains of \mathbb{C} , version 1*) Let Ω be a two-sided extension domain of \mathbb{C} . For $f = f_R + if_I \in \mathcal{B}_{\mathbb{C}}(\partial\Omega)$, we define the Cauchy integral on $\partial\Omega$ as:

$$\begin{cases} \Phi_{\partial\Omega}^1 f_R := -\tilde{\mathcal{D}}_{\partial\Omega} f_R + ih_R, \\ \Phi_{\partial\Omega}^1 (if_I) = \bar{h}_I - i\tilde{\mathcal{D}}_{\partial\Omega}(if_I), \end{cases}$$

where h_R is the harmonic conjugate of $-\tilde{\mathcal{D}}_{\partial\Omega} f_R$ and \bar{h}_I is the anti-harmonic conjugate of $-\tilde{\mathcal{D}}_{\partial\Omega}(if_I)$.

The other definition is as the sum of layer potentials:

Definition 5.4. (*Cauchy integral on two-sided extension domains of \mathbb{C} , version 2*) Let Ω be a two-sided extension domain of \mathbb{C} . For $f = f_R + if_I \in \mathcal{B}_{\mathbb{C}}(\partial\Omega)$, we define the Cauchy integral on $\partial\Omega$ as:

$$\begin{cases} \Phi_{\partial\Omega}^2 f_R := -\tilde{\mathcal{D}}_{\partial\Omega} f_R + i\tilde{\mathcal{S}}_{\partial\Omega} g_R, \\ \Phi_{\partial\Omega}^2 (if_I) = \tilde{\mathcal{S}}_{\partial\Omega} g_I - i\tilde{\mathcal{D}}_{\partial\Omega}(if_I), \end{cases}$$

where $\tilde{\mathcal{S}}_{\partial\Omega} g_R$ is its single layer potential part (Proposition B.7) of the harmonic conjugate of $-\tilde{\mathcal{D}}_{\partial\Omega} f_R$ and $\tilde{\mathcal{S}}_{\partial\Omega} g_I$ is the single layer potential part of the anti-harmonic conjugate of $-\tilde{\mathcal{D}}_{\partial\Omega}(if_I)$.

One conjecture which can be made at this point is that, as in the regular case, those definitions coincide. One way to prove it is to establish the convergence of the layer potential operators as Lipschitz domains converge to an extension domain. In Appendix C.1, we prove that any bounded domain can be approximated by a sequence of Lipschitz domains, from both the inside and the outside. In Appendix C.2, we prove, under some hypotheses, the convergence of the harmonic extension operators defined in Subsection 3.1 when approximating extension domains. What remains to be done is to use

an energy minimization argument to reconstruct the layer potential operators from the harmonic extensions, and establish the convergence of the projectors related to that energy minimization to deduce the convergence of the layer potential operators.

A Bounded problems on extension domains

A.1 Harmonic extension operators

By the Riesz representation theorem, the Neumann boundary value problem (3.2) for $k = 1$ on a one-sided extension domain Ω of \mathbb{R}^n is weakly well-posed, in the sense of the variational formulation on $H^1(\Omega)$ (see also [5]):

$$\forall v \in H^1(\Omega), \quad \langle u, v \rangle_{H^1(\Omega)} = \langle g, \text{Tr}_i v \rangle_{\mathcal{B}', \mathcal{B}} \quad (\text{A.1})$$

with the *a priori* estimate:

$$\|u\|_{H^1(\Omega)} \leq \|g\|_{\mathcal{B}'(\partial\Omega)}. \quad (\text{A.2})$$

The continuity constant in (A.2) is equal to 1, since $\|\text{Tr}_i\|_{\mathcal{L}(H^1(\Omega), \mathcal{B}(\partial\Omega))} = 1$.

Let us define the space of weak solutions to the Neumann boundary value problem for $(-\Delta + 1)$:

$$X_{(-\Delta+1)_N} := \{u \in H^1(\Omega) \mid \exists g \in \mathcal{B}'(\partial\Omega) \text{ s.t. (A.1) holds}\}$$

endowed with the standard H^1 norm. Given the continuity of the solution with respect to $g \in \mathcal{B}'(\partial\Omega)$ (by (A.2)), the space $(X_{(-\Delta+1)_N}, \|\cdot\|_{H^1(\Omega)})$ is closed in $H^1(\Omega)$, *i.e.* it is the Hilbert space. Since any 1-harmonic function is a solution to problem (A.1) associated to its own interior normal derivative, it holds $X_{(-\Delta+1)_N} = V_1(\Omega)$, where $V_1(\Omega)$ is defined by (2.3), and coincides with $\text{Ker}(\text{Tr}_i)^\perp$ by (2.4).

By the well-posedness of (A.1) with the continuity estimate (A.2), the operator of the inverse problem $\mathcal{B}'(\partial\Omega) \rightarrow X_{(-\Delta+1)_N}$ is linear and bounded; it is called the Neumann 1-harmonic extension operator on Ω .

The case $k = 0$ is more tricky since 0 is an eigenvalue of the Laplacian with Neumann boundary condition: any constant is a solution to the homogeneous problem. For Problem (3.2) with $k = 0$ to be well-posed, we need to work on a space which is similar to $H^1(\Omega)$, but in which the only constant is null:

$$H_{\langle 0 \rangle}^1(\Omega) := \left\{ v \in H^1(\Omega) \mid \int_{\Omega} v \, dx = 0 \right\}. \quad (\text{A.3})$$

Note that the constants have been removed in a way which does not involve any boundary measure. That space is endowed with the norm $\|\nabla \cdot\|_{L^2(\Omega)^n}$: since Ω is bounded and $\{0\}$ is closed, then $H_{\langle 0 \rangle}^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, *i.e.*, $(H_{\langle 0 \rangle}^1(\Omega), \|\cdot\|_{H^1(\Omega)})$ is a Hilbert space. As Poincaré's inequality

holds on $H_{\langle 0 \rangle}^1(\Omega)$ by [43], [54, Corollary 3.4], the semi-norm $\|\nabla \cdot\|_{L^2(\Omega)^n}$ on $H^1(\Omega)$ becomes a norm on $H_{\langle 0 \rangle}^1(\Omega)$ equivalent to $\|\cdot\|_{H^1(\Omega)}$. Therefore, we consider the Hilbert space $H_{\langle 0 \rangle}^1(\Omega)$, endowed with the inner product:

$$\langle u, v \rangle_{H_{\langle 0 \rangle}^1(\Omega)} := \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

We prove the restriction of the trace $\text{Tr}_i : H_{\langle 0 \rangle}^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ is not onto and introduce the range of that restriction:

$$\mathcal{B}_0(\partial\Omega) := \text{Tr}_i (H_{\langle 0 \rangle}^1(\Omega)). \quad (\text{A.4})$$

Lemma A.1. *Let Ω be a one-sided extension domain of \mathbb{R}^n . Then, it holds:*

$$\mathcal{B}(\partial\Omega) = \mathcal{B}_0(\partial\Omega) \oplus \mathbb{R}.$$

Proof. By [38, Theorem 7], for every $f \in \mathcal{B}(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ to (3.1) for $k = 0$. Moreover, there exists a unique $u_0 \in H_{\langle 0 \rangle}^1(\Omega)$ such that $\Delta u = 0$ weakly on Ω and $\frac{\partial_i}{\partial \nu} u = \frac{\partial_i}{\partial \nu} u_0$. Since Ω is connected, Green's formula for the harmonic function $u - u_0$ implies there exists $c \in \mathbb{R}$ such that $u - u_0 = c$, hence f is uniquely decomposed as $f = \text{Tr}_i u_0 + c$, with $\text{Tr}_i u_0 \in \mathcal{B}_0(\partial\Omega)$. \square

By analogy with Theorem 2.9, we state the following theorem.

Theorem A.2. *The space $\mathcal{B}_0(\partial\Omega)$ is a Hilbert space endowed with the norm:*

$$\|f\|_{\mathcal{B}_0(\partial\Omega)} := \min \left\{ \|v\|_{H_{\langle 0 \rangle}^1(\Omega)} \mid \text{Tr}_i v = f \text{ on } \partial\Omega \right\}.$$

That minimum is reached for a unique value of $u \in H_{\langle 0 \rangle}^1(\Omega)$, which is the unique element such that $\Delta u = 0$ on Ω and $\text{Tr}_i u = f$ on $\partial\Omega$. Consequently, the trace operator $\text{Tr}_i : V_{\langle 0 \rangle}(\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$ defines an isometry between the subspace of harmonic functions in $H_{\langle 0 \rangle}^1(\Omega)$:

$$V_{\langle 0 \rangle}(\Omega) := \left\{ u \in H_{\langle 0 \rangle}^1(\Omega) \mid \Delta u = 0 \text{ weakly on } \Omega \right\}, \quad (\text{A.5})$$

endowed with the norm $\|\nabla \cdot\|_{L^2(\Omega)^n}$, and $\mathcal{B}_0(\partial\Omega)$. In addition, the adjoint of the trace operator $\text{Tr}_i^ : \mathcal{B}'_0(\partial\Omega) \rightarrow V'_{\langle 0 \rangle}(\Omega)$, and the inverse operators $\text{Tr}_i^{-1} : \mathcal{B}_0(\partial\Omega) \rightarrow V_{\langle 0 \rangle}(\Omega)$ and $(\text{Tr}_i^*)^{-1} : V'_{\langle 0 \rangle}(\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$ are isometric as well. In particular, it holds:*

$$\forall g \in \mathcal{B}'_0(\partial\Omega), \quad \|g\|_{\mathcal{B}'_0(\partial\Omega)} = \|\text{Tr}_i^* g\|_{H^1(\Omega)'}, \quad (\text{A.6})$$

$$\forall u \in V_{\langle 0 \rangle}(\Omega), \quad \|u\|_{H_{\langle 0 \rangle}^1(\Omega)} = \|\text{Tr}_i u\|_{\mathcal{B}_0(\partial\Omega)}. \quad (\text{A.7})$$

The following proposition is a direct consequence of the theorem above.

Proposition A.3. *For every $f \in \mathcal{B}_0(\partial\Omega)$, it holds:*

$$\|f\|_{\mathcal{B}(\partial\Omega)} \leq \|f\|_{\mathcal{B}_0(\partial\Omega)}.$$

Moreover, the equality is strict whenever $f \neq 0$.

In particular, the injection of $\mathcal{B}_0(\partial\Omega)$ into $\mathcal{B}(\partial\Omega)$ is continuous.

Proof. Let us denote by u_1 the only element of $V_1(\Omega)$ such that $\text{Tr}_i u_1 = f$, and by $u_{\langle 0 \rangle}$ the only element of $V_{\langle 0 \rangle}(\Omega)$ such that $\text{Tr}_i u_{\langle 0 \rangle} = f$. Then, by definition of the norm on $\mathcal{B}(\partial\Omega)$ and Theorems 2.9 and A.2, we have:

$$\|u_1\|_{H^1(\Omega)} = \|f\|_{\mathcal{B}(\partial\Omega)} \leq \|u_{\langle 0 \rangle}\|_{H_{\langle 0 \rangle}^1(\Omega)} = \|f\|_{\mathcal{B}_0(\partial\Omega)}.$$

Furthermore, the inequality becomes an equality if and only if $u_1 = u_{\langle 0 \rangle}$, which implies $u_1 = 0$ and $f = 0$. \square

In the theorems defining the interior normal derivative in the weak sense [67], the surjectivity of the trace operator is a key element. Therefore, for every $u \in H_{(0),\Delta}^1(\Omega) := \{u \in H_{(0)}^1(\Omega) \mid \Delta u \in L^2(\Omega)\}$, it appears natural to define its normal derivative as an element of the dual space $\mathcal{B}'_0(\partial\Omega)$, in the same way as it was done in Definitions 2.12 and 2.13.

Definition A.4. (Weak interior normal derivative in $\mathcal{B}'_0(\partial\Omega)$) *The weak interior normal derivative in $\mathcal{B}'_0(\partial\Omega)$ of $u \in H_{(0),\Delta}^1(\Omega)$ is the element $\psi \in \mathcal{B}'_0(\partial\Omega)$ such that for all $v \in H_{(0)}^1(\Omega)$, it holds:*

$$\langle \psi, \text{Tr}_i v \rangle_{\mathcal{B}'_0, \mathcal{B}_0} = \int_{\Omega} (\Delta u) v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx,$$

and we denote $\frac{\partial_i^\circ u}{\partial \nu} = \psi$.

Obviously, for every $u \in H_{(0),\Delta}^1(\Omega)$, it holds $u \in H_{\Delta}^1(\Omega)$, defined by Equation (2.11). Therefore, we could also use Definition 2.12 to define $\frac{\partial_i u}{\partial \nu} \in \mathcal{B}'(\partial\Omega)$. However, that choice could be misleading: since Problem (3.2) for $k = 0$ is set on $H_{(0)}^1(\Omega)$, only the restriction $\frac{\partial_i u}{\partial \nu}|_{\mathcal{B}_0} = \frac{\partial_i^\circ u}{\partial \nu} : \mathcal{B}_0(\partial\Omega, \mu) \rightarrow \mathbb{R}$ intervenes in the weak formulation of the problem (see (A.9) below). Therefore, the identification $\frac{\partial_i u}{\partial \nu} = g$ is only valid when we apply those functions to elements of $\mathcal{B}_0(\partial\Omega)$, which means the Neumann boundary condition of Problem (3.2) for $k = 0$ must be understood in the space $\mathcal{B}'_0(\partial\Omega)$.

One might nonetheless wish to understand that condition in $\mathcal{B}'(\partial\Omega)$. In that case, since it holds (Lemma A.1):

$$\mathcal{B}(\partial\Omega) = \mathcal{B}_0(\partial\Omega) \oplus \mathbb{R},$$

we only need to set the value of $\langle \frac{\partial_i^\circ u}{\partial \nu}, 1 \rangle_{\mathcal{B}', \mathcal{B}}$ when u is (weakly) harmonic. By Green's formula, it must hold:

$$\left\langle \frac{\partial_i^\circ u}{\partial \nu}, 1 \right\rangle_{\mathcal{B}', \mathcal{B}} = \int_{\Omega} \Delta u \, dx = 0.$$

Therefore, setting 0 on the constants is the only possible extension of a Neumann boundary condition $g \in \mathcal{B}'_0(\partial\Omega)$ of problem (3.2) for $k = 0$ to $\mathcal{B}'(\partial\Omega)$. The following theorem states that considering a Neumann boundary condition in $\mathcal{B}'_0(\partial\Omega)$ or extended to an element of $\mathcal{B}'(\partial\Omega)$ in such a way is equivalent.

Theorem A.5. *Let us define the set:*

$$\mathcal{B}'_{0,ext}(\partial\Omega) := \{g \in \mathcal{B}'(\partial\Omega) \mid \langle g, 1 \rangle_{\mathcal{B}', \mathcal{B}} = 0\} \quad (\text{A.8})$$

of extensions of the elements of $\mathcal{B}'_0(\partial\Omega)$ to $\mathcal{B}'(\partial\Omega)$. Then, the restriction operator:

$$\begin{aligned} \cdot|_{\mathcal{B}_0} : \mathcal{B}'_{0,ext}(\partial\Omega) &\rightarrow \mathcal{B}'_0(\partial\Omega) \\ g &\mapsto g|_{\mathcal{B}_0} \end{aligned}$$

defines an isometry between $\mathcal{B}'_{0,ext}(\partial\Omega)$ and $\mathcal{B}'_0(\partial\Omega)$.

Proof. Let $g \in \mathcal{B}'_{0,ext}(\partial\Omega)$. Since $\mathcal{B}(\partial\Omega) = \mathcal{B}_0(\partial\Omega) \oplus \mathbb{R}$, it holds:

$$\begin{aligned} \|g\|_{\mathcal{B}'} &= \sup_{\|f\|_{\mathcal{B}}=1} |\langle g, f \rangle_{\mathcal{B}', \mathcal{B}}| \\ &= \max \left(\sup_{\|f_0\|_{\mathcal{B}_0}=1} |\langle g|_{\mathcal{B}_0}, f_0 \rangle_{\mathcal{B}'_0, \mathcal{B}_0}|, \underbrace{|\langle g, c \rangle_{\mathcal{B}', \mathcal{B}}|}_{=0} \right) \quad \text{for } c \in \mathbb{R}, \|c\|_{\mathcal{B}} = 1 \\ &= \sup_{\|f_0\|_{\mathcal{B}_0}=1} |\langle g|_{\mathcal{B}_0}, f_0 \rangle_{\mathcal{B}'_0, \mathcal{B}_0}| \\ &= \|g|_{\mathcal{B}_0}\|_{\mathcal{B}'_0}. \end{aligned}$$

Therefore $\cdot|_{\mathcal{B}_0}$ defines an isometry between $\mathcal{B}'_{0,ext}(\partial\Omega)$ and $\mathcal{B}'_0(\partial\Omega)$. \square

Problem (3.2) for $k = 0$ is understood in the following variational sense:

$$\forall v \in H^1_{(0)}(\Omega), \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx = \langle g, \text{Tr}_i v \rangle_{\mathcal{B}'_0, \mathcal{B}_0}, \quad (\text{A.9})$$

with $g \in \mathcal{B}'_0(\partial\Omega)$ and $\text{Tr}_i : H^1_{(0)}(\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$.

As for the operator $(-\Delta + 1)$, we can use the Riesz representation theorem and the isometric properties from Theorem A.2, to obtain the following result.

Proposition A.6. *Let Ω be a one-sided extension domain of \mathbb{R}^n . For all $g \in \mathcal{B}'_0(\partial\Omega)$ there exists a unique solution $u \in H^1_{(0)}(\Omega)$ of the variational formulation (A.9). Moreover,*

$$\|\nabla u\|_{L^2(\Omega)^n} \leq \|g\|_{\mathcal{B}'(\partial\Omega)}.$$

We provide the analogous construction for the Neumann value problem (3.2) associated to $-\Delta$ in the class of functions with the mean value zero on the

boundary. The problem is weakly well-posed by Proposition A.6 on $H_{(0)}^1(\Omega)$ (see (A.3)). This time, we set

$$X_{(-\Delta)_N} := \{u \in H_{(0)}^1(\Omega) \mid \exists g \in \mathcal{B}'_0(\partial\Omega) \text{ s.t. (A.9) holds}\}, \quad (\text{A.10})$$

endowed with the norm $\|u\|_{H_{(0)}^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)^n}$. Then $(X_{(-\Delta)_N}, \|\cdot\|_{H_{(0)}^1(\Omega)})$ is a Hilbert subspace of $H^1(\Omega)$. Since any element of $V_{(0)}(\Omega)$ (see (A.5) for the definition) is a solution to problem (A.9) associated to its own interior normal derivative, it holds $X_{(-\Delta)_N} = V_{(0)}(\Omega)$, and $(V_{(0)}(\Omega), \|\cdot\|_{H_{(0)}^1(\Omega)})$ is a Hilbert space.

Definition A.7. (Neumann (1-)harmonic extension operator) For any one-sided extension domain Ω of \mathbb{R}^n , let us define:

(i) the Neumann 1-harmonic extension operator on Ω by:

$$\begin{aligned} \mathcal{S}_\Omega : \mathcal{B}'(\partial\Omega) &\rightarrow \mathcal{S}_\Omega(\mathcal{B}'(\partial\Omega)) = V_1(\Omega) \subset H^1(\Omega) \\ g &\mapsto u_g \end{aligned}$$

where u_g is the unique solution to the variational formulation (A.1) corresponding to $g \in \mathcal{B}'(\partial\Omega)$:

$$\forall v \in H^1(\Omega), \quad \langle \mathcal{S}_\Omega g, v \rangle_{H^1(\Omega)} = \langle g, \text{Tr}_i v \rangle_{\mathcal{B}', \mathcal{B}}. \quad (\text{A.11})$$

(ii) the Neumann harmonic extension operator on Ω by:

$$\begin{aligned} \tilde{\mathcal{S}}_\Omega : \mathcal{B}'_0(\partial\Omega) &\rightarrow \tilde{\mathcal{S}}_\Omega(\mathcal{B}'_0(\partial\Omega)) = V_{(0)}(\Omega) \subset H_{(0)}^1(\Omega) \\ g &\mapsto u_g \end{aligned}$$

where u_g is the unique solution to the variational formulation (A.9) corresponding to $g \in \mathcal{B}'_0(\partial\Omega)$:

$$\forall v \in H_{(0)}^1(\Omega), \quad \int_\Omega \nabla(\tilde{\mathcal{S}}_\Omega g) \cdot \nabla v \, dx = \langle g, \text{Tr}_i v \rangle_{\mathcal{B}'_0, \mathcal{B}_0}, \quad (\text{A.12})$$

As it was shown previously, from the well-posedness of the Neumann value problems for $(-\Delta+1)$ and $-\Delta$, we can also deduce the Neumann (1-)harmonic extension operators are isometric.

Theorem A.8. If Ω is a one-sided extension domain of \mathbb{R}^n , then

(i) the Neumann 1-harmonic extension operator, $\mathcal{S}_\Omega : \mathcal{B}'(\partial\Omega) \rightarrow V_1(\Omega)$, is an isometry:

$$\forall g \in \mathcal{B}'(\partial\Omega), \quad \|\mathcal{S}_\Omega g\|_{H^1(\Omega)} = \|g\|_{\mathcal{B}'(\partial\Omega)}, \quad (\text{A.13})$$

and thus is linear and continuous;

(ii) the Neumann harmonic extension operator, $\tilde{\mathcal{S}}_\Omega : \mathcal{B}'_0(\partial\Omega) \rightarrow V_{\langle 0 \rangle}(\Omega)$, is an isometry:

$$\forall g \in \mathcal{B}'_0(\partial\Omega), \quad \|\tilde{\mathcal{S}}_\Omega g\|_{H^1_{\langle 0 \rangle}(\Omega)} = \|g\|_{\mathcal{B}'_0(\partial\Omega)}, \quad (\text{A.14})$$

and thus is linear and continuous.

Proof. This is a direct consequence of the well-posedness of the variational formulations and Theorems 2.9 and A.2 respectively. \square

Remark A.9. Since $\mathcal{S}_\Omega : \mathcal{B}'(\partial\Omega) \rightarrow V_1(\Omega)$ and $\tilde{\mathcal{S}}_\Omega : \mathcal{B}'(\partial\Omega) \rightarrow V_{\langle 0 \rangle}(\Omega)$ are linear bijections between infinite dimensional Hilbert spaces, they cannot be compact operators. However, since Ω is a Sobolev extension domain, the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact and we can deduce the operators $\mathcal{S}_\Omega : \mathcal{B}'(\partial\Omega) \rightarrow L^2(\Omega)$ and $\tilde{\mathcal{S}}_\Omega : \mathcal{B}'_0(\partial\Omega) \rightarrow L^2(\Omega)$ are compact.

The isometric properties also hold for the trace of the Neumann (1-)harmonic extension operators, by analogy with [5], as a composition of two isometries:

Theorem A.10. Let Ω be a one-sided extension domain of \mathbb{R}^n . Then, the trace of the Neumann (1-)harmonic extension operators on Ω :

- (i) $\mathcal{S}|_{\partial\Omega} := \text{Tr}_i \circ \mathcal{S}_\Omega : g \mapsto \text{Tr}_i u_g$ where u_g is the solution to (A.1) associated to g , defines an isometry from $\mathcal{B}'(\partial\Omega)$ to $\mathcal{B}(\partial\Omega)$ which coincides with its adjoint. In particular, the bilinear form defined by $(g, h) \mapsto \langle g, \mathcal{S}|_{\partial\Omega} h \rangle_{\mathcal{B}', \mathcal{B}}$ is equal to the usual inner product on $\mathcal{B}'(\partial\Omega)$.
- (ii) $\tilde{\mathcal{S}}|_{\partial\Omega} := \text{Tr}_i \circ \tilde{\mathcal{S}}_\Omega$, mapping g to $\text{Tr}_i u_g$ where u_g is the solution to (A.9) associated to $g \in \mathcal{B}'_0(\partial\Omega)$, defines an isometry from $\mathcal{B}'_0(\partial\Omega)$ to $\mathcal{B}_0(\partial\Omega)$ and coincides with its adjoint.

Proof. Given its definition, $\mathcal{S}|_{\partial\Omega}$ is a linear operator. For $g, h \in \mathcal{B}'(\partial\Omega)$, it holds:

$$\begin{aligned} \langle g, \mathcal{S}|_{\partial\Omega} h \rangle_{\mathcal{B}', \mathcal{B}} &= \langle g, \text{Tr}_i u_h \rangle_{\mathcal{B}', \mathcal{B}} \\ &= \langle u_g, u_h \rangle_{H^1(\Omega)} \\ &= \langle h, \mathcal{S}|_{\partial\Omega} g \rangle_{\mathcal{B}', \mathcal{B}} \\ &= \langle \mathcal{S}|_{\partial\Omega} g, h \rangle_{\mathcal{B}'', \mathcal{B}'}, \end{aligned}$$

since $\mathcal{B}(\partial\Omega)$ is a Hilbert space and can be identified to its bidual $\mathcal{B}''(\partial\Omega)$, where u_g and u_h are the unique solutions of (A.1) respectively associated to g and h . Hence $\mathcal{S}|_{\partial\Omega}$ coincides with its adjoint.

Furthermore, by Theorem A.8, for $g \in \mathcal{B}'(\partial\Omega)$:

$$\langle g, \mathcal{S}|_{\partial\Omega} g \rangle_{\mathcal{B}', \mathcal{B}} = \|u_g\|_{H^1(\Omega)}^2 = \|g\|_{\mathcal{B}'(\partial\Omega)}^2,$$

Therefore, $(g, h) \mapsto \langle g, \mathcal{S}|_{\partial\Omega} h \rangle$ is equal to the usual inner product on $\mathcal{B}'(\partial\Omega)$. Hence, as the composition of two isometries, $\mathcal{S}|_{\partial\Omega} : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ is an isometry itself.

In the same way, we prove (ii). \square

As for the Neumann (1-)harmonic extensions, we use the well-posedness of the corresponding Dirichlet boundary problems and define the Dirichlet (1-)extension operators.

Definition A.11. (*Dirichlet 1-harmonic extension operator*) For any one-sided extension domain Ω of \mathbb{R}^n , let us define \mathcal{D}_Ω , the Dirichlet 1-harmonic extension operator on Ω by:

$$\begin{aligned} \mathcal{D}_\Omega : \mathcal{B}(\partial\Omega) &\rightarrow \mathcal{D}_\Omega(\mathcal{B}(\partial\Omega)) = V_1(\Omega) \subset H^1(\Omega) \\ f &\mapsto u^f \end{aligned}$$

where u^f is the unique weak solution to the variational formulation (2.6) associated to f :

$$\forall v \in H_f^1(\Omega), \quad \langle \mathcal{D}_\Omega f, v \rangle_{H^1(\Omega)} = \left\langle \frac{\partial_i(\mathcal{D}_\Omega f)}{\partial \nu}, f \right\rangle_{\mathcal{B}, \mathcal{B}}. \quad (\text{A.15})$$

In Definition A.11, the space $\mathcal{D}_\Omega(\mathcal{B}(\partial\Omega)) = V_1(\Omega)$ is the space of weak solutions of the Dirichlet $(-\Delta + 1)$. It is the same as in Definition A.7 considered as the space of weak solutions for the Neumann $(-\Delta + 1)$.

For the Laplace equation with Dirichlet boundary conditions on a one-sided extension domain Ω of \mathbb{R}^n :

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega, \\ \text{Tr}_i u = f, \end{cases} \quad (\text{A.16})$$

we directly use [38, Theorem 7]: for all $f \in \mathcal{B}(\partial\Omega)$, there exists a unique weak solution $u \in H^1(\Omega)$ with $\text{Tr}_i u = f$, and it is such that:

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \cdot \nabla v \, dx = 0. \quad (\text{A.17})$$

Hence, it allows us to define the Dirichlet harmonic extension operator in the same way:

Definition A.12. (*Dirichlet harmonic extension operator*) For any one-sided extension domain Ω of \mathbb{R}^n , let us define $\tilde{\mathcal{D}}$, the Dirichlet harmonic extension operator by:

$$\begin{aligned} \tilde{\mathcal{D}}_\Omega : \mathcal{B}_0(\partial\Omega) &\rightarrow \tilde{\mathcal{D}}_\Omega(\mathcal{B}_0(\partial\Omega)) = V_{\{0\}}(\Omega) \subset H^1(\Omega) \\ f &\mapsto u^f \end{aligned}$$

where u^f is the unique solution to the variational formulation (A.17) associated to $f \in \mathcal{B}_0(\partial\Omega)$.

As before, the operator $\tilde{\mathcal{D}}_\Omega$ is understood as mapping the Dirichlet boundary data to the associated weak solution, *i.e.* in Definition A.12 the space

$V_{(0)}(\Omega)$ is the space of the weak solutions for the Dirichlet $-\Delta$. As in the case of $(-\Delta + 1)$, it holds $\tilde{\mathcal{D}}_\Omega(\mathcal{B}_0(\partial\Omega)) = \tilde{\mathcal{S}}_\Omega(\mathcal{B}'_0(\partial\Omega))$ (equal to $V_{(0)}(\Omega)$ for $-\Delta$).

We deduce the continuity and the isometric properties of those Dirichlet (1-)harmonic extension operators from the isometry of the corresponding trace operators, $\text{Tr}_i : V_1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ (see Theorem 2.9) and $\text{Tr}_i : V_{(0)}(\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$ respectively.

Theorem A.13. *Let Ω is a one-sided extension domain of \mathbb{R}^n . Then, the Dirichlet (1-)harmonic extension operators $\mathcal{D}_\Omega : \mathcal{B}(\partial\Omega) \rightarrow V_1(\Omega)$ and $\tilde{\mathcal{D}}_\Omega : \mathcal{B}_0(\partial\Omega) \rightarrow V_{(0)}(\Omega)$ are isometries:*

$$\begin{aligned} \forall f \in \mathcal{B}(\partial\Omega), \quad \|f\|_{\mathcal{B}(\partial\Omega)} &= \|\mathcal{D}_\Omega f\|_{H^1(\Omega)}, \\ \forall f \in \mathcal{B}_0(\partial\Omega), \quad \|f\|_{\mathcal{B}_0(\partial\Omega)} &= \|\tilde{\mathcal{D}}_\Omega f\|_{H^1_{(0)}(\Omega)}. \end{aligned} \tag{A.18}$$

In this sense, \mathcal{D}_Ω and $\tilde{\mathcal{D}}_\Omega$ are equal to Tr_i^{-1} (defined on the adequate spaces). Consequently, understood as $\mathcal{D}_\Omega : \mathcal{B}(\partial\Omega) \rightarrow H^1(\Omega)$ and $\tilde{\mathcal{D}}_\Omega : \mathcal{B}(\partial\Omega) \rightarrow H^1(\Omega)$, those operators are linear and continuous.

A.2 Poincaré-Steklov operator

As explained in Subsection 3.2, we generalize the notion of Poincaré-Steklov operator in the framework of one-sided extension domains.

Definition A.14. (Poincaré-Steklov operator) *Let Ω be a one-sided extension domain of \mathbb{R}^n . Let $k \in \mathbb{R} \setminus \sigma(\Delta_D)$ and let $u \in H^1(\Omega)$ be the weak solution to:*

$$\begin{cases} (-\Delta + k)u = 0 & \text{on } \Omega \\ \text{Tr}_i u = f \in \mathcal{B}(\partial\Omega). \end{cases} \tag{A.19}$$

The linear mapping defined as:

$$\begin{aligned} \mathcal{A}_k : \mathcal{B}(\partial\Omega) &\rightarrow \mathcal{B}'(\partial\Omega) \\ \text{Tr}_i u &\mapsto \left. \frac{\partial_i u}{\partial\nu} \right|_{\partial\Omega} \end{aligned}$$

is called the Poincaré-Steklov (or Dirichlet-to-Neumann) operator associated to $(-\Delta + k)$.

Having defined the operator $\mathcal{A}_k : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ in Definition A.14 enables to look into the same properties regarding continuity and invertibility as the ones stated for the Poincaré-Steklov operator based on the Laplace equation (i.e. with $k = 0$; see [4] for the statement on d -sets and the general setting of [13, p. 5904] for Lipschitz domains and also [75, Theorem 4.10], still holding in our general case). Since $k \in \mathbb{R} \setminus \sigma(\Delta_D)$, for all $\lambda \in \mathbb{C}$, there exists a solution to the Dirichlet problem (A.19) if $f \in \mathcal{B}(\partial\Omega)$ satisfies:

$$\left\langle \frac{\partial_i v}{\partial\nu}, f \right\rangle_{\mathcal{B}', \mathcal{B}} = 0,$$

for every solution $v \in H_0^1(\Omega)$ of the homogeneous problem:

$$\begin{cases} -\Delta v = (\lambda - k)v & \text{on } \Omega, \\ \text{Tr}_i v = 0. \end{cases} \quad (\text{A.20})$$

If $\lambda = 0$, since k is not eigenvalues of the Dirichlet Laplacian Δ_D , it follows that the only solution to Problem (A.20) is $v = 0$. This ensures \mathcal{A}_k is well defined on all of $\mathcal{B}(\partial\Omega)$. If k is an eigenvalue of the Neumann Laplacian, then $\text{Ker } \mathcal{A}_k \neq \{0\}$.

Let us consider the adjoint operator $\mathcal{A}_k^* : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ (since $\mathcal{B}(\partial\Omega) \simeq \mathcal{B}''(\partial\Omega)$). Let us show that it coincides with \mathcal{A}_k .

Calculating the difference between two Green formulae for $u, v \in H_\Delta^1(\Omega)$ (defined by Equation (2.11)), we find that:

$$\left\langle \frac{\partial_i u}{\partial \nu}, \text{Tr}_i v \right\rangle_{\mathcal{B}', \mathcal{B}} - \left\langle \text{Tr}_i u, \frac{\partial_i v}{\partial \nu} \right\rangle_{\mathcal{B}, \mathcal{B}'} = \int_{\Omega} (\Delta u)v \, dx - \int_{\Omega} u(\Delta v) \, dx.$$

If $u \in H^1(\Omega)$ is a weak solution to (A.19), then $\Delta u = ku \in L^2(\Omega)$ in the sense of distributions, hence $\Delta u \in L^2(\Omega)$. In other words, weak solutions to (A.19) necessarily belong to $H_\Delta^1(\Omega)$. Therefore, since u and v are solutions to (A.19):

$$\int_{\Omega} (\Delta u)v \, dx - \int_{\Omega} u(\Delta v) \, dx = k \int_{\Omega} uv \, dx - k \int_{\Omega} uv \, dx = 0,$$

and we directly obtain that for all $\phi := \text{Tr}_i u \in \mathcal{B}(\partial\Omega)$ and $f := \text{Tr}_i v \in \mathcal{B}(\partial\Omega)$:

$$\langle \mathcal{A}_k \phi, f \rangle_{\mathcal{B}', \mathcal{B}} = \langle \phi, \mathcal{A}_k f \rangle_{\mathcal{B}, \mathcal{B}'},$$

i.e. the operator \mathcal{A}_k is closed and coincides with its adjoint. Since $\mathcal{B}'(\partial\Omega)$ is a Banach space, by the closed graph theorem, the Poincaré-Steklov operator \mathcal{A}_k is continuous.

The following theorem summarizes the results above.

Theorem A.15. *Let Ω be a one-sided extension domain of \mathbb{R}^n and let $k \in \mathbb{R} \setminus \sigma(\Delta_D)$. Then the Poincaré-Steklov operator $\mathcal{A}_k : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ from Definition A.14 is a linear bounded operator coinciding with its adjoint. It is such that $\text{Ker } \mathcal{A}_k = \{0\}$ if and only if $k \notin \sigma(\Delta_N)$.*

Remark A.16. *If the operator Poincaré-Steklov is considered for a closed space of weak solutions to $(-\Delta + k)$ on which the Neumann eigenvalue problem does not have k as an eigenvalue, then it becomes injective. For example, it happens for $k = 0$ if we only consider solution satisfying $\int_{\Omega} u \, dx = 0$.*

Therefore, choosing spaces on which the trace operator is isometric, we can define the Poincaré-Steklov operator as an isometry in the cases $k = 1$ and $k = 0$.

Theorem A.17. *Let Ω be a one-sided extension domain of \mathbb{R}^n . Then, the (interior) Poincaré-Steklov operator*

$$\mathcal{A}_1 =: \mathcal{A} : \text{Tr}_i u \in \mathcal{B}(\partial\Omega) \mapsto \frac{\partial_i u}{\partial\nu} \Big|_{\partial\Omega} \in \mathcal{B}'(\partial\Omega),$$

(resp. $\mathcal{A}_0 =: \tilde{\mathcal{A}} : \text{Tr}_i u \in \mathcal{B}_0(\partial\Omega) \mapsto \frac{\partial_i u}{\partial\nu} \Big|_{\partial\Omega} \in \mathcal{B}'_0(\partial\Omega)$), associated to the weak solutions in $V_1(\Omega)$, (resp. in $V_{(0)}(\Omega)$) to the problem

$$\begin{cases} (-\Delta + 1)u = 0 & \text{on } \Omega, \\ \text{Tr } u = f \in \mathcal{B}(\partial\Omega), \end{cases} \quad (\text{A.21})$$

(resp. of $-\Delta u = 0$ on Ω with $\text{Tr } u = f \in \mathcal{B}_0(\partial\Omega)$), is an isometry.

For any $k \in \mathbb{R} \setminus (\sigma(\Delta_D) \cup \sigma(\Delta_N))$, the Poincaré-Steklov operator $\mathcal{A}_k : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ from Definition A.14 is bijective. For example, the operator \mathcal{A}_1 (which we denote by \mathcal{A}) is bijective. For such a k , we can define both the Neumann extension operator and the Dirichlet extension operator for the problem $(-\Delta + k)$ on Ω , as the operator reconstructing the solution to the problem from the Neumann or Dirichlet boundary conditions respectively. From that perspective, the action of the Poincaré-Steklov is a shift from a Dirichlet boundary condition to a Neumann boundary condition. Therefore, the (k -)harmonic extension operators and the Poincaré-Steklov operator are deeply connected. The following results, detailed for $k = 1 \notin \sigma(\Delta_N)$ and $k = 0 \in \sigma(\Delta_N)$, highlight that connection. For $k = 0 \in \sigma(\Delta_N)$, it is possible to make \mathcal{A}_0 (denoted by $\tilde{\mathcal{A}}$) bijective as well by working on subspaces in which constants have been removed.

Proposition A.18. *Let Ω be a one-sided extension domain of \mathbb{R}^n . Then it holds:*

- (i) $\mathcal{D}_\Omega = \mathcal{S}_\Omega \circ \mathcal{A}$ and $\tilde{\mathcal{D}}_\Omega = \tilde{\mathcal{S}}_\Omega \circ \tilde{\mathcal{A}}$,
- (ii) $\mathcal{A} = \frac{\partial_i}{\partial\nu} \circ \mathcal{D}_\Omega$ and $\tilde{\mathcal{A}} = \frac{\partial_i}{\partial\nu} \circ \tilde{\mathcal{D}}_\Omega$,
- (iii) $\mathcal{A} = \mathcal{S}_\Omega^{-1} \circ \mathcal{D}_\Omega$, and $\tilde{\mathcal{A}} = \tilde{\mathcal{S}}_\Omega^{-1} \circ \tilde{\mathcal{D}}_\Omega$,

where $\mathcal{S} : \mathcal{B}'(\partial\Omega) \rightarrow V_1(\Omega)$, $\mathcal{D} : \mathcal{B}(\partial\Omega) \rightarrow V_1(\Omega)$, $\mathcal{A} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ on the one hand, as well as $\tilde{\mathcal{S}} : \mathcal{B}'_0(\partial\Omega) \rightarrow V_{(0)}(\Omega)$, $\tilde{\mathcal{D}} : \mathcal{B}_0(\partial\Omega) \rightarrow V_{(0)}(\Omega)$, $\tilde{\mathcal{A}} : \mathcal{B}_0(\partial\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$ on the other hand are isometries, and $\frac{\partial_i}{\partial\nu}$ is an isometry either from $V_1(\Omega)$ to $\mathcal{B}'(\partial\Omega)$ (when applied to \mathcal{D}), or from $V_{(0)}(\Omega)$ to $\mathcal{B}'_0(\partial\Omega)$ (when applied to $\tilde{\mathcal{D}}$).

Proof. Let us prove it for the operators associated with $(-\Delta + 1)$ (the case of $-\Delta$ is analogous).

- (i) Let $f \in \mathcal{B}(\partial\Omega)$. $\mathcal{A}f \in \mathcal{B}'(\partial\Omega)$ therefore $\mathcal{S}_\Omega \circ \mathcal{A}f$ is a weak solution to $(-\Delta + 1)u = 0$ on $H^1(\Omega)$. In addition, it holds:

$$\text{Tr}_i (\mathcal{S} \circ \mathcal{A}f) = \mathcal{A}^{-1} \left(\frac{\partial_i}{\partial\nu} (\mathcal{S}_\Omega \circ \mathcal{A}f) \right) = \mathcal{A}^{-1}(\mathcal{A}f) = f.$$

Therefore, $\mathcal{S}_\Omega \circ \mathcal{A}f \in H^1(\Omega)$ is a weak solution to the transmission problem (2.6), hence $\mathcal{D}_\Omega f = \mathcal{S}_\Omega \circ \mathcal{A}f$.

- (ii) The range of the double layer potential \mathcal{D}_Ω is $V_1(\Omega) \subset H_\Delta^1(\Omega)$, therefore its interior normal derivative can be defined as an element of $\mathcal{B}'(\partial\Omega)$. Let $f \in \mathcal{B}(\partial\Omega)$. Then $u := \mathcal{D}_\Omega f$ is such that $\text{Tr}_i u = f$ and $(-\Delta + 1)u = 0$ weakly on Ω . Therefore, it holds:

$$\mathcal{A}f = \mathcal{A}(\text{Tr}_i u) = \frac{\partial_i u}{\partial \nu} = \frac{\partial_i}{\partial \nu} \circ \mathcal{D}_\Omega f.$$

- (iii) Since $\mathcal{D}_\Omega = \text{Tr}_i^{-1} : \mathcal{B}(\partial\Omega) \rightarrow V_1(\Omega)$ and $\mathcal{A}^{-1} = \text{Tr}_i \circ \mathcal{S}_\Omega$ by Theorem A.10, we directly find $\mathcal{A} = (\text{Tr}_i \circ \mathcal{S})^{-1} = \mathcal{S}_\Omega^{-1} \circ \mathcal{D}_\Omega$.

□

The relations described above are synthesized in Diagram (3.7). In the spirit of point (ii) of Proposition A.18, we use the Dirichlet extension operator to describe the Poincaré-Steklov operator $\tilde{\mathcal{A}} = \mathcal{A}_0$ in the variational sense. For all $f, \varphi \in \mathcal{B}(\partial\Omega)$, since $\text{Tr}_i(\tilde{\mathcal{D}}_\Omega \varphi) = \varphi$, it holds, by Green's formula:

$$\langle \tilde{\mathcal{A}}f, \varphi \rangle_{\mathcal{B}', \mathcal{B}} = \left\langle \frac{\partial_i \tilde{\mathcal{D}}_\Omega f}{\partial \nu}, \varphi \right\rangle_{\mathcal{B}', \mathcal{B}} = \int_\Omega \nabla \tilde{\mathcal{D}}_\Omega f \cdot \nabla \tilde{\mathcal{D}}_\Omega \varphi \, dx. \quad (\text{A.22})$$

From Proposition A.18, we can give an explicit formula for the inverse of the Poincaré-Steklov operator using the Neumann (1-)harmonic extension operators.

Corollary A.19. *Let Ω be a one-sided extension domain of \mathbb{R}^n .*

- (i) *The Poincaré-Steklov operator $\mathcal{A} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ for the 1-harmonic solutions is isometric with inverse isometry $\mathcal{A}^{-1} = \text{Tr}_i \circ \mathcal{S}_\Omega : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$.*
- (ii) *The Poincaré-Steklov operator $\tilde{\mathcal{A}}$ associated to $-\Delta$ is isometric from $\mathcal{B}_0(\partial\Omega)$ to $\mathcal{B}'_0(\partial\Omega)$ with inverse isometry $\tilde{\mathcal{A}}^{-1} = \text{Tr}_i \circ \tilde{\mathcal{S}}_\Omega : \mathcal{B}'_0(\partial\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$.*

B Transmission problems for extension domains

B.1 Layer potential operators

Let Ω be a two-sided extension domain of \mathbb{R}^n , and consider the transmission problem described in Subsection 4.1.

For the operator $(-\Delta+1)$. For $g \in \mathcal{B}'(\partial\Omega)$, let us consider the transmission problem (4.1) with $k = 1$, a jump in normal derivative g , and with no jump in trace: $f = 0$. Let us establish its variational formulation. On Ω , by Green's formula, it holds

$$\forall v \in H^1(\Omega), \quad \langle u, v \rangle_{H^1(\Omega)} = \left\langle \frac{\partial_i u}{\partial \nu}, \text{Tr}_i v \right\rangle_{\mathcal{B}', \mathcal{B}}.$$

On Ω^c , we have

$$\forall w \in H^1(\Omega^c), \quad \langle u, w \rangle_{H^1(\Omega^c)} = - \left\langle \frac{\partial_e u}{\partial \nu}, \text{Tr}_e w \right\rangle_{\mathcal{B}', \mathcal{B}}.$$

Hence, the following variational formulation holds (since $\lambda^{(n)}(\partial\Omega) = 0$, by definition of a two-sided extension domain – Definition 2.7):

$$\forall \varphi \in H_{\partial\Omega, [0]_{\text{Tr}}}^1(\mathbb{R}^n), \quad \langle u, \varphi \rangle_{H_{\partial\Omega}^1(\mathbb{R}^n)} = \langle g, \text{Tr} \varphi \rangle_{\mathcal{B}', \mathcal{B}}, \quad (\text{B.1})$$

where

$$H_{\partial\Omega, [0]_{\text{Tr}}}^1(\mathbb{R}^n) := \{v \in H_{\partial\Omega}^1(\mathbb{R}^n) \mid \llbracket \text{Tr} v \rrbracket = 0\},$$

and $\text{Tr} \varphi := \text{Tr}_i \varphi = \text{Tr}_e \varphi$. Since $\llbracket \text{Tr} \cdot \rrbracket$ is continuous on $H_{\partial\Omega}^1(\mathbb{R}^n)$, the space $H_{\partial\Omega, [0]_{\text{Tr}}}^1(\mathbb{R}^n)$ is a Hilbert space endowed with the same inner product $\langle \cdot, \cdot \rangle_{H_{\partial\Omega}^1(\mathbb{R}^n)}$. Therefore, applying the Riesz representation theorem, we deduce the well-posedness of the variational formulation. This allows to define the single layer potential operator in the variational sense, as the operator mapping the jump in normal derivative to the associated 1-harmonic function on $\Omega \cup \Omega^c$ with no jump in trace across $\partial\Omega$.

Theorem B.1. (Single layer potential operator for $(-\Delta + 1)$) For any two-sided extension domain Ω of \mathbb{R}^n , let

$$\begin{aligned} \mathcal{S}_{\partial\Omega} : \mathcal{B}'(\partial\Omega) &\longrightarrow \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega)) \subset H_{\partial\Omega}^1(\mathbb{R}^n) \\ g = \left[\left[\frac{\partial u}{\partial \nu} \right] \right] &\longmapsto u = \mathcal{S}_{\partial\Omega} g, \end{aligned}$$

be the linear operator uniquely defined by

$$\forall v \in H_{\partial\Omega, [0]_{\text{Tr}}}^1(\mathbb{R}^n), \quad \langle \mathcal{S}_{\partial\Omega} g, v \rangle_{H_{\partial\Omega}^1(\mathbb{R}^n)} = \langle g, \text{Tr} v \rangle_{\mathcal{B}', \mathcal{B}}. \quad (\text{B.2})$$

Then $\mathcal{S}_{\partial\Omega} : \mathcal{B}'(\partial\Omega) \rightarrow H_{\partial\Omega}^1(\mathbb{R}^n)$, called the single layer potential operator for the transmission problem (4.1) with $k = 1$, is a linear bounded operator:

$$\|\mathcal{S}_{\partial\Omega} g\|_{H_{\partial\Omega}^1(\mathbb{R}^n)} \leq C \|g\|_{\mathcal{B}'(\partial\Omega)} \quad (\text{B.3})$$

for some constant $C = C(\Omega) > 0$. In addition, it can be expressed as a convolution with G , the fundamental solution to $(-\Delta + 1)$ on \mathbb{R}^n :

$$\forall g \in \mathcal{B}'(\partial\Omega), \quad \forall x \in \mathbb{R}^n, \quad \mathcal{S}_{\partial\Omega} g(x) = (G * \text{Tr}^* g)(x),$$

where $\text{Tr}^* : \mathcal{B}'(\partial\Omega) \rightarrow [\mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))]'$ and the convolution is on \mathbb{R}^n . Therefore,

$$\forall x \in \mathbb{R}^n, \quad \mathcal{S}_{\partial\Omega} g(x) = \langle g, \text{Tr} G(x - \cdot) \rangle_{\mathcal{B}', \mathcal{B}}. \quad (\text{B.4})$$

Remark B.2. *It will be pointed out in the following proof that for all $g \in \mathcal{B}'(\partial\Omega)$, $\text{Tr}^* g$ can be seen as a distribution with compact support. Therefore, the convolution with the fundamental solution is well-defined, at least in the sense of distributions. We will show it is actually an element of $H_{\partial\Omega}^1(\mathbb{R}^n)$.*

Proof. The well-posedness of (B.1) is sufficient to ensure that $\mathcal{S}_{\partial\Omega} : \mathcal{B}'(\partial\Omega) \rightarrow H_{\partial\Omega}^1(\mathbb{R}^n)$ is a linear bounded operator satisfying (B.3).

Since the interior and exterior traces of the solution to the transmission problem (4.1) with $k = 1$ and $f = 0$ are equal, $\mathcal{S}_{\partial\Omega} g$ can be defined on \mathbb{R}^n , for $g \in \mathcal{B}'(\partial\Omega)$. $\mathcal{S}_{\partial\Omega} : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$ is continuously invertible (with inverse $\llbracket \frac{\partial}{\partial\nu} \rrbracket$), therefore $\mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$ is a Hilbert space. Let us notice that for all $\phi \in [\mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))]'$, it holds $G * \phi \in \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$. It is a direct corollary (adapting [17, Lemma IV.5.14]) of the well-posedness on $\mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$ of the following variational problem:

$$\forall v \in \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega)), \quad \langle u, v \rangle_{H_{\partial\Omega}^1(\mathbb{R}^n)} = \langle \phi, v \rangle_{[\mathcal{S}_{\partial\Omega}(\mathcal{B}')]', \mathcal{S}_{\partial\Omega}(\mathcal{B}')}, \quad (\text{B.5})$$

associated with the elliptic problem (4.1) with $k = 1$, $f = 0$, $g = 0$ and a source term equal to ϕ . Since $u \in \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$ is its unique weak solution, by definition of the fundamental solution, it holds $G * \phi = u \in \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$.

In our case, $\text{Tr}^* g \in [\mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))]'$, hence $G * \text{Tr}^* g \in \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$, which means that $G * \text{Tr}^* g$ does not have a jump in trace across $\partial\Omega$: $\llbracket \text{Tr}(G * \text{Tr}^* g) \rrbracket = 0$.

Since $(-\Delta + 1)G$ is the Dirac distribution, it holds (see Remark B.2):

$$(-\Delta + 1)(G * \text{Tr}^* g) = \text{Tr}^* g.$$

For $v \in \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$ with support disjoint from $\partial\Omega$, $\langle \text{Tr}^* g, v \rangle_{[\mathcal{S}_{\partial\Omega}(\mathcal{B}')]', \mathcal{S}_{\partial\Omega}(\mathcal{B}')} = 0$, hence:

$$(-\Delta + 1)(G * \text{Tr}^* g) = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega.$$

Therefore $\Delta(G * \text{Tr}^* g) \in L^2(\mathbb{R}^n \setminus \partial\Omega)$.

By formula (B.8), it also holds:

$$(-\Delta + 1)(G * \text{Tr}^* g) = \text{Tr}^* \left[\left[\frac{\partial(G * \text{Tr}^* g)}{\partial\nu} \right] \right],$$

therefore, since $\text{Tr}^* : \mathcal{B}'(\partial\Omega) \rightarrow [\mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))]'$ is into,

$$\left[\left[\frac{\partial(G * \text{Tr}^* g)}{\partial\nu} \right] \right] = g,$$

and $G * \text{Tr}^* g$ is a solution (therefore, it is the unique solution) to the transmission problem (4.1) with $k = 1$ and $f = 0$ associated to $g \in \mathcal{B}'(\partial\Omega)$. Moreover, $G * \text{Tr}^* g \in H_{\partial\Omega, [0]_{\text{Tr}}}^1(\mathbb{R}^n)$ for $g \in \mathcal{B}'(\partial\Omega)$. \square

In particular, if μ is a d -upper regular measure for $d \in [n-1, n[$ supported by $\partial\Omega$ (Definition 2.4), then [53] $L^2(\partial\Omega, \mu) \subset \mathcal{B}'(\partial\Omega)$. Therefore, if $g \in L^2(\partial\Omega, \mu)$, Equation (B.4) states that:

$$\mathcal{S}_{\partial\Omega} g(x) = \int_{\partial\Omega} G(x-y)g(y) \mu(dy), \quad x \in \mathbb{R}^n$$

which is identical to the classical formula (see Subsection 4.1) replacing Lebesgue's measure σ with μ .

For the transmission problem (4.1) with $k = 1$, $g = 0$ and with a non-homogeneous jump in trace $-f \in \mathcal{B}(\partial\Omega)$, we consider the space $H_{\partial\Omega, [-f]_{\text{Tr}}}^1(\mathbb{R}^n)$ defined on pages iii–vi.

Proposition B.3. *Let Ω be a two-sided extension domain of \mathbb{R}^n . Then the linear operator*

$$\begin{aligned} \mathcal{D}_{\partial\Omega} : \mathcal{B}(\partial\Omega) &\longrightarrow \mathcal{D}_{\partial\Omega}(\mathcal{B}(\partial\Omega)) \subset H_{\partial\Omega}^1(\mathbb{R}^n) \\ f = -\llbracket \text{Tr } u \rrbracket &\longmapsto u = \mathcal{D}_{\partial\Omega} f, \end{aligned}$$

uniquely defined by the variational formulation

$$\forall v \in H_{\partial\Omega, [-f]}^1(\mathbb{R}^n), \quad \langle \mathcal{D}_{\partial\Omega} f, v \rangle_{H_{\partial\Omega}^1(\mathbb{R}^n)} = - \left\langle \frac{\partial(\mathcal{D}_{\partial\Omega} f)}{\partial\nu}, f \right\rangle_{\mathcal{B}', \mathcal{B}}, \quad (\text{B.6})$$

where $\frac{\partial}{\partial\nu}(\mathcal{D}_{\partial\Omega} f) := \frac{\partial_i}{\partial\nu}(\mathcal{D}_{\partial\Omega} f) = \frac{\partial_e}{\partial\nu}(\mathcal{D}_{\partial\Omega} f)$, is bounded:

$$\|\mathcal{D}_{\partial\Omega} f\|_{H_{\partial\Omega}^1(\mathbb{R}^n)} \leq C \|f\|_{\mathcal{B}(\partial\Omega)}, \quad (\text{B.7})$$

for some constant $C = C(\Omega) > 0$. This operator $\mathcal{D}_{\partial\Omega}$ is called the double layer potential operator for the transmission problem (4.1) with $k = 1$.

By the same type of construction as for the problem on a bounded domain Ω (see [38, Theorem 7]), we obtain that the transmission problem (4.1) with $k = 1$ and $g = 0$ is well-posed on $H_{\partial\Omega}^1(\mathbb{R}^n)$ for all $f \in \mathcal{B}(\partial\Omega)$. This proves Proposition B.3.

To give an explicit formula for the double layer potential operator, we introduce operators which allow to restate Green's formula.

Proposition B.4. *Let Ω be a one-sided extension domain of \mathbb{R}^n . Then, there exists a unique linear continuous operator $\mathfrak{D}_i : H^1(\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ such that, for $v \in H^1(\Omega)$, it holds:*

$$\forall u \in H_{\Delta}^1(\Omega), \quad \langle \mathfrak{D}_i v, \text{Tr}_i u \rangle_{\mathcal{B}', \mathcal{B}} = \left\langle \frac{\partial_i u}{\partial\nu}, \text{Tr}_i v \right\rangle_{\mathcal{B}', \mathcal{B}},$$

where $H_{\Delta}^1(\Omega)$ is defined by Equation (2.11).

If Ω is a two-sided extension domain of \mathbb{R}^n , then there exists a unique linear continuous operator $\mathfrak{D}_e : H^1(\Omega^c) \rightarrow \mathcal{B}'(\partial\Omega)$ such that, for $v \in H^1(\Omega^c)$, it holds:

$$\forall u \in H_{\Delta}^1(\Omega^c), \quad \langle \mathfrak{D}_e v, \text{Tr}_e u \rangle_{\mathcal{B}', \mathcal{B}} = \left\langle \frac{\partial_e u}{\partial\nu}, \text{Tr}_e v \right\rangle_{\mathcal{B}', \mathcal{B}}.$$

Proof. Let $v \in H^1(\Omega)$. If $\phi \in \mathcal{B}'(\partial\Omega)$ is such that:

$$\forall u \in H^1_\Delta(\Omega), \quad \langle \phi, \text{Tr}_i u \rangle_{\mathcal{B}', \mathcal{B}} = \left\langle \frac{\partial_i u}{\partial \nu}, \text{Tr}_i v \right\rangle_{\mathcal{B}', \mathcal{B}},$$

then

$$\text{Tr}_i^* \phi = \frac{\partial_i^*}{\partial \nu} \circ \text{Tr}_i v,$$

where $\frac{\partial_i^*}{\partial \nu} : \mathcal{B}(\partial\Omega) \rightarrow (H^1(\Omega))'$ and $\text{Tr}_i^* : \mathcal{B}'(\partial\Omega) \rightarrow (H^1(\Omega))'$. Diagram (3.7) guarantees the existence and uniqueness of $\mathfrak{D}_i v := \phi$. In the same way, we prove the existence of the linear operator $\mathfrak{D}_e : H^1(\Omega^c) \rightarrow \mathcal{B}'(\partial\Omega)$ and its uniqueness.

Let $v \in H^1(\Omega)$. For all $f \in \mathcal{B}(\partial\Omega)$, there exists $u \in H^1_\Delta(\Omega)$ which depends continuously on f and such that $\text{Tr}_i u = f$. Then it holds:

$$\left| \langle \mathfrak{D}_i v, \text{Tr}_i u \rangle_{\mathcal{B}', \mathcal{B}} \right| = \left| \left\langle \frac{\partial_i u}{\partial \nu}, \text{Tr}_i v \right\rangle_{\mathcal{B}', \mathcal{B}} \right| \leq C \|u\|_{H^1_\Delta(\Omega)} \|v\|_{H^1(\Omega)},$$

where $C > 0$, independent from u and v , is linked to the continuity of the operators $\frac{\partial_i}{\partial \nu} : H^1_\Delta(\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ and $\text{Tr}_i : H^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$, and $\|u\|_{H^1_\Delta(\Omega)}$ is defined in (2.13). Since u depends continuously on f , there exists $C' > 0$ which depends on C but neither on u or f , nor on v such that:

$$\left| \langle \mathfrak{D}_i v, \text{Tr}_i u \rangle_{\mathcal{B}', \mathcal{B}} \right| \leq C' \|f\|_{\mathcal{B}(\partial\Omega)} \|v\|_{H^1(\Omega)},$$

hence the continuity of $\mathfrak{D}_i : H^1(\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$. In the same way, we can prove the continuity of $\mathfrak{D}_e : H^1(\Omega^c) \rightarrow \mathcal{B}'(\partial\Omega)$. \square

The operators \mathfrak{D}_i and \mathfrak{D}_e allow us to state a general relation between an element of $H^1_{\partial\Omega}(\mathbb{R}^n)$ and its behaviour on $\partial\Omega$, which is a generalization of a similar formula which can be found in [75] in the case of a Lipschitz domain.

Proposition B.5. *Let $u \in H^1_{\partial\Omega}(\mathbb{R}^n)$, such that $-\Delta u + u = F \in L^2(\mathbb{R}^n)$ on $\mathbb{R}^n \setminus \partial\Omega$. It holds:*

$$\begin{aligned} \forall v \in H^1(\mathbb{R}^n), \quad \langle (-\Delta + 1)u, v \rangle_{L^2(\mathbb{R}^n)} = \\ \langle F, v \rangle_{L^2(\mathbb{R}^n)} - \langle \mathfrak{D}v, \llbracket \text{Tr } u \rrbracket \rangle_{\mathcal{B}', \mathcal{B}} + \left\langle \left[\left[\frac{\partial u}{\partial \nu} \right] \right], \text{Tr } v \right\rangle_{\mathcal{B}', \mathcal{B}}, \end{aligned}$$

where $\text{Tr } v := \text{Tr}_i v = \text{Tr}_e v$ as well as $\mathfrak{D}v := \mathfrak{D}_i v = \mathfrak{D}_e v$. Therefore, in $(H^1(\mathbb{R}^n))'$, it holds

$$-\Delta u + u = F - \mathfrak{D}^* \llbracket \text{Tr } u \rrbracket + \text{Tr}^* \left[\left[\frac{\partial u}{\partial \nu} \right] \right], \quad (\text{B.8})$$

where $\mathfrak{D}^* : \mathcal{B}(\partial\Omega) \rightarrow (H^1(\mathbb{R}^n))'$ and $\text{Tr}^* : \mathcal{B}'(\partial\Omega) \rightarrow (H^1(\mathbb{R}^n))'$.

Proof. From [75], if $v \in H^1(\mathbb{R}^n)$, then $[\text{Tr } v] = 0$ and $\left[\left[\frac{\partial v}{\partial \nu}\right]\right] = 0$, hence the notations $\text{Tr } v$ and $\mathfrak{D}v$. By Green's formula, it holds:

$$\begin{aligned} \forall v \in H^1(\mathbb{R}^n), \quad \int_{\Omega \cup \Omega^c} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^n} uv \, dx \\ = \langle F, v \rangle_{L^2(\mathbb{R}^n)} + \left\langle \left[\left[\frac{\partial u}{\partial \nu}\right]\right], \text{Tr } v \right\rangle_{\mathcal{B}', \mathcal{B}}. \end{aligned}$$

Since Green's formulae can be restated using the operators \mathfrak{D}_i and \mathfrak{D}_e as:

$$\begin{cases} \langle -\Delta u + u, v \rangle_{L^2(\Omega)} = \langle u, v \rangle_{H^1(\Omega)} - \langle \mathfrak{D}v, \text{Tr}_i u \rangle_{\mathcal{B}', \mathcal{B}} \\ \langle -\Delta u + u, v \rangle_{L^2(\Omega^c)} = \langle u, v \rangle_{H^1(\Omega^c)} + \langle \mathfrak{D}v, \text{Tr}_e u \rangle_{\mathcal{B}', \mathcal{B}} \end{cases},$$

we yield the result. \square

We complete the definition of the double layer potential generalizing the classical formula as a convolution with the fundamental solution to $(-\Delta + 1)$.

Theorem B.6. *Let Ω be a two-sided extension domain of \mathbb{R}^n . If $f \in \mathcal{B}(\partial\Omega)$, then the unique solution u to the transmission problem (4.1) for $k = 1$ and $g = 0$ associated to f can be written as:*

$$\forall x \in \mathbb{R}^n \setminus \partial\Omega, \quad \mathcal{D}_{\partial\Omega} f(x) = (G * \mathfrak{D}^* f)(x) \in H^1_{\partial\Omega, [0]_{\text{Tr}}}(\mathbb{R}^n), \quad (\text{B.9})$$

where G is the fundamental solution to $(-\Delta + 1)$ on \mathbb{R}^n , $\mathfrak{D}^* : \mathcal{B}(\partial\Omega) \rightarrow (H^1(\mathbb{R}^n))'$ and the convolution is on \mathbb{R}^n . Therefore,

$$\forall x \in \mathbb{R}^n \setminus \partial\Omega \quad \mathcal{D}_{\partial\Omega} f(x) = \langle \mathfrak{D}G(x - \cdot), f \rangle_{\mathcal{B}', \mathcal{B}}. \quad (\text{B.10})$$

Proof. First of all, $(-\Delta + 1)G$ is the Dirac distribution δ_0 , therefore it holds, for $f \in \mathcal{B}(\partial\Omega)$:

$$(-\Delta + 1)(G * \mathfrak{D}^* f) = \mathfrak{D}^* f.$$

Let $v \in H^1(\mathbb{R}^n)$ with a support disjoint from $\partial\Omega$. Then it holds, for any $\varphi \in H^1(\mathbb{R}^n)$ such that $\text{Tr } \varphi = f$,

$$\langle \mathfrak{D}^* f, v \rangle_{(H^1)', H^1} = \left\langle \frac{\partial \varphi}{\partial \nu}, \text{Tr } v \right\rangle_{\mathcal{B}', \mathcal{B}} = 0,$$

hence:

$$(-\Delta + 1)(G * \mathfrak{D}^* f) = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega.$$

Then, we proceed in the same way as it is done in [75, Theorem 6.11]. Let $f \in \mathcal{B}(\partial\Omega)$. Let us denote:

$$u := \begin{cases} \mathcal{D}f & \text{on } \Omega \\ 0 & \text{on } \Omega^c \end{cases} \in H^1_{\partial\Omega}(\mathbb{R}^n).$$

Then $-\Delta u + u = 0$ weakly on $\mathbb{R}^n \setminus \partial\Omega$, therefore, using formula (B.8), it holds in $(H^1(\mathbb{R}^n))'$:

$$-\Delta u + u = -\mathfrak{D}^*[\text{Tr } u] + \text{Tr}^* \left[\left[\frac{\partial u}{\partial \nu} \right] \right],$$

hence, since $(-\Delta + 1)G = \delta_0$ and $\mathcal{S}_{\partial\Omega} = G * \text{Tr}^*$, it holds:

$$u = -G * \mathfrak{D}^*[\text{Tr } u] + \mathcal{S}_{\partial\Omega} \left[\left[\frac{\partial u}{\partial \nu} \right] \right].$$

We can deduce:

$$G * \mathfrak{D}^* f = \mathcal{S}_{\partial\Omega} \circ \frac{\partial_i}{\partial \nu}(\mathcal{D}f) - u.$$

Only, for all $g \in \mathcal{B}'(\partial\Omega)$, it holds $[\text{Tr}(\mathcal{S}_{\partial\Omega} g)] = 0$ and $[\left[\frac{\partial}{\partial \nu}(\mathcal{S}_{\partial\Omega} g) \right]] = g$, hence:

$$[\text{Tr}(G * \mathfrak{D}^* f)] = -[\text{Tr } u] = -f,$$

and

$$\left[\left[\frac{\partial}{\partial \nu}(G * \mathfrak{D}^* f) \right] \right] = \frac{\partial_i}{\partial \nu}(\mathcal{D}f) - \left[\left[\frac{\partial u}{\partial \nu} \right] \right] = 0.$$

Therefore, $G * \mathfrak{D}^* f$ is the unique solution to the transmission problem (4.1) with $k = 1$ and $g = 0$ associated to $f \in \mathcal{B}(\partial\Omega)$. Equation (B.10) holds by the definition of the operators involved. \square

For regular boundaries (at least C^1) the classical formula for the double layer potential operator (see Subsection 4.1) is well-defined on the space $L^2(\partial\Omega, \sigma)$: the unique solution to the transmission problem (4.1) with $k = 1$, $g = 0$ and $f = -[\text{Tr } u] \in L^2(\partial\Omega, \sigma)$ is $\mathcal{D}_{\partial\Omega} f(x) \in H^2_{\partial\Omega}(\mathbb{R}^n)$ ((2.15) with H^2 instead of H^1), given by:

$$\begin{aligned} \forall x \in \mathbb{R}^n \setminus \partial\Omega, \quad \mathcal{D}_{\partial\Omega} f(x) &= (G * \mathfrak{D}^* f)(x) = \langle \mathfrak{D}G(x - \cdot), f \rangle_{L^2(\partial\Omega, \sigma)} \\ &= \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x - y) f(y) \sigma(dy). \end{aligned} \tag{B.11}$$

Indeed, in that case, the weak solution to the transmission problem is in the space $H^2_{\partial\Omega}(\mathbb{R}^n)$, and the derivation operators $\frac{\partial_i}{\partial \nu} : H^2(\Omega) \rightarrow L^2(\partial\Omega, \sigma)$ and $\frac{\partial_e}{\partial \nu} : H^2(\Omega^c) \rightarrow L^2(\partial\Omega, \sigma)$ are linear and bounded. Therefore, the C^1 regularity of the boundary (and the fact that $\frac{\partial G}{\partial \nu} \in L^2(\partial\Omega, \sigma)$) ensures enough regularity for the weak solution so the classical integral formula coincides with the operator $\mathcal{D}_{\partial\Omega}$ in our framework. If the boundary is less regular, such as a Lipschitz or non-Lipschitz boundary, the weak solution u does not belong to H^2 anymore, only $\Delta u \in L^2(\mathbb{R}^n)$, which is not sufficient to use the integral formula. By [48], if the boundary is (exactly) Lipschitz, the domain must be convex to ensure the H^2 regularity of the weak solution to the elliptic problem with the Dirichlet boundary condition, yet the complement of a convex Lipschitz domain is not a convex domain. That regularity problem does not occur

for the C^1 -boundaries. For any two-sided extension domain Ω , Formula (B.10) generalizes the classical formula from Equation (B.11) without assuming extra regularity for the solution.

Using the definitions of both the single $\mathcal{S}_{\partial\Omega} : \mathcal{B}'(\partial\Omega) \rightarrow H_{\partial\Omega}^1(\mathbb{R}^n)$ and the double layer potential operator $\mathcal{D}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow H_{\partial\Omega}^1(\mathbb{R}^n)$, we give the following result which is known as Green's third identity [92], generalized here in the case of a bounded two-sided extension domain.

Theorem B.7. *Let Ω be a bounded two-sided extension domain of \mathbb{R}^n . The unique solution on $H_{\partial\Omega}^1(\mathbb{R}^n)$ to the transmission problem (4.1) with $k = 1$, for any $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'(\partial\Omega)$, is given by $u = \mathcal{S}_{\partial\Omega} g - \mathcal{D}_{\partial\Omega} f$.*

Proof. $\mathcal{S}_{\partial\Omega} g$ is the unique solution to the transmission problem (4.1) with $k = 1$ and $f = 0$ associated to $g \in \mathcal{B}'(\partial\Omega)$, and $\mathcal{D}_{\partial\Omega} f$ is the unique solution to the transmission problem (4.1) with $k = 1$ and $g = 0$ associated to $f \in \mathcal{B}(\partial\Omega)$. Since (4.1) is linear, the superimposition principle applies. Since the transmission problem with $k = 1$ is well-posed, and $u := \mathcal{S}_{\partial\Omega} g - \mathcal{D}_{\partial\Omega} f$ is a solution to that problem, it is the unique solution. \square

For the operator $-\Delta$. Analogously, it is possible to define the single and double layer potential operators for the weak solutions to the transmission problem (4.1) related to $-\Delta$ and have an analogous representation of harmonic solutions as 1-harmonic ones, given in Proposition B.7. However, we need to be careful about the functional spaces on which the well-posedness of the transmission problems for $-\Delta$ occurs. The bilinear form associated to $-\Delta$:

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega^c} \nabla u \cdot \nabla v \, dx$$

does not necessarily define an inner product in a Hilbert space resembling $H_{\partial\Omega}^1(\mathbb{R}^n)$ and reflecting the properties of the solution.

The main difference comes from whether $n = 2$ or $n \geq 3$. For $n \geq 3$, it is possible to impose additional conditions on u , such as a mean value zero on the bulk and at infinity, so that $a(\cdot, \cdot)$ be an inner product. Indeed, with those extra conditions, both $\|\nabla \cdot\|_{L^2(\Omega)^n}$ and $\|\nabla \cdot\|_{L^2(\Omega^c)^n}$ become norms, but only when $n \geq 3$ for the latter. In particular, the condition $n \geq 3$ ensures the existence of the limit:

$$\langle u \rangle_{\infty} := \lim_{R \rightarrow +\infty} \frac{1}{\lambda^{(n)}(B_R(0))} \int_{B_R(0)} u \, dx. \quad (\text{B.12})$$

Therefore, we consider the cases of \mathbb{R}^2 , and of \mathbb{R}^n with $n \geq 3$ separately. It is known that for $n \geq 3$ [71], the space:

$$W(\mathbb{R}^n) := \overline{C_0^{\infty}(\mathbb{R}^n)}^{\|\nabla \cdot\|_{L^2(\mathbb{R}^n)^n}} = \{u \in L_{loc}^2(\mathbb{R}^n) \mid \nabla u \in L^2(\mathbb{R}^n)^n, \langle u \rangle_{\infty} = 0\} \quad (\text{B.13})$$

endowed with the inner product $\langle u, v \rangle_{W(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx$ is a Hilbert space. It is important to notice that $H^1(\mathbb{R}^n)$ is dense and continuously embedded in $W(\mathbb{R}^n)$, but $W(\mathbb{R}^n)$ is not contained in $L^2(\mathbb{R}^n)$.

If $n \geq 3$ and Ω is a two-sided extension domain of \mathbb{R}^n , then the space:

$$W_{\partial\Omega}(\mathbb{R}^n) := \left\{ u \in L^2_{loc}(\mathbb{R}^n) \mid \nabla u|_{\Omega} \in L^2(\Omega)^n, \nabla u|_{\Omega^c} \in L^2(\Omega^c)^n, \langle u \rangle_{\infty} = 0 \right\}, \quad (\text{B.14})$$

is a Hilbert space endowed with the norm:

$$\|u\|_{W_{\partial\Omega}}^2 := \|u\|_{H^1(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega^c)^n}^2. \quad (\text{B.15})$$

However, $\|\nabla \cdot \|_{L^2(\Omega)^n}$ is only a semi-norm on $H^1(\Omega)$ and could become a norm if we add a mean value zero property, for instance on Ω . Thus we consider:

$$W_{\partial\Omega, \langle 0 \rangle_{\Omega}}(\mathbb{R}^n) := \left\{ u \in W_{\partial\Omega}(\mathbb{R}^n) \mid \int_{\Omega} u \, dx = 0 \right\} \quad (\text{B.16})$$

which is a Hilbert space endowed with the norm

$$\|u\|_{W_{\partial\Omega, \langle 0 \rangle_{\Omega}}(\mathbb{R}^n)}^2 := \|\nabla u\|_{L^2(\Omega)^n}^2 + \|\nabla u\|_{L^2(\Omega^c)^n}^2. \quad (\text{B.17})$$

Similarly to the bounded case, it holds $\text{Tr}_i(W_{\partial\Omega, \langle 0 \rangle_{\Omega}}(\mathbb{R}^n)) = \mathcal{B}_0(\partial\Omega)$. Given the Dirichlet boundary condition of the transmission problem (4.1) (with $k = 0$ and $g = 0$), that means if $u \in W_{\partial\Omega, \langle 0 \rangle_{\Omega}}(\mathbb{R}^n)$ is a solution to the problem, then $\text{Tr}_i u, \text{Tr}_e u \in \mathcal{B}_0(\partial\Omega)$. Therefore, the Neumann boundary condition is to be understood in $\mathcal{B}'_0(\partial\Omega)$, which requires to define the weak exterior normal derivative $\frac{\partial_e u}{\partial\nu} \in \mathcal{B}'_0(\partial\Omega)$, adapting Definition 2.13:

Definition B.8. (Weak exterior normal derivative in $\mathcal{B}'_0(\partial\Omega)$) Let $u \in H^1_{\langle 0 \rangle, \Delta}(\Omega^c)$ (defined on pages iii–vi). The weak exterior normal derivative in $\mathcal{B}'_0(\partial\Omega)$ of u is the element $\phi \in \mathcal{B}'_0(\partial\Omega)$ such that for all $v \in H^1_{\langle 0 \rangle}(\Omega^c)$, it holds:

$$\langle \phi, \text{Tr}_e v \rangle_{\mathcal{B}'_0, \mathcal{B}_0} = - \int_{\Omega^c} (\Delta u)v \, dx - \int_{\Omega^c} \nabla u \cdot \nabla v \, dx,$$

and we denote $\frac{\partial_e u}{\partial\nu} = \phi$.

Remark B.9. As it is mentioned in [5] initially for $W(\mathbb{R}^n)$, the space $W_{\partial\Omega}(\mathbb{R}^n)$ is not a subspace of $L^2(\mathbb{R}^n)$, but contains $H^1(\mathbb{R}^n)$ for $n \geq 3$ [3]. The main difference between $W_{\partial\Omega}(\mathbb{R}^n)$ and $W(\mathbb{R}^n)$ is that the former allows jumps in gradient across $\partial\Omega$, and its elements do not need to be in H^1 on the subdomains containing $\partial\Omega$: they only need to be in L^2_{loc} . In addition [5], in \mathbb{R}^3 the only difference between $H^1(\mathbb{R}^3)$ and $W(\mathbb{R}^3)$ is the speed of the decay for $|x| \rightarrow +\infty$: in $W(\mathbb{R}^3)$ the elements decay at least as $|x|^{-1/2}$, whereas elements of $H^1(\mathbb{R}^3)$ must decay at least as fast as $|x|^{-3/2}$ as $|x| \rightarrow +\infty$.

If $n = 2$ [4, 3], $\|\nabla \cdot \|_{L^2(\mathbb{R}^n)^n}$ is not a norm on $W(\mathbb{R}^2)$, since the Poincaré-type inequalities [71] do not hold: $\langle u \rangle_\infty$ can be infinite, should the limit even exist. Therefore, it only makes sense to consider a class of functions vanishing at infinity, as it is done [92] for the external elliptic problems:

$$V_{van}(\mathbb{R}^2) = \{u \in H_{\partial\Omega}^1(\mathbb{R}^2) \mid u \rightarrow 0 \text{ as } |x| \rightarrow +\infty\}. \quad (\text{B.18})$$

That means we need to work in $W_{\partial\Omega}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ at least.

We introduce the following notation:

$$V_{\partial\Omega}(\mathbb{R}^n) := \begin{cases} V_{van}(\mathbb{R}^2) & \text{if } n = 2, \\ W_{\partial\Omega, \langle 0 \rangle_\Omega}(\mathbb{R}^n) & \text{if } n \geq 3. \end{cases} \quad (\text{B.19})$$

The definitions of the layer potentials being directly linked to the well-posedness of the transmission problem, we begin with stating the following well-posedness result:

Theorem B.10. *Let Ω be a bounded two-sided extension domain of \mathbb{R}^n . Let \tilde{G} be the fundamental solution to $-\Delta$ on \mathbb{R}^n . Then, for all $g \in \mathcal{B}'_0(\partial\Omega)$, the transmission problem (4.1) with $k = 0$ and $f = 0$ has a unique solution, given by the formula:*

$$u = \tilde{G} * \text{Tr}^* g \quad (\text{B.20})$$

which goes to zero as $|x| \rightarrow +\infty$. Here, $\text{Tr}^* : \mathcal{B}'_0(\partial\Omega) \rightarrow V'_{\partial\Omega, [0]_{\partial\nu}}(\mathbb{R}^n)$, where $V_{\partial\Omega, [0]_{\partial\nu}}(\mathbb{R}^n)$ is defined on pages iii–vi. If $n = 2$, then $u \in V_{van}(\mathbb{R}^2)$. If $n \geq 3$, then $u \in W_{\partial\Omega, \langle 0 \rangle_\Omega}(\mathbb{R}^n)$ and it is the unique solution to the variational formulation

$$\forall v \in W_{\partial\Omega, \langle 0 \rangle_\Omega}(\mathbb{R}^n), \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega^c} \nabla u \cdot \nabla v \, dx = \langle g, \text{Tr} v \rangle_{\mathcal{B}'_0, \mathcal{B}_0}. \quad (\text{B.21})$$

In addition, for all $n \geq 2$, the following a priori estimate holds:

$$\|u\|_{W_{\partial\Omega, \langle 0 \rangle_\Omega}(\mathbb{R}^n)} \leq C \|g\|_{\mathcal{B}'_0(\partial\Omega)}, \quad (\text{B.22})$$

for some constant $C := C(n, \Omega) > 0$.

Proof. Let us start with the case $n = 2$. Firstly we verify that $\tilde{G} * \text{Tr}^* g$ solves the transmission problem for all $g \in \mathcal{B}'_0(\partial\Omega)$. Since $-\Delta \tilde{G}$ is the Dirac distribution, it holds:

$$-\Delta(\tilde{G} * \text{Tr}^* g) = \text{Tr}^* g.$$

For all $v \in V_{\partial\Omega, [0]_{\partial\nu}}(\mathbb{R}^n)$ which has a support disjoint from $\partial\Omega$, it holds $\langle \text{Tr}^* g, v \rangle_{V'_{\partial\Omega, [0]_{\partial\nu}}, V_{\partial\Omega, [0]_{\partial\nu}}} = 0$, hence:

$$-\Delta(\tilde{G} * \text{Tr}^* g) = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega.$$

By formula (B.8), it also holds:

$$-\Delta(\tilde{G} * \text{Tr}^* g) = \text{Tr}^* \left[\left[\frac{\partial(\tilde{G} * \text{Tr}^* g)}{\partial \nu} \right] \right],$$

therefore, since $\text{Tr}^* : \mathcal{B}'_0(\partial\Omega) \rightarrow V'_{\partial\Omega, [0]_{\partial\nu}}(\mathbb{R}^n)$ is into,

$$\left[\left[\frac{\partial(\tilde{G} * \text{Tr}^* g)}{\partial \nu} \right] \right] = g,$$

and $\tilde{G} * \text{Tr}^* g$ is a solution to the transmission problem associated to $g \in \mathcal{B}'_0(\partial\Omega)$.

Then, using the Kelvin transform and the uniqueness of the solution on Ω as soon as $\int_{\Omega} u \, dx = 0$, it also follows the uniqueness of the exterior problem. In addition, by [92, p. 379], its limit for $|x| \rightarrow +\infty$ is a constant. Therefore, the unique constant compatible by the Kelvin transformation with $\int_{\Omega} u \, dx = 0$, is zero (see also [92, Theorem 1, V.28.2, p. 418] ensuring that a solution to the Poisson equation is unique in the class of distributions $(C_0^\infty(\mathbb{R}^n))'$ with the assumption of taking the value zero at infinity).

The formula given by that theorem yields the *a priori* estimate.

For $n \geq 3$ the space $W_{\partial\Omega, \langle 0 \rangle_{\Omega}}(\mathbb{R}^n)$ is a Hilbert space corresponding to the inner product associated to the norm (B.17). Then, we apply the Riesz representation theorem to the variational formulation (B.21) and obtain the well-posedness result. From the uniqueness, formula (B.20) allows to calculate the solution. \square

From the well-posedness of the transmission problem, we may define the single layer potential operator for the transmission problem (4.1) for $k = 0$ as a linear continuous operator.

Proposition B.11. *Let Ω be a bounded two-sided extension domain of \mathbb{R}^n . Let \tilde{G} be the fundamental solution to $-\Delta$ on \mathbb{R}^n . Let the single layer potential operator be defined as the operator mapping the jump in normal derivative $\left[\left[\frac{\partial u}{\partial \nu} \right] \right] = g \in \mathcal{B}'_0(\partial\Omega)$ to the unique solution $u \in V_{\partial\Omega}(\mathbb{R}^n)$ to the transmission problem (4.1) with $k = 0$ and $f = 0$, which can be expressed by the convolution formula (B.20):*

$$\tilde{\mathcal{S}}_{\partial\Omega} : g \in \mathcal{B}'_0(\partial\Omega) \longmapsto \tilde{\mathcal{S}}_{\partial\Omega} g := \tilde{G} * \text{Tr}^* g \in V_{\partial\Omega}(\mathbb{R}^n). \quad (\text{B.23})$$

Then this operator is linear and continuous.

For the transmission problem (4.1) with $k = 0$ and $g = 0$, we proceed as in the bounded case. Note that since the solution u is harmonic on both Ω and Ω^c , then both $\frac{\partial_i u}{\partial \nu}$ and $\frac{\partial_e u}{\partial \nu}$ are elements of $\mathcal{B}'_{0, \text{ext}}(\partial\Omega)$ – or equivalently (Theorem A.5) elements of $\mathcal{B}'_0(\partial\Omega)$. However, though $\text{Tr}_i u \in \mathcal{B}_0(\partial\Omega)$ by definition, that does not imply that $\text{Tr}_e u$ must be in $\mathcal{B}_0(\partial\Omega)$ as well (a counter-example can be $u = cte$). Therefore, it makes sense to consider a Dirichlet boundary condition in $\mathcal{B}(\partial\Omega)$, even when it is not an element of $\mathcal{B}_0(\partial\Omega)$.

For a fixed $f \in \mathcal{B}(\partial\Omega)$, we choose $\varphi \in H_{\partial\Omega}^1(\mathbb{R}^n)$ with $\Delta\varphi|_{\Omega} \in L^2(\Omega)$ and $\Delta\varphi|_{\Omega^c} \in L^2(\Omega^c)$ such that $[[\text{Tr } \varphi]] = -f$ and $[[\frac{\partial\varphi}{\partial\nu}]] = 0$. Then $u \in H_{\partial\Omega}^1(\mathbb{R}^n)$ is a solution to the transmission problem if and only if $w := u - \varphi$ is a solution to:

$$\begin{cases} -\Delta w = \Delta\varphi & \text{on } \mathbb{R}^n \setminus \partial\Omega, \\ [[\text{Tr } w]] = 0, \quad \left[\left[\frac{\partial w}{\partial\nu} \right] \right] = 0, \end{cases}$$

the variational form of which is:

$$\forall v \in V_{\partial\Omega, [0]_{\text{Tr}}}(\mathbb{R}^n), \quad \int_{\Omega \cup \Omega^c} \nabla w \cdot \nabla v \, dx = \int_{\Omega \cup \Omega^c} (\Delta\varphi)v \, dx,$$

where $V_{\partial\Omega, [0]_{\text{Tr}}}(\mathbb{R}^n)$ is defined on pages iii–vi. Using Green's formula, it can be written as:

$$\begin{aligned} \forall v \in V_{\partial\Omega, [0]_{\text{Tr}}}(\mathbb{R}^n), \quad \int_{\Omega \cup \Omega^c} \nabla w \cdot \nabla v \, dx &= \int_{\Omega} (\Delta\varphi)v \, dx \\ &\quad - \int_{\Omega^c} \nabla\varphi \cdot \nabla v \, dx - \left\langle \frac{\partial\varphi}{\partial\nu}, \text{Tr } v \right\rangle_{\mathcal{B}', \mathcal{B}}. \end{aligned} \quad (\text{B.24})$$

If $n \geq 3$, the left-hand side of the variational formulation (B.24) is none other than an inner product on $V_{\partial\Omega}(\mathbb{R}^n)$, therefore Riesz representation theorem applies. For $n = 2$, we proceed as in the case of the jump in normal derivative, constructing the solution defined by a convolution with the fundamental solution to $-\Delta$ on \mathbb{R}^n . We summarize the results in the following theorem.

Proposition B.12. *Let Ω be a bounded two-sided extension domain of \mathbb{R}^n . If $f \in \mathcal{B}(\partial\Omega)$, then the unique solution $u \in V_{\partial\Omega}(\mathbb{R}^n)$ to the transmission problem (4.1) for $k = 0$ and $g = 0$ associated to f can be written as:*

$$\forall x \in \mathbb{R}^n \setminus \partial\Omega, \quad u(x) = \tilde{\mathcal{D}}_{\partial\Omega} f(x) := (\tilde{G} * \mathfrak{D}^* f)(x), \quad (\text{B.25})$$

where \tilde{G} is the fundamental solution to $-\Delta$ on \mathbb{R}^n , $\mathfrak{D}^* : \mathcal{B}(\partial\Omega) \rightarrow (H^1(\mathbb{R}^n))'$ and the convolution is on \mathbb{R}^n . Then the mapping $\tilde{\mathcal{D}}_{\partial\Omega} : -[[\text{Tr } u]] = f \in \mathcal{B}(\partial\Omega) \mapsto \tilde{\mathcal{D}}_{\partial\Omega} f := u \in V_{\partial\Omega}(\mathbb{R}^n)$ is linear bounded operator, called the double layer potential operator, and it holds:

$$\tilde{\mathcal{D}}_{\partial\Omega} f = \langle \mathfrak{D} \tilde{G}(x - \cdot), f \rangle_{\mathcal{B}', \mathcal{B}}. \quad (\text{B.26})$$

Proof. The boundedness of the double layer potential operator follows from the *a priori* estimate and Riesz representation theorem for $n \geq 3$ and from the direct estimate of the convolution formula. All other statements are straightforward. \square

As for equation (B.11), for regular boundaries (at least C^1), the solution $\tilde{\mathcal{D}}_{\partial\Omega}f$ to the transmission problem (4.1) for $k = 0$ and $g = 0$ is in $V_{\partial\Omega}(\mathbb{R}^n) \cap H_{\partial\Omega}^2(\mathbb{R}^n)$ and the classical formula is well-defined in $L^2(\partial\Omega, \sigma)$:

$$\begin{aligned} \forall x \in \mathbb{R}^n \setminus \partial\Omega, \quad \tilde{\mathcal{D}}_{\partial\Omega}f(x) &= (\tilde{G} * \mathfrak{D}^*f)(x) = (\mathfrak{D}\tilde{G}(x - \cdot), f)_{L^2(\partial\Omega, \sigma)} \\ &= \int_{\partial\Omega} \frac{\partial\tilde{G}(x - y)}{\partial\nu_y} f(y) \sigma(dy). \end{aligned} \tag{B.27}$$

We establish the general relation between an element of $W_{\partial\Omega, (0)\Omega}(\mathbb{R}^n)$ and its behaviour on $\partial\Omega$ by the analogous way as in Proposition B.5.

Proposition B.13. (Representation formula for $-\Delta$) *Let $u \in W_{\partial\Omega, (0)\Omega}(\mathbb{R}^n)$, such that $-\Delta u = f$ on $\mathbb{R}^n \setminus \partial\Omega$ with $f \in L^2(\mathbb{R}^n)$. It holds, for all $v \in H^1(\mathbb{R}^n)$:*

$$\langle -\Delta u, v \rangle_{L^2(\mathbb{R}^n)} = \langle f, v \rangle_{L^2(\mathbb{R}^n)} - \langle \mathfrak{D}v, \llbracket \text{Tr } u \rrbracket \rangle_{\mathcal{B}', \mathcal{B}} + \left\langle \left[\left[\frac{\partial u}{\partial \nu} \right] \right], \text{Tr } v \right\rangle_{\mathcal{B}', \mathcal{B}},$$

where $\text{Tr } v := \text{Tr}_i v = \text{Tr}_e v$ as well as $\mathfrak{D}v := \mathfrak{D}_i v = \mathfrak{D}_e v$.

Finally, as previously for $(-\Delta+1)$ in Theorem B.7, we also establish Green's third identity for $-\Delta$:

Theorem B.14. *Let Ω be a two-sided extension domain \mathbb{R}^n . The unique solution on $V_{\partial\Omega}(\mathbb{R}^n)$ to transmission problem (4.1) with $k = 0$ for all $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'_0(\partial\Omega)$, is given by $u = \tilde{\mathcal{S}}_{\partial\Omega}g - \tilde{\mathcal{D}}_{\partial\Omega}f$.*

B.2 Approximation by truncated problems

For the operator $(-\Delta+1)$. First, let us assume $k = 1$. Since problem (4.10) is understood in the weak sense, the solutions are sought in an H^1 -type space. Allowing jumps across $\partial\Omega$ and considering the homogeneous Dirichlet boundary condition on ∂U , we may define the space of solutions:

$$H_{0, \partial\Omega}^1(U) := \{u \in L^2(U) \mid \nabla u|_{U \setminus \partial\Omega} \in L^2(U \setminus \partial\Omega)^n, \text{Tr}^{\partial U} u = 0\} \tag{B.28}$$

endowed with the norm product:

$$\|\cdot\|_{H_{0, \partial\Omega}^1(U)} := \sqrt{\|\cdot\|_{H^1(\Omega)}^2 + \|\cdot\|_{H^1(U \setminus \bar{\Omega})}^2}. \tag{B.29}$$

Remark B.15. *The homogeneous Dirichlet boundary condition on U allows to extend elements of $H_{0, \partial\Omega}^1(U)$ to $H_{\partial\Omega}^1(\mathbb{R}^n)$ by setting 0 outside of U . If $u \in H_{0, \partial\Omega}^1(U)$, we will keep denoting that extension by u and it holds $\|u\|_{H_{0, \partial\Omega}^1(U)} = \|u\|_{H_{\partial\Omega}^1(\mathbb{R}^n)}$.*

Let us start with defining the single layer potential operator for the truncated problem (4.10) with $k = 1$ and $f = 0$. Proceeding in the same way as for (B.2), the variational formulation associated to the truncated transmission problem with no jump in trace is given by:

$$\forall v \in H_{0,\partial\Omega,[0]_{\text{Tr}}}^1(U), \quad \langle u, v \rangle_{H_{0,\partial\Omega}^1(U)} = \langle g, \text{Tr}^{\partial\Omega} v \rangle_{\mathcal{B}', \mathcal{B}}, \quad (\text{B.30})$$

where $H_{0,\partial\Omega,[0]_{\text{Tr}}}^1(U)$ is defined on pages iii–vi. By the Riesz representation theorem, the problem is well-posed.

Proposition B.16. *Let Ω be a two-sided extension domain of \mathbb{R}^n , and let U be a one-sided extension domain such that $\Omega \subset\subset U$. Let*

$$\begin{aligned} \mathcal{S}_{\partial\Omega}^U : \mathcal{B}'(\partial\Omega) &\longrightarrow \mathcal{S}_{\partial\Omega}^U(\mathcal{B}'(\partial\Omega)) \subset H_{0,\partial\Omega}^1(U) \\ g = \left[\left[\frac{\partial u}{\partial \nu} \right] \right]_{\partial\Omega} &\longmapsto u = \mathcal{S}_{\partial\Omega}^U g, \end{aligned}$$

be the linear operator uniquely defined by

$$\forall v \in H_{0,\partial\Omega,[0]_{\text{Tr}}}^1(U), \quad \langle \mathcal{S}_{\partial\Omega}^U g, v \rangle_{H_{0,\partial\Omega}^1(U)} = \langle g, \text{Tr} v \rangle_{\mathcal{B}', \mathcal{B}}. \quad (\text{B.31})$$

Then $\mathcal{S}_{\partial\Omega}^U : \mathcal{B}'(\partial\Omega) \rightarrow H_{0,\partial\Omega}^1(U)$, called the single layer potential operator for the transmission problem truncated at U , is a linear bounded operator:

$$\|\mathcal{S}_{\partial\Omega}^U g\|_{H_{0,\partial\Omega}^1(U)} \leq C \|g\|_{\mathcal{B}'(\partial\Omega)} \quad (\text{B.32})$$

for some constant $C := C(n, \Omega) > 0$.

In the same way, consider problem (4.10) with $k = 1$ and $g = 0$ to define the double layer potential operator truncated at U . The variational formulation for the truncated transmission problem with no jump in normal derivative is given by (as mentioned before, all normal derivatives are on $\partial\Omega$):

$$\forall v \in H_{0,\partial\Omega,[-f]_{\text{Tr}}}^1(U), \quad \langle u, v \rangle_{H_{0,\partial\Omega}^1(U)} = - \left\langle \frac{\partial u}{\partial \nu}, f \right\rangle_{\mathcal{B}', \mathcal{B}(\partial\Omega)}. \quad (\text{B.33})$$

Again, using the same method as in [38, Theorem 7] and the Riesz representation theorem, this problem is well-posed.

Proposition B.17. *Let Ω be a two-sided extension domain of \mathbb{R}^n , and let U be a bounded one-sided extension domain such that $\Omega \subset\subset U$. Let*

$$\begin{aligned} \mathcal{D}_{\partial\Omega}^U : \mathcal{B}(\partial\Omega) &\longrightarrow \mathcal{D}_{\partial\Omega}^U(\mathcal{B}(\partial\Omega)) \subset H_{0,\partial\Omega}^1(U) \\ f = - \llbracket \text{Tr} u \rrbracket_{\partial\Omega} &\longmapsto u = \mathcal{D}_{\partial\Omega}^U f, \end{aligned}$$

be the linear operator uniquely defined by

$$\forall v \in H_{0,\partial\Omega,[-f]_{\text{Tr}}}^1(U), \quad \langle \mathcal{D}_{\partial\Omega}^U f, v \rangle_{H_{0,\partial\Omega}^1(U)} = - \left\langle \frac{\partial(\mathcal{D}_{\partial\Omega}^U f)}{\partial \nu}, f \right\rangle_{\mathcal{B}', \mathcal{B}(\partial\Omega)}, \quad (\text{B.34})$$

where $\frac{\partial}{\partial \nu}(\mathcal{D}_{\partial\Omega}^U f) := \frac{\partial_i}{\partial \nu}(\mathcal{D}_{\partial\Omega}^U f) = \frac{\partial_{\bar{i}}}{\partial \nu}(\mathcal{D}_{\partial\Omega}^U f)$.

Then $\mathcal{D}_{\partial\Omega}^U : \mathcal{B}(\partial\Omega) \rightarrow H_{0,\partial\Omega}^1(U)$, called the double layer potential operator for the transmission problem truncated at U , is a linear bounded operator:

$$\|\mathcal{D}_{\partial\Omega}^U f\|_{H_{0,\partial\Omega}^1(U)} \leq C \|f\|_{\mathcal{B}(\partial\Omega)} \quad (\text{B.35})$$

for some constant $C := C(n, \Omega) > 0$.

We may now prove the truncated transmission problem (4.10) is an approximation of the transmission problem on the whole space (4.1) for $k = 1$, in the sense of the strong convergence of the associated solutions. As it is done in the Lipschitz case [3], let us consider an increasing sequence of bounded one-sided extension domains $(U_k)_{k \in \mathbb{N}}$ such that there exists an increasing sequence $(B_k)_{k \in \mathbb{N}}$ of balls such that:

$$\forall k \in \mathbb{N}, \quad \Omega \subset\subset B_k \subset U_k \quad \text{and} \quad \bigcup_{k \in \mathbb{N}} B_k = \mathbb{R}^n. \quad (\text{B.36})$$

We will denote $\mathcal{S}_{\partial\Omega}^k := \mathcal{S}_{\partial\Omega}^{U_k}$ and $\mathcal{D}_{\partial\Omega}^k := \mathcal{D}_{\partial\Omega}^{U_k}$ for $k \in \mathbb{N}$ the single and double layer potentials for the problem truncated at U_k respectively. For $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'(\partial\Omega)$, $\mathcal{D}_{\partial\Omega}^k f$ and $\mathcal{S}_{\partial\Omega}^k g$ will be seen as elements of $H_{\partial\Omega}^1(\mathbb{R}^n)$ by setting 0 outside of U_k .

Theorem B.18. *Let Ω be a two-sided extension domain of \mathbb{R}^n , and $(U_k)_{k \in \mathbb{N}}$ be an increasing sequence of bounded one-sided extension domains such that there exists an increasing sequence $(B_k)_{k \in \mathbb{N}}$ of balls satisfying (B.36). Then,*

(i) *for any $g \in \mathcal{B}'(\partial\Omega)$, it holds:*

$$\mathcal{S}_{\partial\Omega}^k g \xrightarrow[k \rightarrow \infty]{} \mathcal{S}_{\partial\Omega} g \quad \text{in } H_{\partial\Omega}^1(\mathbb{R}^n).$$

(ii) *for any $f \in \mathcal{B}(\partial\Omega)$, it holds:*

$$\mathcal{D}_{\partial\Omega}^k f \xrightarrow[k \rightarrow \infty]{} \mathcal{D}_{\partial\Omega} f \quad \text{in } H_{\partial\Omega}^1(\mathbb{R}^n).$$

Proof. For point (i), let $g \in \mathcal{B}'(\partial\Omega)$, and let us denote $u_k := \mathcal{S}_{\partial\Omega}^k g$ and $u := \mathcal{S}_{\partial\Omega} g$. We follow the steps of the proof of [3, Theorem 4.3]. For all $k \in \mathbb{N}$, it holds:

$$\begin{aligned} \|u_k\|_{H_{\partial\Omega}^1(\mathbb{R}^n)}^2 &= \|u_k\|_{H_{\partial\Omega}^1(U_k)}^2 = \langle g, \text{Tr}^{\partial\Omega} u_k \rangle_{\mathcal{B}', \mathcal{B}} \\ &\leq \|g\|_{\mathcal{B}'} \|u_k\|_{H_{\partial\Omega}^1(U_k)} = \|g\|_{\mathcal{B}'} \|u_k\|_{H_{\partial\Omega}^1(\mathbb{R}^n)} \end{aligned}$$

Therefore, $(u_k)_{k \in \mathbb{N}}$ is bounded in $H_{\partial\Omega}^1(\mathbb{R}^n)$ and has a weak accumulation point, denoted by $u_\infty \in H_{\partial\Omega}^1(\mathbb{R}^n)$. Since $(U_k)_{k \in \mathbb{N}}$ is an increasing sequence, considering the extensions of elements of $H_{\partial\Omega}^1(U_k)$ by setting 0 outside of U_k for all $k \in \mathbb{N}$, we can say $(H_{\partial\Omega}^1(U_k))_{k \in \mathbb{N}}$ forms an increasing sequence of Hilbert

subspaces of $H_{\partial\Omega}^1(\mathbb{R}^n)$. Therefore, passing to the limit in the variational formulation (B.39) (the left hand side can be seen as an inner product on $H_{\partial\Omega}^1(\mathbb{R}^n)$), it holds:

$$\forall v \in \bigcup_{k \in \mathbb{N}} H_{\partial\Omega, [0]_{\text{Tr}}}^1(U_k), \quad \langle u_\infty, v \rangle_{H_{\partial\Omega}^1(\mathbb{R}^n)} = \langle g, \text{Tr}^{\partial\Omega} v \rangle_{\mathcal{B}', \mathcal{B}}.$$

By density of $\bigcup_{k \in \mathbb{N}} H_{\partial\Omega, [0]_{\text{Tr}}}^1(U_k)$ in $H_{\partial\Omega, [0]_{\text{Tr}}}^1(\mathbb{R}^n)$, the previous inequality holds for all $v \in H_{\partial\Omega, [0]_{\text{Tr}}}^1(\mathbb{R}^n)$, hence $u = u_\infty$. Since u is the only weak accumulation point of the bounded sequence $(u_k)_{k \in \mathbb{N}}$ in $H_{\partial\Omega}^1(\mathbb{R}^n)$, it holds $u_k \rightharpoonup u$ in $H_{\partial\Omega}^1(\mathbb{R}^n)$ as $k \rightarrow \infty$. By the variational formulations (B.2) for the problem on the whole space and (B.31) for the truncated problems, it holds:

$$\forall k \in \mathbb{N}, \quad \|u_k\|_{H_{\partial\Omega}^1(\mathbb{R}^n)} = \|u_k\|_{H_{\partial\Omega}^1(U_k)} = \|u\|_{H_{\partial\Omega}^1(\mathbb{R}^n)},$$

hence the strong convergence.

Point (ii) is proved in a similar way, using the boundedness of the interior normal derivation operator and straightforward norm inequalities to yield the boundedness of the sequence of solutions to the truncated problem. \square

Consequently, Green's third identity (Proposition B.7) for the truncated problem shows the solutions to the truncated problem strongly converge in $H_{\partial\Omega}^1(\mathbb{R}^n)$ to the solution of the transmission problem on the whole space as the truncation domain U grows to \mathbb{R}^n in the sense of (B.36).

For the operator $-\Delta$. Let us now focus on the case $k = 0$. As for the transmission problem on the whole space, uniqueness conditions need to be added to problem (4.10) for it to be well-posed. Since we will be considering the variational formulation of the transmission problems (truncated and on the whole space), we assume $n \geq 3$. Let us introduce the space on which the problem is to be solved:

$$H_{0, \partial\Omega, \langle 0 \rangle_\Omega}^1(U) := \left\{ u \in L^2(U) \left| \begin{array}{l} \nabla u|_{U \setminus \partial\Omega} \in L^2(U \setminus \partial\Omega)^n, \\ \int_\Omega u \, dx = 0 \quad \text{and} \quad \text{Tr}^{\partial U} u = 0 \end{array} \right. \right\}, \quad (\text{B.37})$$

endowed with the inner product:

$$\| \cdot \|_{H_{0, \partial\Omega, \langle 0 \rangle_\Omega}^1(U)}^2 := \|\nabla \cdot\|_{L^2(\Omega)^n}^2 + \|\nabla \cdot\|_{L^2(U \setminus \bar{\Omega})^n}^2 \quad (\text{B.38})$$

Remark B.19. Similarly to the case $k = 1$, the homogeneous Dirichlet boundary condition on U allows to extend elements of $H_{0, \partial\Omega, \langle 0 \rangle_\Omega}^1(U)$ to $W_{\partial\Omega, \langle 0 \rangle_\Omega}(\mathbb{R}^n)$ by setting 0 outside of U . If $u \in H_{0, \partial\Omega, \langle 0 \rangle_\Omega}^1(U)$, we will keep denoting that extension by u and it holds $\|u\|_{H_{0, \partial\Omega, \langle 0 \rangle_\Omega}^1(U)} = \|u\|_{W_{\partial\Omega, \langle 0 \rangle_\Omega}(\mathbb{R}^n)}$.

Let us start with defining the single layer potential operator for the truncated problem (4.10) with $k = 0$ and $f = 0$. The variational formulation associated to the truncated transmission problem with no jump in trace is given by:

$$\forall v \in H_{0,\partial\Omega,(0)\Omega,[0]_{\text{Tr}}}^1(U), \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{U \setminus \bar{\Omega}} \nabla u \cdot \nabla v \, dx = \langle g, \text{Tr}^{\partial\Omega} v \rangle_{\mathcal{B}'_0, \mathcal{B}_0}, \quad (\text{B.39})$$

where $H_{0,\partial\Omega,(0)\Omega,[0]_{\text{Tr}}}^1(U)$ is defined on pages iii–vi. By the Riesz representation theorem, the problem is well-posed.

Proposition B.20. *Let Ω be a two-sided extension domain of \mathbb{R}^n , $n \geq 3$, and let U be a bounded one-sided extension domain such that $\Omega \subset\subset U$. Let*

$$\begin{aligned} \tilde{\mathcal{S}}_{\partial\Omega}^U : \mathcal{B}'_0(\partial\Omega) &\longrightarrow \tilde{\mathcal{S}}_{\partial\Omega}^U(\mathcal{B}'_0(\partial\Omega)) \subset H_{0,\partial\Omega,(0)\Omega}^1(U) \\ g = \left[\begin{array}{c} \partial u \\ \partial \nu \end{array} \right] &\longmapsto u = \tilde{\mathcal{S}}_{\partial\Omega}^U g, \end{aligned}$$

be the linear operator uniquely defined by

$$\forall v \in H_{0,\partial\Omega,(0)\Omega,[0]_{\text{Tr}}}^1(U), \quad \langle \tilde{\mathcal{S}}_{\partial\Omega}^U g, v \rangle_{H_{0,\partial\Omega,(0)\Omega}^1(U)} = \langle g, \text{Tr} v \rangle_{\mathcal{B}'_0, \mathcal{B}_0}. \quad (\text{B.40})$$

Then $\tilde{\mathcal{S}}_{\partial\Omega}^U : \mathcal{B}'_0(\partial\Omega) \rightarrow H_{0,\partial\Omega,(0)\Omega}^1(U)$, called the single layer potential operator for the transmission problem truncated at U , is a linear bounded operator:

$$\|\tilde{\mathcal{S}}_{\partial\Omega}^U g\|_{H_{0,\partial\Omega,(0)\Omega}^1(U)} \leq C \|g\|_{\mathcal{B}'_0(\partial\Omega)} \quad (\text{B.41})$$

for some constant $C := C(\Omega) > 0$.

In the same way, consider problem (4.10) with $k = 0$ and $g = 0$. The variational formulation for the truncated transmission problem with no jump in normal derivative is given by:

$$\forall v \in H_{0,\partial\Omega,(0)\Omega,[-f]_{\text{Tr}}}^1(U), \quad \langle u, v \rangle_{H_{0,\partial\Omega,(0)\Omega}^1(U)} = - \left\langle \frac{\partial u}{\partial \nu}, f \right\rangle_{\mathcal{B}'_0, \mathcal{B}_0}. \quad (\text{B.42})$$

Using the same method as in [38, Theorem 7], this problem is well-posed.

Proposition B.21. *Let Ω be a two-sided extension domain of \mathbb{R}^n , $n \geq 3$, and let U be a bounded one-sided extension domain such that $\Omega \subset\subset U$. Let*

$$\begin{aligned} \tilde{\mathcal{D}}_{\partial\Omega}^U : \mathcal{B}(\partial\Omega) &\longrightarrow \tilde{\mathcal{D}}_{\partial\Omega}^U(\mathcal{B}(\partial\Omega)) \subset H_{0,\partial\Omega,(0)\Omega}^1(U) \\ f = - \llbracket \text{Tr} u \rrbracket_{\partial\Omega} &\longmapsto u = \tilde{\mathcal{D}}_{\partial\Omega}^U f, \end{aligned}$$

be the linear operator uniquely defined by

$$\forall v \in H_{0,\partial\Omega,(0)\Omega,[-f]_{\text{Tr}}}^1(U), \quad \langle \tilde{\mathcal{D}}_{\partial\Omega}^U f, v \rangle_{H_{0,\partial\Omega,(0)\Omega}^1(U)} = - \left\langle \frac{\partial(\tilde{\mathcal{D}}_{\partial\Omega}^U f)}{\partial \nu}, f \right\rangle_{\mathcal{B}'_0, \mathcal{B}_0}, \quad (\text{B.43})$$

where $\frac{\partial}{\partial \nu}(\tilde{\mathcal{D}}_{\partial\Omega}^U f) := \frac{\partial_i}{\partial \nu}(\tilde{\mathcal{D}}_{\partial\Omega}^U f) = \frac{\partial_e}{\partial \nu}(\tilde{\mathcal{D}}_{\partial\Omega}^U f)$.

Then $\tilde{\mathcal{D}}_{\partial\Omega}^U : \mathcal{B}(\partial\Omega) \rightarrow H_{0,\partial\Omega,(0)\Omega}^1(U)$, called the double layer potential operator for the transmission problem truncated at U , is a linear bounded operator:

$$\|\tilde{\mathcal{D}}_{\partial\Omega}^U f\|_{H_{0,\partial\Omega,(0)\Omega}^1(U)} \leq C \|f\|_{\mathcal{B}(\partial\Omega)} \quad (\text{B.44})$$

for some constant $C := C(\Omega) > 0$.

We may now prove the truncated transmission problem (4.10) is an approximation of (4.1) for $k = 0$, in the sense of the strong convergence of the associated solutions. Let us consider an increasing sequence of bounded one-sided extension domains $(U_k)_{k \in \mathbb{N}}$ in the sense of (B.36). We will denote $\tilde{\mathcal{S}}_{\partial\Omega}^k := \tilde{\mathcal{S}}_{\partial\Omega}^{U_k}$ and $\tilde{\mathcal{D}}_{\partial\Omega}^k := \tilde{\mathcal{D}}_{\partial\Omega}^{U_k}$ for $k \in \mathbb{N}$ the single and double layer potentials for the problem truncated at U_k respectively. For $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'_0(\partial\Omega)$, $\tilde{\mathcal{D}}_{\partial\Omega}^k f$ and $\tilde{\mathcal{S}}_{\partial\Omega}^k g$ will be seen as elements of $W_{\partial\Omega,(0)\Omega}(\mathbb{R}^n)$ by setting 0 outside of U_k .

Theorem B.22. *Let Ω be a two-sided extension domain of \mathbb{R}^n , $n \geq 3$, and $(U_k)_{k \in \mathbb{N}}$ be an increasing sequence of bounded one-sided extension domains such that there exists an increasing sequence of balls satisfying (B.36). Then,*

(i) *for any $g \in \mathcal{B}'_0(\partial\Omega)$, it holds:*

$$\tilde{\mathcal{S}}_{\partial\Omega}^k g \longrightarrow \tilde{\mathcal{S}}_{\partial\Omega} g \quad \text{in } W_{\partial\Omega,(0)\Omega}(\mathbb{R}^n) \quad \text{as } k \rightarrow +\infty.$$

(ii) *for any $f \in \mathcal{B}(\partial\Omega)$, it holds:*

$$\tilde{\mathcal{D}}_{\partial\Omega}^k f \longrightarrow \tilde{\mathcal{D}}_{\partial\Omega} f \quad \text{in } W_{\partial\Omega,(0)\Omega}(\mathbb{R}^n) \quad \text{as } k \rightarrow +\infty.$$

Proof. We proceed in the same way as for Theorem B.18. The density of $\bigcup_{k \in \mathbb{N}} H_{0,\partial\Omega,(0)\Omega}^1(U_k)$ in $W_{\partial\Omega,(0)\Omega}(\mathbb{R}^n)$ comes from [3, Theorem 4.3]. \square

B.3 Neumann-Poincaré operators and boundary values

Unlike the usual definition of the Neumann-Poincaré operator as a singular integral on the boundary [77, 91], we choose to define it as the mean value of the interior and the exterior traces of the solution to the Dirichlet transmission problem.

Definition B.23. (Neumann-Poincaré operator) *If Ω is a bounded two-sided extension domain of \mathbb{R}^n , let us define:*

(i) $\mathcal{K}_{\partial\Omega}$, the Neumann-Poincaré operator for the transmission problem (4.1), associated to $(-\Delta+1)$ for $g = 0$, as the average of the traces of $u = \mathcal{D}_{\partial\Omega} f$:

$$\begin{aligned} \mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) &\longrightarrow \mathcal{B}(\partial\Omega) \\ f = -\llbracket \text{Tr } u \rrbracket &\longmapsto \frac{1}{2}(\text{Tr}_i + \text{Tr}_e)u = \frac{1}{2}(\text{Tr}_i + \text{Tr}_e) \circ \mathcal{D}_{\partial\Omega} f. \end{aligned}$$

(ii) $\tilde{\mathcal{K}}_{\partial\Omega}$, the Neumann-Poincaré operator for the transmission problem (4.1) associated to $-\Delta$ for $g = 0$ as the average of the traces of $u = \tilde{\mathcal{D}}_{\partial\Omega}f$:

$$\begin{aligned} \tilde{\mathcal{K}}_{\partial\Omega} &: \mathcal{B}(\partial\Omega) \longrightarrow \mathcal{B}(\partial\Omega) \\ f = -\llbracket \text{Tr } u \rrbracket &\longmapsto \frac{1}{2}(\text{Tr}_i + \text{Tr}_e)u = \frac{1}{2}(\text{Tr}_i + \text{Tr}_e) \circ \tilde{\mathcal{D}}_{\partial\Omega}f. \end{aligned}$$

In other words, since for all $f \in \mathcal{B}(\partial\Omega)$, $\llbracket \text{Tr}(\mathcal{D}_{\partial\Omega}f) \rrbracket = -f = \llbracket \text{Tr}(\tilde{\mathcal{D}}_{\partial\Omega}f) \rrbracket$, it holds:

$$\left\{ \begin{array}{l} \text{Tr}_i(\mathcal{D}_{\partial\Omega}f) = \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right) f, \\ \text{Tr}_e(\mathcal{D}_{\partial\Omega}f) = \left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right) f, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \text{Tr}_i(\tilde{\mathcal{D}}_{\partial\Omega}f) = \left(-\frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega}\right) f, \\ \text{Tr}_e(\tilde{\mathcal{D}}_{\partial\Omega}f) = \left(\frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega}\right) f. \end{array} \right. \quad (\text{B.45})$$

Given that definition, the Neumann-Poincaré operators are connected to the jumps in trace of the double layer potentials. The following proposition states their adjoints play the same part for the jumps in normal derivatives of the single layer potentials. Note that the sign convention $\llbracket \text{Tr}(\mathcal{D}_{\partial\Omega}f) \rrbracket = -f$ we chose earlier induces those adjoint operators are equal to the average of the normal derivatives of the single layer potentials, instead of the opposite of that quantity.

Proposition B.24. *Let Ω be a bounded two-sided extension domain of \mathbb{R}^n . Then:*

(i) *for all $g \in \mathcal{B}'(\partial\Omega)$, it holds:*

$$\left\{ \begin{array}{l} \frac{\partial_i(\mathcal{S}_{\partial\Omega}g)}{\partial\nu} = \left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^*\right) g \\ \frac{\partial_e(\mathcal{S}_{\partial\Omega}g)}{\partial\nu} = \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^*\right) g \end{array} \right. , \quad (\text{B.46})$$

where $\mathcal{K}_{\partial\Omega}^* : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ denotes the adjoint operator to $\mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$. Hence, it holds:

$$\begin{aligned} \mathcal{K}_{\partial\Omega}^* &: \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega) \\ g &\mapsto \frac{1}{2} \left(\frac{\partial_i}{\partial\nu} + \frac{\partial_e}{\partial\nu} \right) \circ \mathcal{S}_{\partial\Omega}g. \end{aligned}$$

Moreover, the boundary operators $\mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ and $\mathcal{K}_{\partial\Omega}^* : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ are linear and continuous;

(ii) *For all $g \in \mathcal{B}'_{0,\text{ext}}(\partial\Omega)$, it holds:*

$$\left\{ \begin{array}{l} \frac{\partial_i(\tilde{\mathcal{S}}_{\partial\Omega}g)}{\partial\nu} = \left(\frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega}^*\right) g \\ \frac{\partial_e(\tilde{\mathcal{S}}_{\partial\Omega}g)}{\partial\nu} = \left(-\frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega}^*\right) g \end{array} \right. , \quad (\text{B.47})$$

where $\tilde{\mathcal{K}}_{\partial\Omega}^* : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ denotes the adjoint operator to $\tilde{\mathcal{K}}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$. Thus, it holds:

$$\begin{aligned} \tilde{\mathcal{K}}_{\partial\Omega}^* : \mathcal{B}'_0(\partial\Omega) &\rightarrow \mathcal{B}'_0(\partial\Omega) \\ g &\mapsto \frac{1}{2} \left(\frac{\partial_i}{\partial\nu} + \frac{\partial_e}{\partial\nu} \right) \circ \tilde{\mathcal{S}}_{\partial\Omega} g. \end{aligned}$$

Moreover, the boundary operators $\tilde{\mathcal{K}}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ and $\tilde{\mathcal{K}}_{\partial\Omega}^* : \mathcal{B}'_0(\partial\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$ are linear and continuous.

Proof. We proceed as it is done in [66]. Let $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'(\partial\Omega)$. Let us denote $\hat{\mathcal{K}}_{\partial\Omega} := \frac{1}{2} \left(\frac{\partial_i}{\partial\nu} + \frac{\partial_e}{\partial\nu} \right) \circ \mathcal{S}_{\partial\Omega}$. Then, given the definitions of the layer potential operators $\mathcal{S}_{\partial\Omega}$ and $\mathcal{D}_{\partial\Omega}$, it holds, by Green's formula:

$$\begin{aligned} 0 &= \left\langle \frac{\partial_i(\mathcal{D}_{\partial\Omega}f)}{\partial\nu}, \text{Tr}_i(\mathcal{S}_{\partial\Omega}g) \right\rangle_{\mathcal{B}',\mathcal{B}} - \left\langle \frac{\partial_e(\mathcal{D}_{\partial\Omega}f)}{\partial\nu}, \text{Tr}_e(\mathcal{S}_{\partial\Omega}g) \right\rangle_{\mathcal{B}',\mathcal{B}} \\ &= \int_{\mathbb{R}^n} \nabla(\mathcal{D}_{\partial\Omega}f) \cdot \nabla(\mathcal{S}_{\partial\Omega}g) \, dx + \int_{\mathbb{R}^n} (\mathcal{D}_{\partial\Omega}f)(\mathcal{S}_{\partial\Omega}g) \, dx \\ &= \left\langle \frac{\partial_i(\mathcal{S}_{\partial\Omega}g)}{\partial\nu}, \text{Tr}_i(\mathcal{D}_{\partial\Omega}f) \right\rangle_{\mathcal{B}',\mathcal{B}} - \left\langle \frac{\partial_e(\mathcal{S}_{\partial\Omega}g)}{\partial\nu}, \text{Tr}_e(\mathcal{D}_{\partial\Omega}f) \right\rangle_{\mathcal{B}',\mathcal{B}}. \end{aligned}$$

Hence:

$$\begin{cases} \left\langle \frac{\partial_i(\mathcal{D}f)}{\partial\nu}, \text{Tr}_i(\mathcal{S}_{\partial\Omega}g) \right\rangle_{\mathcal{B}',\mathcal{B}} = \left\langle \frac{\partial_e(\mathcal{D}_{\partial\Omega}f)}{\partial\nu}, \text{Tr}_e(\mathcal{S}_{\partial\Omega}g) \right\rangle_{\mathcal{B}',\mathcal{B}}, \\ \left\langle \frac{\partial_i(\mathcal{S}_{\partial\Omega}g)}{\partial\nu}, \text{Tr}_i(\mathcal{D}f) \right\rangle_{\mathcal{B}',\mathcal{B}} = \left\langle \frac{\partial_e(\mathcal{S}_{\partial\Omega}g)}{\partial\nu}, \text{Tr}_e(\mathcal{D}_{\partial\Omega}f) \right\rangle_{\mathcal{B}',\mathcal{B}}, \end{cases}$$

therefore we deduce:

$$\begin{aligned} \left\langle \left[\frac{\partial(\mathcal{S}_{\partial\Omega}g)}{\partial\nu} \right], \text{Tr}_i(\mathcal{D}_{\partial\Omega}f) + \text{Tr}_e(\mathcal{D}_{\partial\Omega}f) \right\rangle_{\mathcal{B}',\mathcal{B}} \\ = \left\langle \frac{\partial_i(\mathcal{S}_{\partial\Omega}g)}{\partial\nu} + \frac{\partial_e(\mathcal{S}_{\partial\Omega}g)}{\partial\nu}, -[\text{Tr}(\mathcal{D}_{\partial\Omega}f)] \right\rangle_{\mathcal{B}',\mathcal{B}}, \end{aligned}$$

which means:

$$\langle g, \mathcal{K}_{\partial\Omega}f \rangle_{\mathcal{B}',\mathcal{B}} = \langle \hat{\mathcal{K}}_{\partial\Omega}g, f \rangle_{\mathcal{B}',\mathcal{B}}.$$

Therefore, $\hat{\mathcal{K}}_{\partial\Omega} = \mathcal{K}_{\partial\Omega}^*$, and the fact that $\left[\frac{\partial(\mathcal{S}_{\partial\Omega}g)}{\partial\nu} \right] = g$ yields the formula.

The double layer potential $\mathcal{D}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow H_{\partial\Omega}^1(\mathbb{R}^n)$ is continuous, and the trace operators $\text{Tr}_i, \text{Tr}_e : H_{\partial\Omega}^1(\mathbb{R}^n) \rightarrow \mathcal{B}(\partial\Omega)$ are continuous. Therefore, the operator $\mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ and its adjoint operator $\mathcal{K}_{\partial\Omega}^* : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ are continuous, given Definition B.23.

We proceed in the same way to obtain the relations on the operator $\tilde{\mathcal{K}}_{\partial\Omega}^*$. If $g \in \mathcal{B}'_{0,ext}(\partial\Omega)$, then $\tilde{\mathcal{S}}_{\partial\Omega} g$ is harmonic on Ω and Ω^c , and $\tilde{\mathcal{K}}_{\partial\Omega}^* g \in \mathcal{B}'_{0,ext}(\partial\Omega)$. Therefore, $\tilde{\mathcal{K}}_{\partial\Omega}^*$ can be seen as an endomorphism of $\mathcal{B}'_{0,ext}(\partial\Omega)$. By Theorem A.5, the spaces $\mathcal{B}'_0(\partial\Omega)$ and $\mathcal{B}'_{0,ext}(\partial\Omega)$ are isomorphic, which means $\tilde{\mathcal{K}}_{\partial\Omega}^*$ can be seen as a continuous endomorphism of $\mathcal{B}'_0(\partial\Omega)$ instead. \square

The generalization of the Neumann-Poincaré operator from Definition B.23 allows to have the same connection to the boundary values of the layer potential operator as in the Lipschitz case, when that operator is defined as a singular integral.

Remark B.25. *We can give a formal definition of the operator $\tilde{\mathcal{K}}_{\partial\Omega}^*$ from $\mathcal{B}'(\partial\Omega)$ to $\mathcal{B}'(\partial\Omega)$. However, if we wish to establish the link between $\tilde{\mathcal{K}}_{\partial\Omega}^*$ and the solution to the transmission problem (4.1) for $g = 0$ and $k = 0$, as we did in the case of $(-\Delta + 1)$, then we need to restrict the domain of $\tilde{\mathcal{K}}_{\partial\Omega}^*$ to $\mathcal{B}'_{0,ext}(\partial\Omega)$, or equivalently to $\mathcal{B}'_0(\partial\Omega)$.*

From that definition of $\tilde{\mathcal{K}}_{\partial\Omega}^$ as a continuous endomorphism on $\mathcal{B}'_0(\partial\Omega)$, we can deduce its adjoint $\tilde{\mathcal{K}}_{\partial\Omega}$ defines a continuous endomorphism on $\mathcal{B}_0(\partial\Omega)$.*

We adapt [2, Lemma 2.5, p. 17] (see also [44]) from the L^2 -framework to our case:

Lemma B.26. *Let Ω be a bounded two-sided extension domain of \mathbb{R}^n . Then, for $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \frac{1}{2}$ it holds*

(i) *the operator $\lambda I + \mathcal{K}_{\partial\Omega}^*$ is an injection from $\mathcal{B}'(\partial\Omega)$ to $\mathcal{B}'(\partial\Omega)$;*

(ii) *the operator $\lambda I + \tilde{\mathcal{K}}_{\partial\Omega}^*$ is an injection from $\mathcal{B}'_0(\partial\Omega)$ to $\mathcal{B}'_0(\partial\Omega)$;*

Proof. Let $\lambda \in \mathbb{R}$. Let us assume there exists $\varphi \in \text{Ker}(\lambda I + \mathcal{K}_{\partial\Omega}^*) \setminus \{0\}$. Then, it holds $\mathcal{S}_{\partial\Omega}\varphi \in L^2(\mathbb{R}^n \setminus \partial\Omega)$ and $\nabla \mathcal{S}_{\partial\Omega}\varphi \in L^2(\mathbb{R}^n \setminus \partial\Omega)^n$. Since $\varphi \neq 0$, it holds:

$$\int_{\Omega} |\nabla \mathcal{S}_{\partial\Omega}\varphi|^2 dx > 0 \quad \text{or} \quad \int_{\Omega^c} |\nabla \mathcal{S}_{\partial\Omega}\varphi|^2 dx > 0,$$

that is, using Green's formula:

$$\underbrace{\left\langle \left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^* \right) \varphi, \text{Tr}(\mathcal{S}_{\partial\Omega}\varphi) \right\rangle_{\mathcal{B}', \mathcal{B}}}_{A} > 0 \quad \text{or} \quad \underbrace{- \left\langle \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^* \right) \varphi, \text{Tr}(\mathcal{S}_{\partial\Omega}\varphi) \right\rangle_{\mathcal{B}', \mathcal{B}}}_{B} > 0.$$

Since $\mathcal{K}_{\partial\Omega}^* \varphi = -\lambda \varphi$, it holds:

$$A = \left(\frac{1}{2} - \lambda \right) \langle \varphi, \text{Tr}(\mathcal{S}_{\partial\Omega}\varphi) \rangle_{\mathcal{B}', \mathcal{B}} \quad \text{and} \quad B = \left(\frac{1}{2} + \lambda \right) \langle \varphi, \text{Tr}(\mathcal{S}_{\partial\Omega}\varphi) \rangle_{\mathcal{B}', \mathcal{B}}.$$

Note that, since $\varphi \neq 0$ and by definition of $\mathcal{S}_{\partial\Omega}\varphi$, it holds $\langle \varphi, \text{Tr}(\mathcal{S}_{\partial\Omega}\varphi) \rangle_{\mathcal{B}', \mathcal{B}} \neq 0$.

If $\lambda = -\frac{1}{2}$, then $B = 0$, hence $\mathcal{S}_{\partial\Omega}\varphi$ is constant on Ω^c . Since $\mathcal{S}_{\partial\Omega}\varphi$ is the solution to (B.1) associated to φ , then $\mathcal{S}_{\partial\Omega}\varphi = 0$ and $\varphi = 0$, which is contradictory.

If $\lambda \neq -\frac{1}{2}$, then $A \geq 0$ and $B > 0$, therefore it holds:

$$\lambda = \frac{1}{2} \frac{A - B}{A + B} \in \left] -\frac{1}{2}, \frac{1}{2} \right[.$$

In the same way, we prove the result for $\tilde{\mathcal{K}}_{\partial\Omega}^* : \mathcal{B}'_0(\partial\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega)$. \square

Proposition B.24 highlights the link between the weak solution to our transmission problem, and the Neumann-Poincaré operators and their adjoints, which motivates our definition of those operators. Since (by definition) the single layer potential has no jump in trace and the double layer potential has no jump in normal derivative across the boundary, we may introduce the following operators:

$$\begin{aligned} \mathcal{V}_{\partial\Omega} &:= \text{Tr} \circ \mathcal{S}_{\partial\Omega} : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega), \\ \mathcal{W}_{\partial\Omega} &:= -\frac{\partial}{\partial\nu} \circ \mathcal{D}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega), \end{aligned} \quad (\text{B.48})$$

for the operator $(-\Delta + 1)$, as well as their equivalent for the operator $-\Delta$:

$$\begin{aligned} \tilde{\mathcal{V}}_{\partial\Omega} &:= \text{Tr} \circ \mathcal{S}_{\partial\Omega} : \mathcal{B}'_0(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega), \\ \tilde{\mathcal{W}}_{\partial\Omega} &:= -\frac{\partial}{\partial\nu} \circ \mathcal{D}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'_0(\partial\Omega), \end{aligned} \quad (\text{B.49})$$

Those operators allow to prove symmetrization formulae for the Neumann-Poincaré operators, generalizing [80, Theorem 3.1.3].

Theorem B.27. *Let Ω be a two-sided extension domain of \mathbb{R}^n . Consider the following operators:*

$$\mathcal{C}_i := \frac{1}{2}I + \mathcal{M} \quad \text{and} \quad \mathcal{C}_e := \frac{1}{2}I - \mathcal{M},$$

where

$$\mathcal{M} := \begin{pmatrix} -\mathcal{K}_{\partial\Omega} & \mathcal{V}_{\partial\Omega} \\ \mathcal{W}_{\partial\Omega} & \mathcal{K}_{\partial\Omega}^* \end{pmatrix} : \mathcal{B}(\partial\Omega) \times \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega) \times \mathcal{B}'(\partial\Omega).$$

Then, the operators \mathcal{C}_i and \mathcal{C}_e defined above are linear and continuous projectors, referred to as ‘Calderón projectors’ (interior and exterior respectively).

Proof. Let $u = \mathcal{S}_{\partial\Omega}g - \mathcal{D}_{\partial\Omega}f$ be a solution to the transmission problem on \mathbb{R}^n , with $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'(\partial\Omega)$. The interior boundary values of u are given by:

$$\mathcal{C}_i \begin{pmatrix} \llbracket \text{Tr} u \rrbracket \\ \llbracket \frac{\partial u}{\partial\nu} \rrbracket \end{pmatrix} = \begin{pmatrix} \text{Tr}_i u \\ \frac{\partial_i u}{\partial\nu} \end{pmatrix} = \begin{pmatrix} \mathcal{V}_{\partial\Omega}g - \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right)f \\ \left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^*\right)g + \mathcal{W}_{\partial\Omega}f \end{pmatrix}.$$

We aim at proving that $\mathcal{C}_i^2 = \mathcal{C}_i$. Let v be a solution to the transmission problem such that $\llbracket \text{Tr } v \rrbracket = \text{Tr}_i u$ and $\llbracket \frac{\partial v}{\partial \nu} \rrbracket = \frac{\partial_i u}{\partial \nu}$. Since the transmission problem is well-posed and $u\mathbb{1}_\Omega$ is a solution with the same boundary jump values as v , it holds $v = u\mathbb{1}_\Omega$. Therefore, it holds:

$$\mathcal{C}_i^2 \begin{pmatrix} \llbracket \text{Tr } u \rrbracket \\ \llbracket \frac{\partial u}{\partial \nu} \rrbracket \end{pmatrix} = \begin{pmatrix} \text{Tr}_i v \\ \frac{\partial_i v}{\partial \nu} \end{pmatrix} = \begin{pmatrix} \text{Tr}_i u \\ \frac{\partial_i u}{\partial \nu} \end{pmatrix} = \mathcal{C}_i \begin{pmatrix} \llbracket \text{Tr } u \rrbracket \\ \llbracket \frac{\partial u}{\partial \nu} \rrbracket \end{pmatrix}.$$

Hence, \mathcal{C}_i is a projector. In the same way, we prove \mathcal{C}_e is a projector. \square

Since $\mathcal{C}_i^2 = \mathcal{C}_i$ and $\mathcal{C}_i = \frac{1}{2}I + \mathcal{M}$, we can deduce:

$$\mathcal{M}^2 = \frac{1}{4}I,$$

which can be rewritten as:

$$\begin{cases} \mathcal{K}_{\partial\Omega} \mathcal{V}_{\partial\Omega} = \mathcal{V}_{\partial\Omega} \mathcal{K}_{\partial\Omega}^*, \\ \mathcal{W}_{\partial\Omega} \mathcal{K}_{\partial\Omega} = \mathcal{K}_{\partial\Omega}^* \mathcal{W}_{\partial\Omega}, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{K}_{\partial\Omega}^2 + \mathcal{V}_{\partial\Omega} \mathcal{W}_{\partial\Omega} = \frac{1}{4}I, \\ (\mathcal{K}_{\partial\Omega}^*)^2 + \mathcal{W}_{\partial\Omega} \mathcal{V}_{\partial\Omega} = \frac{1}{4}I. \end{cases} \quad (\text{B.50})$$

In the same way, we prove the symmetrization relations for the operators associated to $-\Delta$, stated in Subsection 4.3.

B.4 Poincaré-Steklov operators for transmission problems

As explained in Subsection 4.4, we wish to generalize notion of Poincaré-Steklov operator for the transmission problem (Equation (4.1)) from [89] in the case of extension domains, using the operator we have defined in Appendix B.3. The following theorem proves both definitions given in Subsection 4.4 are equivalent and explains the connection between the solutions u and v involved.

Theorem B.28. *Let Ω be a two-sided extension domain of \mathbb{R}^n .*

(i) *It holds:*

$$\mathcal{V}_{\partial\Omega}^{-1} \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega} \right) = - \left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^* \right)^{-1} \mathcal{W}_{\partial\Omega}.$$

That operator, denoted by $\mathcal{A}_{\partial\Omega,i}$ is called interior Poincaré-Steklov operator for the transmission problem associated to $(-\Delta + 1)$. It is linear continuous invertible and is described by:

$$\begin{aligned} \mathcal{A}_{\partial\Omega,i} : \mathcal{B}(\partial\Omega) &\longrightarrow \mathcal{B}'(\partial\Omega) \\ -\llbracket \text{Tr } u \rrbracket &\longmapsto \llbracket \frac{\partial v}{\partial \nu} \rrbracket, \end{aligned}$$

where $u = -\mathcal{D}_{\partial\Omega} \llbracket \text{Tr } u \rrbracket$, and $v = \mathcal{S}_{\partial\Omega} \llbracket \frac{\partial v}{\partial \nu} \rrbracket$ is characterized by $u|_\Omega = v|_\Omega$.

(ii) It holds:

$$\mathcal{V}_{\partial\Omega}^{-1} \left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega} \right) = - \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^* \right)^{-1} \mathcal{W}_{\partial\Omega}.$$

That operator, denoted by $\mathcal{A}_{\partial\Omega,e}$ is called exterior Poincaré-Steklov operator for the transmission problem associated to $(-\Delta + 1)$. It is linear continuous invertible and is described by:

$$\begin{aligned} \mathcal{A}_{\partial\Omega,e} : \mathcal{B}(\partial\Omega) &\longrightarrow \mathcal{B}'(\partial\Omega) \\ -\llbracket \text{Tr } u \rrbracket &\longmapsto \left[\left[\frac{\partial w}{\partial \nu} \right] \right], \end{aligned}$$

where $u = -\mathcal{D}_{\partial\Omega} \llbracket \text{Tr } u \rrbracket$, and $w = \mathcal{S}_{\partial\Omega} \llbracket \frac{\partial w}{\partial \nu} \rrbracket$ is characterized by $u|_{\Omega^c} = w|_{\Omega^c}$.

Proof. Let us prove point (i) (the proof of point (ii) is similar). Let $u = -\mathcal{D}_{\partial\Omega} \llbracket \text{Tr } u \rrbracket$ and $v = \mathcal{S}_{\partial\Omega} \llbracket \frac{\partial v}{\partial \nu} \rrbracket$ be as in (4.15). Then it holds $\text{Tr}_i u = \text{Tr}_i v$, which implies $u|_{\Omega} = v|_{\Omega}$ since the problem

$$\begin{cases} (-\Delta + 1)\varphi = 0 & \text{on } \Omega, \\ \text{Tr } \varphi = f \in \mathcal{B}(\partial\Omega), \end{cases}$$

is well-posed. The fact that $u|_{\Omega} = v|_{\Omega}$ and $\llbracket \text{Tr } v \rrbracket = 0$ uniquely determines the solution v on \mathbb{R}^n (since the Dirichlet problem for $(-\Delta + 1)$ on Ω^c is well-posed). Therefore, the operator $\mathcal{A}_{\partial\Omega,i}$ is well-defined and $\mathcal{A}_{\partial\Omega,i} = \mathcal{V}_{\partial\Omega}^{-1}(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega})$. The linearity, continuity and invertibility follow from that relation. Using the well-posedness of the Neumann problems for $(-\Delta + 1)$ on Ω and on Ω^c yields $\mathcal{A}_{\partial\Omega,i} = -(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^*)^{-1} \mathcal{W}_{\partial\Omega}$. \square

The definition of the Poincaré-Steklov operators for the transmission problem can be understood in terms of the following equivalences, where $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'(\partial\Omega)$:

$$\begin{aligned} \mathcal{D}_{\partial\Omega} f|_{\Omega} = \mathcal{S}_{\partial\Omega} g|_{\Omega} &\iff \mathcal{A}_{\partial\Omega,i} f = g, \\ \mathcal{D}_{\partial\Omega} f|_{\Omega^c} = \mathcal{S}_{\partial\Omega} g|_{\Omega^c} &\iff \mathcal{A}_{\partial\Omega,e} f = g. \end{aligned}$$

Those relations allow to prove the Poincaré-Steklov operator for the bounded problem (Subsection 3.2) and for the transmission problem are connected: $\mathcal{A} = -\mathcal{A}_{\partial\Omega,e}$.

B.5 Norm estimates and Neumann series

Using the boundary operators defined in Subsection 4.3, let us define the following norms on $\mathcal{B}'(\partial\Omega)$ and $\mathcal{B}(\partial\Omega)$ respectively:

$$\| \cdot \|_{\mathcal{V}_{\partial\Omega}}^2 := \langle \cdot, \mathcal{V}_{\partial\Omega} \cdot \rangle_{\mathcal{B}', \mathcal{B}} \quad \text{and} \quad \| \cdot \|_{\mathcal{V}_{\partial\Omega}^{-1}}^2 := \langle \mathcal{V}_{\partial\Omega}^{-1} \cdot, \cdot \rangle_{\mathcal{B}', \mathcal{B}}, \quad (\text{B.51})$$

well-defined using the variational formulations of the transmission problems:

$$\forall g \in \mathcal{B}'(\partial\Omega), \quad \|g\|_{\mathcal{V}_{\partial\Omega}}^2 = \langle g, \mathcal{V}_{\partial\Omega}g \rangle_{\mathcal{B}', \mathcal{B}} = \|\mathcal{S}_{\partial\Omega}g\|_{H_{\partial\Omega}^1(\mathbb{R}^n)}^2. \quad (\text{B.52})$$

By continuity of the single layer potential operator and of its inverse (which is non other than $\llbracket \frac{\partial \cdot}{\partial \nu} \rrbracket$), it holds, for some $\alpha > 0$:

$$\forall g \in \mathcal{B}'(\partial\Omega), \quad \frac{1}{\alpha} \|g\|_{\mathcal{B}'(\partial\Omega)}^2 \leq \|g\|_{\mathcal{V}_{\partial\Omega}}^2 \leq \alpha \|g\|_{\mathcal{B}'(\partial\Omega)}^2. \quad (\text{B.53})$$

In the same way, let us define the following norm on $\mathcal{B}(\partial\Omega)$:

$$\|\cdot\|_{\mathcal{W}_{\partial\Omega}}^2 := \langle \mathcal{W}_{\partial\Omega}\cdot, \cdot \rangle_{\mathcal{B}', \mathcal{B}}.$$

It holds:

$$\forall f \in \mathcal{B}(\partial\Omega), \quad \|f\|_{\mathcal{W}_{\partial\Omega}}^2 = \langle \mathcal{W}_{\partial\Omega}f, f \rangle_{\mathcal{B}', \mathcal{B}} = \|\mathcal{D}_{\partial\Omega}f\|_{H_{\partial\Omega}^1(\mathbb{R}^n)}^2. \quad (\text{B.54})$$

By continuity of the double layer potential operator and of its inverse (which is non other than $-\llbracket \text{Tr} \cdot \rrbracket$), it holds, for some $\beta > 0$:

$$\forall f \in \mathcal{B}(\partial\Omega), \quad \frac{1}{\beta} \|f\|_{\mathcal{B}(\partial\Omega)}^2 \leq \|f\|_{\mathcal{W}_{\partial\Omega}}^2 \leq \beta \|f\|_{\mathcal{B}(\partial\Omega)}^2. \quad (\text{B.55})$$

We can also define the norms $\|\cdot\|_{\tilde{\mathcal{V}}_{\partial\Omega}}$ on $\mathcal{B}(\partial\Omega)$, and $\|\cdot\|_{\tilde{\mathcal{V}}_{\partial\Omega}^{-1}}$ and $\|\cdot\|_{\tilde{\mathcal{W}}_{\partial\Omega}}$ on $\mathcal{B}'_0(\partial\Omega)$ for the problem associated to $-\Delta$, and yield the same kind of inequalities as (B.53) and (B.55) with constants $\tilde{\alpha}, \tilde{\beta} > 0$.

The norm equivalences (B.53) and (B.55) (and their restatement for $-\Delta$) place us in the framework of [89], provided the notations are adapted to the case of extension domains, which allows to state the following theorem.

Theorem B.29. *Let Ω be a two-sided extension domain of \mathbb{R}^n .*

- (i) *The layer potential operators $\mathcal{S}_{\partial\Omega} : (\mathcal{B}'(\partial\Omega), \|\cdot\|_{\mathcal{V}_{\partial\Omega}}) \rightarrow \mathcal{S}_{\partial\Omega}(\mathcal{B}'(\partial\Omega))$ and $\mathcal{D}_{\partial\Omega} : (\mathcal{B}(\partial\Omega), \|\cdot\|_{\mathcal{W}_{\partial\Omega}}) \rightarrow \mathcal{D}_{\partial\Omega}(\mathcal{B}(\partial\Omega))$ define isometries;*
- (ii) *The layer potential operators $\tilde{\mathcal{S}}_{\partial\Omega} : (\mathcal{B}'_0(\partial\Omega), \|\cdot\|_{\tilde{\mathcal{V}}_{\partial\Omega}}) \rightarrow \tilde{\mathcal{S}}_{\partial\Omega}(\mathcal{B}'_0(\partial\Omega))$ and $\tilde{\mathcal{D}}_{\partial\Omega} : (\mathcal{B}(\partial\Omega), \|\cdot\|_{\tilde{\mathcal{W}}_{\partial\Omega}}) \rightarrow \tilde{\mathcal{D}}_{\partial\Omega}(\mathcal{B}(\partial\Omega))$ define isometries;*
- (iii) *For all $f \in \mathcal{B}(\partial\Omega)$ and $\tilde{f} \in \mathcal{B}_0(\partial\Omega)$, it holds:*

$$(1 - c) \|f\|_{\mathcal{V}_{\partial\Omega}^{-1}} \leq \left\| \left(\pm \frac{1}{2}I + \mathcal{K}_{\partial\Omega} \right) f \right\|_{\mathcal{V}_{\partial\Omega}^{-1}} \leq c \|f\|_{\mathcal{V}_{\partial\Omega}^{-1}},$$

and

$$(1 - \tilde{c}) \|\tilde{f}\|_{\tilde{\mathcal{V}}_{\partial\Omega}^{-1}} \leq \left\| \left(\pm \frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega} \right) \tilde{f} \right\|_{\tilde{\mathcal{V}}_{\partial\Omega}^{-1}} \leq \tilde{c} \|\tilde{f}\|_{\tilde{\mathcal{V}}_{\partial\Omega}^{-1}},$$

where

$$c = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\alpha\beta}} < 1 \quad \text{and} \quad \tilde{c} = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\tilde{\alpha}\tilde{\beta}}} < 1,$$

with the constants $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ from (B.53), (B.55) and their restatements for $-\Delta$ chosen large enough ($\alpha\beta, \tilde{\alpha}\tilde{\beta} > 4$).

(iv) The operators $\pm\frac{1}{2}I + \mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ and $\pm\frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega} : \mathcal{B}_0(\partial\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$ are isomorphisms.

Proof. Points (i) and (ii) follow from equations (B.52) and (B.54), as well as their restatements for $-\Delta$. Point (iii) is obtained following [89]: the norm equivalences (B.53) and (B.55) as well as the relations between the layer potentials and the Neumann-Poincaré operators yield the result. The difference between $\frac{1}{2}$ and $-\frac{1}{2}$ simply comes from the consideration of either the interior trace or the exterior trace operators. Point (iii) implies the operators $\pm\frac{1}{2}I + \mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ and $\pm\frac{1}{2}I + \tilde{\mathcal{K}}_{\partial\Omega} : \mathcal{B}_0(\partial\Omega) \rightarrow \mathcal{B}_0(\partial\Omega)$ are injective and have a closed range. Since their adjoints are injective by Lemma B.26, Point (iv) holds by Banach's closed range theorem. \square

We use a result from perturbation theory [37, Theorem 1.2.9] (Lemma B.30 below) to establish spectral properties of the Neumann-Poincaré operators.

Lemma B.30. *Let X be a Banach space and $A \in \mathcal{L}(X)$ be an isomorphism. Let $\varphi \in \mathcal{L}(X)$ be such that $\|\varphi\|_{\mathcal{L}(X)} < \|A^{-1}\|_{\mathcal{L}(X)}^{-1}$. Then $A + \varphi \in \mathcal{L}(X)$ is an isomorphism.*

Theorem B.31. *Let Ω be a two-sided extension domain of \mathbb{R}^n . Then, for all $\lambda \in]-\infty, \frac{1}{2}] \cup [\frac{1}{2}, +\infty[$:*

- (i) *the operators $\lambda I + \mathcal{K}_{\partial\Omega}$ and $\lambda I + \mathcal{K}_{\partial\Omega}^*$ are invertible on $\mathcal{B}(\partial\Omega)$ and $\mathcal{B}'(\partial\Omega)$ respectively;*
- (ii) *the operators $\lambda I + \tilde{\mathcal{K}}_{\partial\Omega}$ and $\lambda I + \tilde{\mathcal{K}}_{\partial\Omega}^*$ are invertible on $\mathcal{B}_0(\partial\Omega)$ and $\mathcal{B}'_0(\partial\Omega)$ respectively.*

Proof. To extend the result from Theorem B.29, Point (iv) to other values of λ , we use Lemma B.30 in two steps. For any $\lambda \neq -\frac{1}{2}$, it holds:

$$\lambda I + \mathcal{K}_{\partial\Omega} = \left(\lambda + \frac{1}{2}\right)I + \left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right). \quad (\text{B.56})$$

By Theorem B.29, $\|-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\| < 1$. Hence, by Lemma B.30, thinking of $-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}$ as a perturbation φ of $(\lambda + \frac{1}{2})I$, $\lambda I + \mathcal{K}_{\partial\Omega}$ is invertible anytime $|\lambda + \frac{1}{2}| \geq 1$, which is valid when $\lambda \geq \frac{1}{2}$. Proceeding in the same way with $\frac{1}{2}I + \mathcal{K}_{\partial\Omega}$ instead, we find that $\lambda I + \mathcal{K}_{\partial\Omega}$ is invertible for $\lambda \leq -\frac{1}{2}$. Therefore, the real spectrum of $\mathcal{K}_{\partial\Omega}$ is included in $]-\frac{1}{2}, \frac{1}{2}[$. By Banach's closed range theorem, the same holds for $\mathcal{K}_{\partial\Omega}^*$.

Proceeding in the same way, we prove the result for $\tilde{\mathcal{K}}_{\partial\Omega}$ and $\tilde{\mathcal{K}}_{\partial\Omega}^*$. \square

Let us now look into the constants from the norm equivalences (B.53) and (B.55). The optimal constants for those inequalities are none other than the norms of the operators $\mathcal{V}_{\partial\Omega}$, $\mathcal{W}_{\partial\Omega}$ and their inverses. Consider the following norm on $\mathcal{B}(\partial\Omega)$:

$$\|\cdot\|_{\text{Tr}} = \min \{ \|v\|_{H^1(\mathbb{R}^n)} \mid \text{Tr } v = \cdot \}.$$

By energy minimization, it follows that:

$$\forall v \in V_{1,\partial\Omega,[0]_{\text{Tr}}}(\mathbb{R}^n), \quad \|\text{Tr } v\|_{\text{Tr}} = \|v\|_{H^1(\mathbb{R}^n)},$$

where $V_{1,\partial\Omega,[0]_{\text{Tr}}}(\mathbb{R}^n)$ is defined on pages iii–vi. Therefore, it holds:

$$\|\mathcal{V}_{\partial\Omega}\|_{\mathcal{L}(\mathcal{B}',(\mathcal{B},\|\cdot\|_{\text{Tr}}))} = \|\mathcal{S}_{\partial\Omega}\|_{\mathcal{L}(\mathcal{B}',H^1_{\partial\Omega})}.$$

For all $g \in \mathcal{B}'(\partial\Omega)$, it holds:

$$\forall v \in H^1(\mathbb{R}^n), \quad \langle \mathcal{S}_{\partial\Omega} g, v \rangle_{H^1(\mathbb{R}^n)} = \langle g, \text{Tr } v \rangle_{\mathcal{B}'(\partial\Omega),\mathcal{B}(\partial\Omega)},$$

hence:

$$\|\mathcal{V}_{\partial\Omega}\|_{\mathcal{L}(\mathcal{B}',(\mathcal{B},\|\cdot\|_{\text{Tr}}))} = \|\mathcal{S}_{\partial\Omega}\|_{\mathcal{L}(\mathcal{B}',H^1_{\partial\Omega})} = 1.$$

It holds $\mathcal{S}_{\partial\Omega}^{-1} = \llbracket \frac{\partial}{\partial \nu} \rrbracket : V_{1,\partial\Omega,[0]_{\text{Tr}}}(\mathbb{R}^n) \rightarrow \mathcal{B}'(\partial\Omega)$. Let $u \in V_{1,\partial\Omega,[0]_{\text{Tr}}}(\mathbb{R}^n)$. Then, for all $v \in V_{1,\partial\Omega,[0]_{\text{Tr}}}(\mathbb{R}^n)$, it holds:

$$\begin{aligned} \left| \left\langle \llbracket \frac{\partial u}{\partial \nu} \rrbracket, \text{Tr } v \right\rangle_{\mathcal{B}',\mathcal{B}} \right| &= |\langle u, v \rangle_{H^1(\mathbb{R}^n)}| \\ &\leq \|u\|_{H^1(\mathbb{R}^n)} \|v\|_{H^1(\mathbb{R}^n)} = \|u\|_{H^1(\mathbb{R}^n)} \|\text{Tr } v\|_{\text{Tr}} \end{aligned}$$

Hence (taking $u = v$), $\|\mathcal{S}_{\partial\Omega}^{-1}\|_{\mathcal{L}(H^1_{\partial\Omega},\mathcal{B}')} = 1$.

We give optimal constants for the equivalence between the norms $\|\cdot\|_{\mathcal{B}(\partial\Omega)}$ and $\|\cdot\|_{\text{Tr}}$.

Proposition B.32. *Let Ω be a two-sided extension domain. Let E_{Ω} and E_{Ω^c} be the 1-harmonic extension operators from Ω and Ω^c respectively to \mathbb{R}^n . Then it holds:*

$$\sqrt{\left(1 + \frac{1}{\|E_{\Omega^c}\|^2 - 1}\right)} \|\cdot\|_{\mathcal{B}(\partial\Omega)} \leq \|\cdot\|_{\text{Tr}} \leq \|E_{\Omega}\| \|\cdot\|_{\mathcal{B}(\partial\Omega)}, \quad (\text{B.57})$$

with optimal constants.

Proof. Using the isometric properties of both norms, if $v \in V_{1,\partial\Omega,[0]_{\text{Tr}}}(\mathbb{R}^n)$, then (B.57) is equivalent to:

$$\sqrt{\left(1 + \frac{1}{\|E_{\Omega^c}\|^2 - 1}\right)} \|v|_{\Omega}\|_{H^1(\Omega)} \leq \|v\|_{H^1(\mathbb{R}^n)} \leq \|E_{\Omega}\| \|v|_{\Omega}\|_{H^1(\Omega)}, \quad (\text{B.58})$$

The second inequality and optimality of the constant follow directly from the boundedness of E_{Ω} . Moreover, it holds:

$$\frac{\|v|_{\Omega^c}\|_{H^1(\Omega^c)}^2}{\|v\|_{H^1(\mathbb{R}^n)}^2} \geq \frac{1}{\|E_{\Omega^c}\|^2} \quad \text{and} \quad \|v\|_{H^1(\mathbb{R}^n)}^2 = \|v|_{\Omega}\|_{H^1(\Omega)}^2 + \|v|_{\Omega^c}\|_{H^1(\Omega^c)}^2,$$

hence

$$1 - \frac{\|v|_{\Omega}\|_{H^1(\Omega)}^2}{\|v\|_{H^1(\mathbb{R}^n)}^2} \geq \frac{1}{\|E_{\Omega^c}\|^2}.$$

The first inequality and the optimality of the constant follow. \square

For all $f \in \mathcal{B}(\partial\Omega)$, it holds:

$$\|\mathcal{D}_{\partial\Omega}f\|_{H^1_{\partial\Omega}}^2 = \left| \left\langle \frac{\partial}{\partial\nu}\mathcal{D}_{\partial\Omega}f, f \right\rangle_{\mathcal{B}, \mathcal{B}} \right| \leq \left\| \frac{\partial}{\partial\nu}\mathcal{D}_{\partial\Omega}f \right\|_{\mathcal{B}'} \|f\|_{\mathcal{B}}$$

Moreover, since the interior Poincaré-Steklov operator is an isometry for the norm $\|\cdot\|_{\mathcal{B}(\partial\Omega)}$, then it holds:

$$\begin{aligned} \left\| \frac{\partial}{\partial\nu}\mathcal{D}_{\partial\Omega}f \right\|_{\mathcal{B}'} &= \|\mathrm{Tr}_i \mathcal{D}_{\partial\Omega}f\|_{\mathcal{B}} = \|\mathcal{D}_{\partial\Omega}f|_{\Omega}\|_{H^1(\Omega)} \\ &\leq \sqrt{\left(1 + \frac{1}{\|\mathbf{E}_{\Omega^c}\|^2 - 1}\right)} \|\mathcal{D}_{\partial\Omega}f\|_{H^1(\mathbb{R}^n)}, \end{aligned}$$

by (B.58), hence $\|\mathcal{D}_{\partial\Omega}\| \leq \sqrt{\left(1 + \frac{1}{\|\mathbf{E}_{\Omega^c}\|^2 - 1}\right)}$.

It holds:

$$\begin{aligned} \left| \left\langle \frac{\partial u}{\partial\nu}, \llbracket \mathrm{Tr} v \rrbracket \right\rangle_{\mathcal{B}, \mathcal{B}} \right| &= |\langle u, v \rangle_{H^1(\mathbb{R}^n)}| \\ &\leq \|u\|_{H^1(\mathbb{R}^n)} \|v\|_{H^1(\mathbb{R}^n)} \\ &\leq \|\mathbf{E}_{\Omega}\| \left\| \frac{\partial u}{\partial\nu} \right\|_{\mathcal{B}'} \|v\|_{H^1(\mathbb{R}^n)} \end{aligned}$$

hence $\|\llbracket \mathrm{Tr} \cdot \rrbracket\| \leq \|\mathbf{E}_{\Omega}\|$. It holds for u harmonic on $\mathbb{R}^n \setminus \partial\Omega$:

$$\left\| \frac{\partial u}{\partial\nu} \right\|_{\mathcal{B}'} = \|\mathrm{Tr}_i u\|_{\mathcal{B}} = \|u|_{\Omega}\|_{H^1(\Omega)} \leq \sqrt{\left(1 + \frac{1}{\|\mathbf{E}_{\Omega^c}\|^2 - 1}\right)} \|u\|_{H^1(\mathbb{R}^n)}$$

and

$$\|u\|_{H^1(\mathbb{R}^n)} \leq \|\mathbf{E}_{\Omega}\| \left\| \frac{\partial u}{\partial\nu} \right\|_{\mathcal{B}'}.$$

Hence:

$$\|\mathcal{W}_{\partial\Omega}\| \leq 1 + \frac{1}{\|\mathbf{E}_{\Omega^c}\|^2 - 1},$$

and

$$\|\mathcal{W}_{\partial\Omega}^{-1}\| \leq \|\mathbf{E}_{\Omega}\|^2.$$

Similar estimates hold for the operators associated to $-\Delta$.

B.6 Imaging

B.6.1 Representation formula

In this subsection, we aim at generalizing to two-sided extension domains imaging results established in the case of bounded Lipschitz domains [1, 61]. Since we will be considering two two-sided extension domains of \mathbb{R}^n , Ω and D , we shall use the following notations:

- $\text{Tr}_i^{\partial\Omega} : H^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ and $\text{Tr}_e^{\partial\Omega} : H^1(\Omega^c) \rightarrow \mathcal{B}(\partial\Omega)$ for the interior and exterior trace operators on the boundary of Ω , according to Definitions 2.8 and 2.10,
- $\frac{\partial_i u}{\partial\nu}\Big|_{\partial\Omega} \in \mathcal{B}'(\partial\Omega)$ and $\frac{\partial_e u}{\partial\nu}\Big|_{\partial\Omega} \in \mathcal{B}'(\partial\Omega)$ for the interior and exterior normal derivatives of $u \in H^1_{\Delta}(\mathbb{R}^n \setminus \partial\Omega)$ on the boundary of Ω , according to Definitions 2.12 and 2.13,
- $[[\text{Tr } u]]_{\partial\Omega} := \text{Tr}_i^{\partial\Omega} u - \text{Tr}_e^{\partial\Omega} u \in \mathcal{B}(\partial\Omega)$ for the jump in trace across $\partial\Omega$, according to Definition 2.14,
- $\left[\left[\frac{\partial u}{\partial\nu} \right] \right]_{\partial\Omega} := \frac{\partial_i u}{\partial\nu}\Big|_{\partial\Omega} - \frac{\partial_e u}{\partial\nu}\Big|_{\partial\Omega} \in \mathcal{B}'(\partial\Omega)$ for the jump in normal derivative across $\partial\Omega$, according to Definition 2.14,

as well as similar notations for other two-sided extension domains, such as D .

Let Ω and D be two two-sided extension domains of \mathbb{R}^n such that $D \subset\subset \Omega$. For the transmission problem (4.1) for $k = 0$ across ∂D , we can define the single layer potential operator $\tilde{\mathcal{S}}_D : \mathcal{B}'_0(\partial D) \rightarrow H^1_{\partial D}(\mathbb{R}^n)$ and the double layer potential operator $\tilde{\mathcal{D}}_D : \mathcal{B}(\partial D) \rightarrow H^1(\mathbb{R}^n \setminus \partial D)$ on D , as well as the associated Neumann-Poincaré operators $\tilde{\mathcal{K}}_{\partial D} : \mathcal{B}(\partial D) \rightarrow \mathcal{B}(\partial D)$ and $\tilde{\mathcal{K}}_{\partial D}^* : \mathcal{B}'_0(\partial D) \rightarrow \mathcal{B}'_0(\partial D)$. Let $k \in]0, 1[\cup]1, +\infty[$ be the conductivity of the domain D (the conductivity of Ω is assumed to be equal to 1). On the space:

$$H^1_{\partial D}(\Omega) := \left\{ u \in L^2(\Omega) \mid u|_D \in H^1(D), u|_{\Omega \setminus \bar{D}} \in H^1(\Omega \setminus \bar{D}) \right\},$$

we will consider the following two-phased transmission problem:

$$\begin{cases} \nabla \cdot \left((1 + (k-1)\mathbf{1}_D) \nabla u \right) = 0 & \text{on } \Omega, \\ \frac{\partial_i u}{\partial\nu}\Big|_{\partial\Omega} = g, \\ \int_{\Omega} u \, dx = 0, \end{cases} \quad (\text{B.59})$$

where $g \in \mathcal{B}'_0(\partial\Omega)$. The last condition ensures the uniqueness of the solution. The associated variational formulation is:

$$\begin{aligned} \forall v \in H^1_{\partial D, \langle 0 \rangle}(\Omega), \quad & \int_{\Omega} (1 + (k-1)\mathbf{1}_D) \nabla u \cdot \nabla v \, dx \\ & = \left\langle (1 + (k-1)\mathbf{1}_D)g, \text{Tr}_i^{\partial\Omega} v \right\rangle_{\mathcal{B}', \mathcal{B}}, \end{aligned} \quad (\text{B.60})$$

where $H^1_{\partial D, \langle 0 \rangle}(\Omega)$ is defined on pages iii–vi. Let us start by proving the well-posedness of the variational formulation (B.60).

Proposition B.33. *The variational formulation (B.60) of the two-phased transmission problem is well-posed on $H^1_{\partial D, \langle 0 \rangle}(\Omega)$.*

Proof. $H_{\partial D, \langle 0 \rangle}^1(\Omega)$ is a Hilbert space with $(u, v) \mapsto \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$ as an inner product which, by Poincaré's inequality, is equivalent on $H_{\partial D, \langle 0 \rangle}^1(\Omega)$ to the usual inner product on $H^1(\Omega)$. Since $(1 + (k - 1)\mathbf{1}_D)$ is bounded on \mathbb{R}^n , the mapping:

$$v \longmapsto \left\langle (1 + (k - 1)\mathbf{1}_D)g, \text{Tr}_i^{\partial\Omega} v \right\rangle_{\mathcal{B}', \mathcal{B}},$$

is continuous on $H_{\partial D, \langle 0 \rangle}^1(\Omega)$, and there exists $C > 0$ such that:

$$\forall v \in H_{\partial D, \langle 0 \rangle}^1(\Omega), \quad \left| \left\langle (1 + (k - 1)\mathbf{1}_D)g, \text{Tr}_i^{\partial\Omega} v \right\rangle_{\mathcal{B}', \mathcal{B}} \right| \leq C \|g\|_{\mathcal{B}'} \|v\|_{H_{\partial D, \langle 0 \rangle}^1}.$$

In addition, for $u, v \in H_{\partial D, \langle 0 \rangle}^1(\Omega)$, it holds:

$$\left| \int_{\Omega} (1 + (k - 1)\mathbf{1}_D) \nabla u \cdot \nabla v \, dx \right| \leq (k + 1) \lambda^{(n)}(\Omega) \|\nabla u\|_{L^2(\Omega)^n} \|\nabla v\|_{L^2(\Omega)^n}$$

one the one hand, and

$$\int_{\Omega} (1 + (k - 1)\mathbf{1}_D) \nabla u \cdot \nabla u \, dx \geq \min(1, k) \lambda^{(n)}(\Omega) \|\nabla u\|_{L^2(\Omega)^n}^2$$

on the other hand. Therefore, applying Lax-Milgram's theorem, we can deduce the existence of a unique $u \in H_{\partial D, \langle 0 \rangle}^1(\Omega)$ solution to the variational formulation (B.60). Choosing $v = u$ in the variational formulation, it holds:

$$\min(1, k) \lambda^{(n)}(\Omega) \|u\|_{H_{\partial D, \langle 0 \rangle}^1}^2 \leq C \|g\|_{\mathcal{B}'} \|u\|_{H_{\partial D, \langle 0 \rangle}^1},$$

where $C > 0$ is the previous continuity constant (which does not depend on u), that is:

$$\min(1, k) \lambda^{(n)}(\Omega) \|u\|_{H_{\partial D, \langle 0 \rangle}^1} \leq C \|g\|_{\mathcal{B}'},$$

hence the well-posedness of (B.60). \square

Knowing that the two-phased transmission problem (B.60) is well-posed, we seek a representation formula for the solution of that problem and update [1, Theorem 2.17] (see also [59, 60]).

Theorem B.34. (Representation formula) *If Ω and D are two-sided extension domains of \mathbb{R}^n with $D \subset\subset \Omega$, and if $k \in]0, 1[\cup]1, +\infty[$, then the unique weak solution to the two-phased transmission problem (B.59) associated to $g \in \mathcal{B}'_0(\partial\Omega)$ is:*

$$u = H - \tilde{\mathcal{S}}_{\partial D} \varphi \tag{B.61}$$

where

$$H = \tilde{\mathcal{S}}_{\partial\Omega} g - \tilde{\mathcal{D}}_{\partial\Omega} f$$

with $\tilde{\mathcal{S}}_{\partial D}$ defined on $\mathcal{B}'_0(\partial D)$, $\tilde{\mathcal{S}}_{\partial\Omega}$ on $\mathcal{B}'_0(\partial\Omega)$, $\tilde{\mathcal{D}}_{\partial\Omega}$ on $\mathcal{B}(\partial\Omega)$, $f := \text{Tr}_i^{\partial\Omega} u \in \mathcal{B}(\partial\Omega)$, and $\varphi \in \mathcal{B}'_0(\partial D)$ such that:

$$\left(\frac{k + 1}{2(k - 1)} I + \tilde{\mathcal{K}}_{\partial D}^* \right) \varphi = \frac{\partial H}{\partial \nu} \Big|_{\partial D}, \tag{B.62}$$

with $\tilde{\mathcal{K}}_{\partial D}^* : \mathcal{B}'_0(\partial D) \rightarrow \mathcal{B}'_0(\partial D)$.

H is the harmonic part of the solution, while $-\tilde{\mathcal{S}}_{\partial D}\varphi$ corresponds to the refraction part. A such decomposition is unique.

In addition, it holds:

$$H - \tilde{\mathcal{S}}_{\partial D}\varphi = 0 \quad \text{on } \mathbb{R}^n \setminus \bar{\Omega}.$$

Proof. Let us consider the following two-phased transmission problem on the space $W_{\partial D, \partial \Omega}(\mathbb{R}^n)$, which is defined by:

$$W_{\partial D, \partial \Omega}(\mathbb{R}^n) := \left\{ u \in L^2_{loc}(\mathbb{R}^n) \mid \nabla u|_D \in L^2(D), \nabla u|_{\Omega \setminus \bar{D}} \in L^2(\Omega \setminus \bar{D}), \right. \\ \left. \nabla u|_{\Omega^c} \in L^2(\Omega^c), \langle u \rangle_\infty = 0 \right\},$$

$$\begin{cases} \nabla \cdot \left((1 + (k-1)\mathbf{1}_D) \nabla u \right) = 0 & \text{on } \mathbb{R}^n \setminus \partial \Omega, \\ \llbracket \text{Tr } u \rrbracket_{\partial \Omega} = f, \\ \left\llbracket \frac{\partial u}{\partial \nu} \right\rrbracket_{\partial \Omega} = g, \end{cases} \quad (\text{B.63})$$

where $k \in]0, 1[\cup]1, +\infty[$, $g \in \mathcal{B}'_0(\partial \Omega)$, and $f := \text{Tr}_i^{\partial \Omega} u_\Omega \in \mathcal{B}(\partial \Omega)$, with u_Ω the solution to the transmission problem on Ω (B.59) associated to g .

Problem (B.63) has a unique solution on $W_{\partial D, \partial \Omega}(\mathbb{R}^n)$. Indeed, if v is a solution of that problem for homogeneous transmission conditions, then for all $r > 0$, it holds:

$$\begin{aligned} \int_{B_r(0)} |\nabla v|^2 dx &\leq \frac{1+k}{k} \int_{B_r(0)} (1 + (k-1)\mathbf{1}_D) |\nabla v|^2 dx \\ &\leq \frac{1+k}{k} \left\langle (1 + (k-1)\mathbf{1}_D) \frac{\partial_i v}{\partial \nu}, \text{Tr}_i^{B_r(0)} v \right\rangle_{\mathcal{B}', \mathcal{B}} \\ &\leq -\frac{1+k}{k} \int_{B_r(0)^c} (1 + (k-1)\mathbf{1}_D) |\nabla v|^2 dx \\ &\leq -\frac{1+k}{k} \int_{B_r(0)^c} |\nabla v|^2 dx \\ &\leq 0, \end{aligned}$$

applying Green's formula on $B_r(0)$ and $B_r(0)^c$. That being true for all $r > 0$, v is constant. Since $v \in W_{\partial D, \partial \Omega}(\mathbb{R}^n)$, it holds $v = 0$. Hence the uniqueness of the solution on $W_{\partial D, \partial \Omega}(\mathbb{R}^n)$. On the one hand, the function u_0 defined as $u_0|_\Omega = u_\Omega$ and $u_0|_{\Omega^c} = 0$ is a solution to the transmission problem (B.63). On the other hand, $\tilde{\mathcal{S}}_{\partial \Omega} g - \tilde{\mathcal{D}}_{\partial \Omega} f - \tilde{\mathcal{S}}_{\partial D} \varphi$ is also a solution of that problem. Indeed, if u is a solution to the problem, then Green's formula on Ω states that:

$$\forall v \in H^1(\Omega), \quad \left\langle \frac{\partial_i u}{\partial \nu} \Big|_{\partial \Omega}, \text{Tr}_i^{\partial \Omega} v \right\rangle_{\mathcal{B}', \mathcal{B}(\partial \Omega)} - \int_{\Omega \setminus \bar{D}} \nabla u \cdot \nabla v dx = k \int_D \nabla u \cdot \nabla v dx \quad (\text{B.64})$$

Since Green's formulae on D and $\Omega \setminus \overline{D}$ respectively state that, for all $v \in H^1(\Omega)$,

$$\begin{cases} \int_D \nabla u \cdot \nabla v \, dx = \left\langle \frac{\partial_i u}{\partial \nu} \Big|_{\partial D}, \text{Tr}^{\partial D} v \right\rangle_{\mathcal{B}', \mathcal{B}(\partial D)} \\ \int_{\Omega \setminus \overline{D}} \nabla u \cdot \nabla v \, dx = \left\langle \frac{\partial_i u}{\partial \nu} \Big|_{\partial \Omega}, \text{Tr}_i^{\partial \Omega} v \right\rangle_{\mathcal{B}', \mathcal{B}(\partial \Omega)} - \left\langle \frac{\partial_\epsilon u}{\partial \nu} \Big|_{\partial D}, \text{Tr}^{\partial D} v \right\rangle_{\mathcal{B}', \mathcal{B}(\partial D)} \end{cases},$$

then (B.64) becomes:

$$\forall v \in H^1(\Omega), \quad \left\langle k \frac{\partial_i u}{\partial \nu} \Big|_{\partial \Omega} - \frac{\partial_\epsilon u}{\partial \nu} \Big|_{\partial \Omega}, \text{Tr}^{\partial D} v \right\rangle_{\mathcal{B}', \mathcal{B}(\partial D)} = 0,$$

that is, since the trace operator $\text{Tr}^{\partial D} : H^1(\Omega) \rightarrow \mathcal{B}(\partial D)$ is onto:

$$k \frac{\partial_i u}{\partial \nu} \Big|_{\partial D} - \frac{\partial_\epsilon u}{\partial \nu} \Big|_{\partial D} = 0$$

Since $u = H - \tilde{\mathcal{S}}_{\partial D} \varphi$, it holds:

$$k \frac{\partial_i(\tilde{\mathcal{S}}_{\partial D} \varphi)}{\partial \nu} \Big|_{\partial D} - \frac{\partial_\epsilon(\tilde{\mathcal{S}}_{\partial D} \varphi)}{\partial \nu} \Big|_{\partial D} = (k-1) \frac{\partial H}{\partial \nu} \Big|_{\partial D},$$

since H has no jump in normal derivative across D by Weyl's lemma, hence:

$$\left(\frac{k+1}{2(k-1)} I + \mathcal{K}_{\partial D}^* \right) \varphi = \frac{\partial H}{\partial \nu} \Big|_{\partial D}.$$

H is harmonic on D , therefore, by Green's formula, $\frac{\partial H}{\partial \nu} \Big|_{\partial D} \in \mathcal{B}'_0(\partial D)$. Since $\left(\frac{k+1}{2(k-1)} I + \mathcal{K}_{\partial D}^* \right)$ is invertible on $\mathcal{B}'_0(\partial D)$, such a $\varphi \in \mathcal{B}'_0(\partial D)$ exists.

We will now prove the uniqueness of the decomposition. Let us suppose that $H' \in H^1_{\partial \Omega}(\mathbb{R}^n)$ is harmonic in Ω , $\varphi' \in \mathcal{B}'_0(\partial D)$, and that it holds:

$$H - \tilde{\mathcal{S}}_{\partial D} \varphi = H' - \tilde{\mathcal{S}}_{\partial D} \varphi' \quad \text{in } \Omega.$$

Then $\tilde{\mathcal{S}}_{\partial D}(\varphi - \varphi')$ is harmonic on Ω , therefore, by Weyl's lemma it is real analytic: $\left[\frac{\partial}{\partial \nu} \tilde{\mathcal{S}}_{\partial D}(\varphi - \varphi') \right]_{\partial D} = 0$, hence $\varphi = \varphi'$ and $H = H'$. \square

That representation formula can be simplified if $n = 2$ in the case of a disk.

Corollary B.35. *If Ω is a two-sided extension domain of \mathbb{R}^2 , if D is a disk compactly included in Ω and if $k \in]0, 1[\cup]1, +\infty[$, then the solution to the two-phased transmission problem (B.59) is given by:*

$$u = H - \frac{2(k-1)}{k+1} \tilde{\mathcal{S}}_{\partial D} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D} \right) \quad (\text{B.65})$$

where

$$H = \tilde{\mathcal{S}}_{\partial \Omega} g - \tilde{\mathcal{D}}_{\partial \Omega} f$$

with $\tilde{\mathcal{S}}_{\partial D}$ defined on $L^2_0(\partial D, \sigma) \subset \mathcal{B}'_0(\partial D)$, $\tilde{\mathcal{S}}_{\partial \Omega}$ defined on $\mathcal{B}'_0(\partial \Omega)$, $\tilde{\mathcal{D}}_{\partial \Omega}$ defined on $\mathcal{B}(\partial \Omega)$ and $f = \text{Tr}_i u \in \mathcal{B}(\partial \Omega)$.

Proof. As explained in [1], since D is smooth, $\frac{\partial H}{\partial \nu}|_{\partial D} \in L_0^2(\partial D, \sigma)$. Therefore, φ defined by (B.62) is an element of $L_0^2(\partial D, \sigma)$, and it holds:

$$\tilde{\mathcal{K}}_{\partial D}^* \varphi = 0.$$

Therefore, the representation formula (B.61) can be restated as (B.65) if D is a disk of \mathbb{R}^2 . \square

B.6.2 Subdomain identification

We state here a generalized version of the theorem about the uniqueness of the inclusion with one measurement in the monotonous case which can be found in [1] for Lipschitz boundaries, in the case of a two-sided extension domain Ω , and a two-sided extension subdomain D .

Theorem B.36. (*Subdomain identification in the monotonous case*)
Let Ω be a two-sided extension domain of \mathbb{R}^n . Let $D_1 \subset D_2$ be two two-sided extension domains compactly included in Ω , and let $k \in]0, 1[\cup]1, +\infty[$. Let us denote by u_1 and u_2 the solutions to the two-phased transmission problem (B.59), respectively associated to $D = D_1$ and $D = D_2$. If, for some Neumann condition $g \in \mathcal{B}'(\partial\Omega) \setminus \{0\}$, $\text{Tr}_i^{\partial\Omega} u_1 = \text{Tr}_i^{\partial\Omega} u_2$, then $D_1 = D_2$.

Proof. It holds, for all $v \in H^1(\Omega)$:

$$\int_{\Omega} \nabla \cdot \left((1 + (k-1)\mathbf{1}_{D_1}) \nabla u_1 \right) v \, dx = \int_{\Omega} \nabla \cdot \left((1 + (k-1)\mathbf{1}_{D_2}) \nabla u_2 \right) v \, dx.$$

Since $\frac{\partial_i u_1}{\partial \nu}|_{\partial\Omega} = \frac{\partial_i u_2}{\partial \nu}|_{\partial\Omega} = g$ and $D_1, D_2 \subset \Omega$, it holds:

$$\begin{aligned} \left\langle (1 + (k-1)\mathbf{1}_{D_1}) \frac{\partial_i u_1}{\partial \nu} \Big|_{\partial\Omega}, \text{Tr}_i^{\partial\Omega} v \right\rangle_{\mathcal{B}', \mathcal{B}} \\ = \left\langle (1 + (k-1)\mathbf{1}_{D_2}) \frac{\partial_i u_2}{\partial \nu} \Big|_{\partial\Omega}, \text{Tr}_i^{\partial\Omega} v \right\rangle_{\mathcal{B}', \mathcal{B}}, \end{aligned}$$

hence, by Green's formula:

$$\int_{\Omega} (1 + (k-1)\mathbf{1}_{D_1}) \nabla u_1 \cdot \nabla v \, dx = \int_{\Omega} (1 + (k-1)\mathbf{1}_{D_2}) \nabla u_2 \cdot \nabla v \, dx, \quad (\text{B.66})$$

that is:

$$\int_{\Omega} (1 + (k-1)\mathbf{1}_{D_1}) \nabla(u_1 - u_2) \cdot \nabla v \, dx = (k-1) \int_{D_2 \setminus D_1} \nabla u_2 \cdot \nabla v \, dx. \quad (\text{B.67})$$

Only, $\text{Tr}_i^{\partial\Omega} u_1 = \text{Tr}_i^{\partial\Omega} u_2$, so by Green's formula,

$$\int_{\Omega} (1 + (k-1)\mathbf{1}_{D_1}) \nabla(u_1 - u_2) \cdot \nabla u_1 \, dx = 0.$$

Applying (B.67) for $v = u_1$ and $v = u_1 - u_2$, we get:

$$0 = (k - 1) \int_{D_2 \setminus D_1} \nabla u_2 \cdot \nabla u_1 \, dx,$$

and

$$\begin{aligned} \int_{\Omega} (1 + (k - 1)\mathbb{1}_{D_1}) |\nabla(u_1 - u_2)|^2 \, dx \\ = (k - 1) \int_{D_2 \setminus D_1} \nabla u_2 \cdot \nabla(u_1 - u_2) \, dx, \end{aligned}$$

hence:

$$\int_{\Omega} (1 + (k - 1)\mathbb{1}_{D_1}) |\nabla(u_1 - u_2)|^2 \, dx + (k - 1) \int_{D_2 \setminus D_1} |\nabla u_2|^2 \, dx = 0$$

If $k > 1$, then $u_1 = u_2$ as the transmission problem (B.60) is well-posed. Therefore, since $g \neq 0$, it holds $D_1 = D_2$.

If $k < 1$, equation (B.66) lets us show that it holds:

$$\int_{\Omega} (1 + (k - 1)\mathbb{1}_{D_2}) |\nabla(u_1 - u_2)|^2 \, dx + (1 - k) \int_{D_2 \setminus D_1} |\nabla u_1|^2 \, dx = 0, \quad (\text{B.68})$$

and, for the same reason, $D_1 = D_2$. \square

Using Theorem B.36 and Corollary B.35, we can deduce the uniqueness of the inclusion in the case of disks when Ω is a two-sided extension domain of \mathbb{R}^2 .

Theorem B.37. (Subdomain identification in the case of disks) *Let Ω be a two-sided extension domain of \mathbb{R}^2 . Let D_1 and D_2 be two disks compactly included in Ω , and let $k \in]0, 1[\cup]1, +\infty[$. Let us denote by u_1 and u_2 the solutions to the two-phased transmission problem (B.59), respectively associated to $D = D_1$ and $D = D_2$. If, for some Neumann condition $g \in \mathcal{B}'_0(\partial\Omega) \setminus \{0\}$, $\text{Tr}_i^{\partial\Omega} u_1 = \text{Tr}_i^{\partial\Omega} u_2$, then $D_1 = D_2$.*

Proof. Let us denote $f := \text{Tr}_i^{\partial\Omega} u_1 = \text{Tr}_i^{\partial\Omega} u_2$. The representation formula (B.65) states that, for $p \in \{1, 2\}$:

$$u_p = H - \frac{2(k - 1)}{k + 1} \tilde{\mathcal{S}}_{\partial D_p} \left(\left. \frac{\partial H}{\partial \nu} \right|_{\partial D_p} \right)$$

with $H = \tilde{\mathcal{S}}_{\partial\Omega} g - \tilde{\mathcal{D}}_{\partial\Omega} f$.

• If $D_1 \subset D_2$ (or the other way around), then it has been proved in Theorem B.36 that $D_1 = D_2$.

On $\mathbb{R}^2 \setminus (D_1 \cup D_2)$, u_1 and u_2 are both solution to the same problem:

$$\begin{cases} \Delta u_p = 0 & \text{on } \mathbb{R}^2 \setminus (D_1 \cup D_2) \\ \llbracket \text{Tr } u_p \rrbracket_{\partial\Omega} = f, \\ \llbracket \frac{\partial u_p}{\partial \nu} \rrbracket_{\partial\Omega} = g, \end{cases}$$

which has a unique solution on $V_{van}(\mathbb{R}^2 \setminus (D_1 \cup D_2)) := \{u|_{\mathbb{R}^2 \setminus (D_1 \cup D_2)} \mid u \in V_{van}(\mathbb{R}^2)\}$, therefore $u_1 = u_2$ on $\mathbb{R}^2 \setminus (D_1 \cup D_2)$, hence it holds:

$$\tilde{\mathcal{S}}_{\partial D_1} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_1} \right) = \tilde{\mathcal{S}}_{\partial D_2} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_2} \right) \quad \text{on } \mathbb{R}^2 \setminus (D_1 \cup D_2). \quad (\text{B.69})$$

• If $D_1 \cap D_2 = \emptyset$, then, by (B.69), it holds that:

$$\left[\left[\frac{\partial}{\partial \nu} \circ \tilde{\mathcal{S}}_{\partial D_1} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_1} \right) \right] \right]_{\partial D_1} = \left[\left[\frac{\partial}{\partial \nu} \circ \tilde{\mathcal{S}}_{\partial D_2} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_2} \right) \right] \right]_{\partial D_1} = 0, \quad (\text{B.70})$$

since $\tilde{\mathcal{S}}_{\partial D_2} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_2} \right)$ has no jump in normal derivative across ∂D_1 . Therefore, $\frac{\partial H}{\partial \nu} \Big|_{\partial D_1} = 0$ which means that H is constant \rightarrow ABSURD.

• If $D_1 \cap D_2 \notin \{\emptyset, D_1, D_2\}$, since $\frac{\partial H}{\partial \nu} \Big|_{\partial D_p} \in L_0^2(\partial D_p, \sigma)$ for $p \in \{1, 2\}$, it holds $\tilde{\mathcal{K}}_{D_p}^* \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_p} \right) = 0$, therefore:

$$\frac{\partial_i}{\partial \nu} \circ \tilde{\mathcal{S}}_{\partial D_p} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_p} \right) = \frac{1}{2} \frac{\partial H}{\partial \nu} \Big|_{\partial D_p},$$

hence there exists $c_p \in \mathbb{R}$ such that:

$$\tilde{\mathcal{S}}_{\partial D_p} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_p} \right) = \frac{1}{2} H + c_p \quad \text{on } D_p.$$

Therefore, denoting $c := c_1 - c_2$, it holds:

$$\tilde{\mathcal{S}}_{\partial D_1} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_1} \right) = \tilde{\mathcal{S}}_{\partial D_2} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_2} \right) + c \quad \text{on } D_1 \cap D_2.$$

Since the single layer potential operator is continuous on \mathbb{R}^2 , this equality remains valid on $\partial(D_1 \cap D_2)$. Since equation (B.69) is also valid on $\partial(D_1 \cap D_2)$, we can deduce that $c = 0$:

$$\tilde{\mathcal{S}}_{\partial D_1} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_1} \right) = \tilde{\mathcal{S}}_{\partial D_2} \left(\frac{\partial H}{\partial \nu} \Big|_{\partial D_2} \right) \quad \text{on } \mathbb{R}^2 \setminus (D_1 \Delta D_2),$$

where $D_1 \Delta D_2$ denotes the symmetric difference between D_1 and D_2 . Therefore, equation (B.70) holds once more and H is constant \rightarrow ABSURD.

All in all, the only possible configuration is $D_1 = D_2$. \square

C Hilbert transform and Cauchy integral on extension domains

In Subsection 5.2, we introduce two possible definitions for the Cauchy integral (or equivalently, the Hilbert transform) for two-sided extension domains of \mathbb{C} . Aiming towards proving those definitions coincide, we prove any domain can be approximated by a sequence of Lipschitz domains in Appendix C.1, and in Appendix C.2, we prove the convergence of the Dirichlet (1-)harmonic extension operators, in the framework from [65] which we remind.

C.1 Lipschitz approximation of arbitrary domains

Let us consider a dyadic grid on \mathbb{R}^n :

$$\Pi_k := \{\pi_j, j \in \mathbb{Z}^n\},$$

where

$$\pi_j^k = \left\{ x \in \mathbb{R}^n \mid \forall m \in \llbracket 0, n \rrbracket, x \in [2^{-k}j_m, 2^{-k}(j_m + 1)] \right\},$$

for $k \in \mathbb{N}$ and denoting $j = (j_m)_{m \in \llbracket 0, n \rrbracket} \in \mathbb{Z}^n$. If $\mathcal{P} \subset \Pi_k$, denote:

$$\cup \mathcal{P} := \bigcup_{\pi \in \mathcal{P}} \pi.$$

Let us define:

$$\begin{aligned} \overline{\Omega}_k^- &:= \max_{\mathcal{P} \subset \Pi_k} \left\{ A = \cup \mathcal{P} \mid A \subset \Omega, \mathring{A} \text{ is connected} \right\}, \\ \overline{\Omega}_k^+ &:= \max_{\mathcal{P} \subset \Pi_k} \left\{ B = \cup \mathcal{P} \mid B \subset \Omega^c, \mathring{B} \text{ is connected} \right\}, \end{aligned}$$

for $k \in \mathbb{N}$ large enough (see Lemma C.1), where the maximum is understood in terms of volume. Denote their interiors by:

$$\Omega_k^- := (\overline{\Omega}_k^-)^\circ \quad \text{and} \quad \Omega_k^+ := (\overline{\Omega}_k^+)^\circ. \quad (\text{C.1})$$

Those domains are referred to as interior and exterior dyadic approximations (of order k) of Ω , respectively. The connectedness condition can be written as:

$$\forall \pi_j^k \in \mathcal{P}, \exists i \in \mathbb{Z}^d, \quad |j - i| = 1 \quad \text{and} \quad \pi_i^k \in \mathcal{P}.$$

As an example, Figure 4 represents a dyadic approximation of the Von Koch snowflake.

Naturally, the first step is to make sure the dyadic approximations are well-defined, at least for a grid thin enough (for we are interested in the limit when the grid becomes infinitely thin)

Lemma C.1. *Let Ω be an arbitrary bounded domain of \mathbb{R}^n such that $\overline{\Omega}^c$ is connected too. For $k \in \mathbb{N}$ large enough, Ω_k^- and Ω_k^+ are well-defined.*

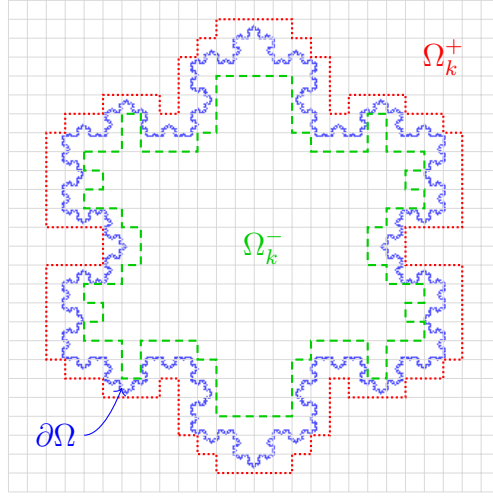


Figure 4: Dyadic approximations of a Von Koch snowflake ($\partial\Omega$, in blue) in \mathbb{R}^2 . The interior approximation Ω_k^- lies inside the green dashed line; the exterior approximation Ω_k^+ lies outside the red dotted line.

Proof. Since Ω is bounded, there exists a cube \mathcal{Q} drawn on Π_0 such that $\Omega \subset \mathcal{Q}$. Then, $\overline{\Omega}_k^+$ is the largest union of cubes of Π_k of connected interior, containing \mathcal{Q} , and such that $\overline{\Omega}_k^+ \subset \Omega^c$.

For all $k \in \mathbb{N}$, denote by \mathcal{C}_k^1 any union of cubes of Π_k included in Ω , of connected interior and such that:

$$\lambda^{(n)}(\mathcal{C}_k^1) = \max_{\mathcal{P} \subset \Pi_k} \left\{ \lambda^{(n)}(A) \mid A = \cup \mathcal{P}, A \subset \Omega, \mathring{A} \text{ is connected} \right\}.$$

Since any dyadic cube from Π_k can be divided into 2^k dyadic cubes from Π_{k+1} , the sequence $(\lambda^{(n)}(\mathcal{C}_k^1))_{k \in \mathbb{N}}$ is non-decreasing. Therefore, there exists $\ell \leq \lambda^{(n)}(\Omega)$ positive such that:

$$\lambda^{(n)}(\mathcal{C}_k^1) \underset{k \rightarrow \infty}{\nearrow} \ell.$$

There exists $k_0 \in \mathbb{N}$ such that:

$$\forall k \geq k_0, \quad \frac{\ell}{2} < \lambda^{(n)}(\mathcal{C}_k^1) \leq \ell.$$

Then, for all $k \geq k_0$, \mathcal{C}_k^1 is uniquely defined. Otherwise, assume for some $k \geq k_0$ there were two possible choices for \mathcal{C}_k^1 , denoted by C_1 and C_2 . Since Ω is connected, there exists a continuous path $\gamma \subset \Omega$ joining C_1 and C_2 . Since Ω is open, it holds:

$$\forall x \in \gamma, \quad d(x, \partial\Omega) > 0 \quad \text{i.e.} \quad \min_{x \in \gamma} d(x, \partial\Omega) > 0, \quad (\text{C.2})$$

by continuity with respect to x of the distance. Therefore, for $\tilde{k} \geq k$ large enough, there exists a set \mathcal{P} of cubes included in Ω from $\Pi_{\tilde{k}}$ joining C_1 and C_2

in the sense that $C_1 \cup \mathcal{P} \cup C_2$ has a connected interior. By definition of $\mathcal{C}_{\tilde{k}}^1$, we can deduce:

$$\lambda^{(n)}(\mathcal{C}_{\tilde{k}}^1) \geq \lambda^{(n)}(C_1) + \lambda^{(n)}(C_2) > \ell,$$

which is absurd since $\tilde{k} \geq k_0$. Therefore, \mathcal{C}_k^1 is uniquely defined, which means Ω_k^- is well-defined. \square

Remark C.2. *In the proof of Lemma C.1, it is also shown $(\Omega_k^-)_{k \in \mathbb{N}}$ and $(\Omega_k^+)_{k \in \mathbb{N}}$ form non-decreasing sequences for k large enough.*

We may now state that the dyadic approximations go to the approximated domain as the grid becomes thinner.

Theorem C.3. *Let Ω be an arbitrary bounded domain of \mathbb{R}^n such that $\overline{\Omega}^c$ is connected too. Then $\Omega_k^- \sqcup \Omega_k^+ \rightarrow \mathbb{R}^n \setminus \partial\Omega$.*

Proof. From Remark C.2, let $k_0 \in \mathbb{N}$ be such that $(\Omega_k^-)_{k \geq k_0}$ is non-decreasing. Then it holds:

$$\Omega_k^- \xrightarrow[k \rightarrow \infty]{} \Omega_\infty^- := \bigcup_{k \geq k_0} \Omega_k^- \subset \Omega.$$

Let $x \in \Omega$. Then $d(x, \partial\Omega) > 0$. In the same way as for (C.2), there exists $\tilde{k} \geq k_0$ so that:

$$\{x\} \cup \Omega_{\tilde{k}}^- \subset \Omega_{\tilde{k}}^-.$$

Hence $x \in \Omega_\infty^-$, and $\Omega_\infty^- = \Omega$.

To show $\Omega_k^+ \rightarrow \overline{\Omega}^c$, we proceed in the same way considering $x \in \overline{\Omega}^c$ a path on $\overline{\Omega}^c$ linking x and a large square containing Ω . \square

Remark C.4. *Lemma C.1, Remark C.2 and Proposition C.3 can easily be generalized in the case of an arbitrary (potentially unbounded) open set Ω such that Ω and $\overline{\Omega}^c$ have a finite number of connected components.*

C.2 Convergence of the Dirichlet harmonic extension operators

C.2.1 Convergence along a sequence of Hilbert spaces

In this subsection, we recall the definition of convergence of Hilbert spaces from [65] and of elements along a converging sequence of Hilbert spaces. In addition, we establish a few results to have a better understanding of that notion.

Definition C.5. (Convergence of Hilbert spaces) *A sequence $(H_k)_{k \in \mathbb{N}}$ of Hilbert spaces is said to converge towards a Hilbert space H through $(C, (\Phi_k)_{k \in \mathbb{N}})$ (in the sense of Kuwae-Shioya [65]) where C is dense in H and $(\Phi_k : C \rightarrow H_k)_{k \in \mathbb{N}}$ is a sequence of linear maps if it holds:*

$$\forall f \in C, \quad \lim_{k \rightarrow +\infty} \|\Phi_k f\|_{H_k} = \|f\|_H. \quad (\text{C.3})$$

Such a convergence will be denoted by: $H_k \xrightarrow[k \rightarrow \infty]{ks} H$ through $(C, (\Phi_k)_{k \in \mathbb{N}})$.

As we could expect, that notion of convergence in all generality is not very strong. Let us point out two distasteful peculiarities of that notion of limit:

1. if $(H_k)_{k \in \mathbb{N}} \xrightarrow{ks} H$, then $(H_k)_{k \in \mathbb{N}} \xrightarrow{ks} V$ for all V isometric to H ;
2. if H_k is a Hilbert subspace of V_k for all $k \in \mathbb{N}$ and $(H_k)_{k \in \mathbb{N}} \xrightarrow{ks} H$ through $(C, (\Phi_k)_{k \in \mathbb{N}})$, then $(V_k)_{k \in \mathbb{N}} \xrightarrow{ks} H$ through $(C, (\Phi_k)_{k \in \mathbb{N}})$ as well (up to changing the target of the Φ_k).

However, the convergence of Definition C.5 is compatible with certain structures such as dual spaces and direct sums of Hilbert spaces.

Lemma C.6. *Assume $(H_k)_{k \in \mathbb{N}} \xrightarrow{ks} H$ through $(C, (\Phi_k)_{k \in \mathbb{N}})$ in the sense of Definition C.5. Denote by $j_k : H_k \rightarrow H'_k$ and $j : H \rightarrow H'$ the Riesz isometries between the Hilbert spaces and their dual spaces. Then it holds:*

$$(H'_k)_{k \in \mathbb{N}} \xrightarrow{ks} H' \quad \text{through} \quad (j(C), (j_k \circ \Phi_k \circ j^{-1})_{k \in \mathbb{N}}).$$

Lemma C.7. *Let $(H_k)_{k \in \mathbb{N}}$, $(V_k)_{k \in \mathbb{N}}$ be sequences of Hilbert spaces such that $H_k \xrightarrow{ks} H$ and $V_k \xrightarrow{ks} V$. Assume, for all $k \in \mathbb{N}$, $H_k + V_k = H_k \oplus V_k$ and $H + V = H \oplus V$, and consider the norms on those sums defined by:*

$$\|\cdot\|_{H_k \oplus V_k} := \sqrt{\|\cdot\|_{H_k}^2 + \|\cdot\|_{V_k}^2} \quad \text{and} \quad \|\cdot\|_{H \oplus V} := \sqrt{\|\cdot\|_A^2 + \|\cdot\|_B^2}.$$

Then it holds: $H_k \oplus V_k \xrightarrow{ks} H \oplus V$.

Proof. Assume that $H_k \xrightarrow{ks} H$ through $(\mathcal{C}_H, (\Phi_k^H)_{k \in \mathbb{N}})$, and that $V_k \xrightarrow{ks} V$ through $(\mathcal{C}_V, (\Phi_k^V)_{k \in \mathbb{N}})$. Then $\mathcal{C}_H \oplus \mathcal{C}_V$ is dense in $H \oplus V$ and defining:

$$\begin{aligned} \Psi_k : \mathcal{C}_H \oplus \mathcal{C}_V &\longmapsto H_k \oplus V_k \\ u + v &\longrightarrow \Phi_k^H u + \Phi_k^V v, \end{aligned}$$

then it holds: $\forall w \in \mathcal{C}_H \oplus \mathcal{C}_V, \|\Psi_k w\|_{H_k \oplus V_k} \longrightarrow \|w\|_{H \oplus V}$. \square

In our study, we are interested in the convergence of H^1 spaces of converging domains.

Proposition C.8. *Let $(\Omega_k)_{k \in \mathbb{N}}$ be a sequence of two-sided extension domains and Ω be another extension domain such that $\Omega_k \xrightarrow{\mathbb{1}} \Omega$ (in the sense of characteristic functions). Then it holds:*

$$H^1(\Omega_k) \oplus H^1(\Omega_k^c) \xrightarrow[k \rightarrow \infty]{ks} H^1(\Omega) \oplus H^1(\Omega^c).$$

Proof. Let E_Ω denote the extension operator from $H^1(\Omega)$ to $H^1(\mathbb{R}^n)$. Let $u \in H^1(\Omega)$. From the convergence in the sense of characteristic functions, it holds:

$$\mathbf{1}_{\Omega_k} |E_\Omega u|^2 \xrightarrow{a.e.} \mathbf{1}_\Omega |E_\Omega u|^2.$$

Since $E_\Omega u \in L^2(\mathbb{R}^n)$, by dominated convergence, it holds:

$$\int_{\mathbb{R}^n} \mathbf{1}_{\Omega_k} |E_\Omega u|^2 dx \longrightarrow \int_{\mathbb{R}^n} \mathbf{1}_\Omega |E_\Omega u|^2 dx.$$

In the same way:

$$\int_{\mathbb{R}^n} \mathbf{1}_{\Omega_k} |\nabla E_\Omega u|^2 dx \longrightarrow \int_{\mathbb{R}^n} \mathbf{1}_\Omega |\nabla E_\Omega u|^2 dx,$$

hence $\|E_\Omega u\|_{H^1(\Omega_k)} \rightarrow \|E_\Omega u\|_{H^1(\Omega)}$. Therefore, $H^1(\Omega_k) \xrightarrow{ks} H^1(\Omega)$.

In the same way, $H^1(\Omega_k^c) \xrightarrow{ks} H^1(\Omega^c)$. By Lemma C.7, we can deduce:

$$H^1(\Omega_k) \oplus H^1(\Omega_k^c) \xrightarrow[k \rightarrow \infty]{ks} H^1(\Omega) \oplus H^1(\Omega^c).$$

□

In the case of a converging sequence of Hilbert spaces, we may consider a converging vectors along that sequence.

Definition C.9. (Strong convergence of vectors in moving Hilbert spaces) Let $(H_k)_{k \in \mathbb{N}}$ be a sequence of Hilbert spaces converging towards a Hilbert space H through $(C, (\Phi_k)_{k \in \mathbb{N}})$. A sequence $(f_k)_{k \in \mathbb{N}}$ with $f_k \in H_k$ – denoted by $(f_k \in H_k)_{k \in \mathbb{N}}$ – is said to converge (strongly) towards an element $f \in H$ (in the sense of Kuwae-Shioya [65]) if there exists a sequence $(g_j) \in C^{\mathbb{N}}$ with $g_j \xrightarrow{H} f$, such that:

$$\lim_{j \rightarrow +\infty} \left(\limsup_{k \rightarrow +\infty} \|\Phi_k g_j - f_k\|_{H_k} \right) = 0.$$

We shall denote such convergence by: $(f_k \in H_k)_{k \in \mathbb{N}} \xrightarrow{ks} f \in H$.

Although the notion of convergence described in Definition C.9 can seem rather complex, it becomes less intricate if we assume the convergence of the corresponding Hilbert spaces is ‘nice enough’.

Lemma C.10. Let $(H_k)_{k \in \mathbb{N}}$ be a sequence of Hilbert spaces and H be another Hilbert space such that $(H_k)_{k \in \mathbb{N}} \xrightarrow{ks} H$ through $(H, (\Phi_k)_{k \in \mathbb{N}})$ where $(\Phi_k \in \mathcal{L}(H, H_k))_{k \in \mathbb{N}}$ is uniformly bounded. Then for all $(u_k \in H_k)_{k \in \mathbb{N}}$ and $u \in H$, it holds:

$$u_k \xrightarrow[k \rightarrow \infty]{ks} u \iff \|\Phi_k u - u_k\|_{H_k} \xrightarrow[k \rightarrow \infty]{} 0$$

Proof. The converse follows directly from the definition of convergence of vectors. Assume $(u_k \in H_k)_{k \in \mathbb{N}} \xrightarrow{ks} u \in H$. There exists $(v_m) \in H^{\mathbb{N}}$ such that:

$$v_m \xrightarrow[m \rightarrow \infty]{H} u \quad \text{and} \quad \limsup_{k \rightarrow \infty} \|\Phi_k v_m - u_k\|_{H_k} \xrightarrow[m \rightarrow \infty]{} 0.$$

From

$$\|\Phi_k u - u_k\|_{H_k} \leq C_\Phi \|v_m - u\|_H + \|\Phi_k v_m - u_k\|_{H_k},$$

where C_Φ comes from the uniform boundedness of $(\Phi_k)_{k \in \mathbb{N}}$, the implication holds. □

The notion of convergence of vectors allows to define the convergence of bounded operators in a natural way.

Definition C.11. (Strong convergence of bounded operators in moving Hilbert spaces) Let $(H_k)_{k \in \mathbb{N}}$ be a sequence of Hilbert spaces converging towards a Hilbert space H in the sense of Definition C.5. Let $(B_k \in \mathcal{L}(H_k))_{k \in \mathbb{N}}$ and $B \in \mathcal{L}(H)$ be linear bounded operators. It is said that the sequence $(B_k)_{k \in \mathbb{N}}$ converges towards the operator B (in the sense of Kuwae-Shioya [65]) if for every converging sequence $(f_k \in H_k)_{k \in \mathbb{N}} \xrightarrow{ks} f \in H$, it holds:

$$B_k f_k \xrightarrow[k \rightarrow \infty]{ks} B f.$$

We shall denote such convergence by: $(B_k \in \mathcal{L}(H_k))_{k \in \mathbb{N}} \xrightarrow{ks} B \in \mathcal{L}(H)$.

In the case of a single Hilbert space, the convergence of unbounded operators can be established by means of the associated quadratic forms [78]. That notion can be generalized to a sequence of Hilbert spaces as well.

Definition C.12. (Weak convergence of vectors in moving Hilbert spaces) Let $(H_k)_{k \in \mathbb{N}}$ be a sequence of Hilbert spaces converging towards a Hilbert space H in the sense of Definition C.5. A sequence $(f_k \in H_k)_{k \in \mathbb{N}}$ is said to weakly converge towards an element $f \in H$ (in the sense of Kuwae-Shioya [65]) if for any converging sequence $(g_k \in H_k)_{k \in \mathbb{N}}$ with limit $g \in H$, it holds:

$$\langle f_k, g_k \rangle_{H_k} \longrightarrow \langle f, g \rangle_H \quad \text{as } k \rightarrow +\infty.$$

We shall denote such convergence by: $(f_k \in H_k)_{k \in \mathbb{N}} \xrightarrow{ks} f \in H$.

Definition C.13. (Convergence of quadratic forms in the sense of Kuwae-Shioya-Mosco) Let $(H_k)_{k \in \mathbb{N}}$ be a sequence of Hilbert spaces converging towards a Hilbert space H in the sense of Definition C.5. Let $(Q_k)_{k \in \mathbb{N}}$ be a sequence of quadratic forms such that Q_k be defined on H_k , and Q be a quadratic form on H . $(Q_k)_{k \in \mathbb{N}}$ is said to converge towards Q in the sense of Kuwae-Shioya-Mosco if it holds:

$$(i) \quad \forall f \in H, \forall (f_k \in H_k)_{k \in \mathbb{N}}, \quad \left(f_k \xrightarrow{ks} f \implies \liminf_k Q_k(f_k) \geq Q(f) \right),$$

$$(ii) \quad \forall f \in H, \exists (f_k \in H_k)_{k \in \mathbb{N}}, \quad \left(f_k \xrightarrow{ks} f \text{ and } \limsup_k Q_k(f_k) \leq Q(f) \right).$$

We shall denote such convergence by: $Q_k \xrightarrow[k \rightarrow \infty]{ksm} Q$ as $k \rightarrow +\infty$.

We refer to [65] for properties of the convergence of quadratic forms in the sense of Kuwae-Shioya-Mosco.

C.2.2 Case of the Dirichlet harmonic extension operators

If we are to consider the convergence of the Dirichlet harmonic extension operators from Subsection 3.1 along a sequence of converging domains, then all of them are defined on different spaces (the trace space on each domain). To do so, we must start with proving the convergence of the trace spaces under our convergence hypotheses.

Proposition C.14. *Let $(\Omega_k)_{k \in \mathbb{N}}$ be a sequence of two-sided extension domains and let Ω be another two-sided extension domain such that:*

$$\forall v \in H^1(\mathbb{R}^n), \quad \|\mathrm{Tr}^{\partial\Omega_k} v\|_{\mathcal{B}(\partial\Omega_k)} \xrightarrow[k \rightarrow \infty]{} \|\mathrm{Tr}^{\partial\Omega} v\|_{\mathcal{B}(\partial\Omega)}. \quad (\text{C.4})$$

Then $(\mathcal{B}(\partial\Omega_k))_{k \in \mathbb{N}} \xrightarrow[k \rightarrow \infty]{ks} \mathcal{B}(\partial\Omega)$ through $(\mathcal{B}(\partial\Omega), (\mathrm{Tr}^{\partial\Omega_k} \circ H_\Omega)_{k \in \mathbb{N}})$, where $H_\Omega : \mathcal{B}(\partial\Omega) \rightarrow H^1(\mathbb{R}^n)$ denotes the harmonic extension from $\partial\Omega$ to \mathbb{R}^n vanishing at infinity.

Proof. For all $f \in \mathcal{B}(\partial\Omega)$, it holds:

$$\|\mathrm{Tr}^{\partial\Omega_k} H_\Omega f\|_{\mathcal{B}(\partial\Omega_k)} \xrightarrow[k \rightarrow \infty]{} \|\mathrm{Tr}^{\partial\Omega} H_\Omega f\|_{\mathcal{B}(\partial\Omega)} = \|f\|_{\mathcal{B}(\partial\Omega)}.$$

□

An important remark at this point is that, by Theorem 2.9, all the trace operators $\mathrm{Tr}^{\partial\Omega_k} : H^1(\mathbb{R}^n) \rightarrow \mathcal{B}(\partial\Omega_k)$ have norm at most 1 (note the domains of the operators discussed here). Therefore, Lemma C.10 applies to the convergence of the \mathcal{B} spaces established in the previous proposition. Since we are using the harmonic extension H_Ω , we prove harmonicity is preserved along a sequence of converging domains

Lemma C.15. *Let $(\Omega_k)_{k \in \mathbb{N}}$ be a sequence of two-sided extension domains of \mathbb{R}^n and Ω be another two-sided extension domain, such that $\Omega_k \xrightarrow{1, K} \Omega$. Let $(u_k)_{k \in \mathbb{N}} \in H^1(\mathbb{R}^n)^{\mathbb{N}}$ such that u_k is harmonic on $\Omega_k \cup \Omega_k^c$ for all k . Then, if $u_k \rightarrow u \in H^1(\mathbb{R}^n)$, then u is harmonic on $\Omega \cup \Omega^c$.*

Proof. From the convergence in the sense of characteristic functions, it holds $H^1(\Omega_k) \rightarrow H^1(\Omega)$ and the statement makes sense. Let $\phi \in C_0^\infty(\Omega)$. From the compact convergence, $\mathrm{supp} \phi \subset \Omega_k$ for k large enough, and it holds:

$$\langle u, \Delta\phi \rangle_{H^1(\Omega)} = \lim_{k \rightarrow \infty} \langle u_k, \Delta\phi \rangle_{H^1(\Omega_k)} = 0,$$

hence the harmonicity of u on Ω . In the same way, we prove the harmonicity of u on Ω^c . □

The final result which we need to prove is the convergence of the Dirichlet (1-)harmonic extension operators is the convergence of the trace operators on harmonic functions.

Proposition C.16. *Let $(\Omega_k)_{k \in \mathbb{N}}$ be a sequence of two-sided extension domains of \mathbb{R}^n , and Ω be another two-sided extension domain satisfying the boundary norm convergence (C.4) and such that $\Omega_k \xrightarrow{1,K} \Omega$. Then, if $(u_k)_{k \in \mathbb{N}} \in H^1(\mathbb{R}^n)^{\mathbb{N}}$ such that u_k is harmonic on $\Omega_k \cup \Omega_k^c$ for all k , and if $u_k \rightarrow u \in H^1(\mathbb{R}^n)$, then it holds $\text{Tr}^{\partial\Omega_k} u_k \xrightarrow{ks} \text{Tr}^{\partial\Omega} u$.*

Proof. If $u_k \xrightarrow{ks} u \in H^1(\Omega)$, then, by Lemma C.15, u is harmonic on Ω . It holds:

$$\left\| \underbrace{\text{Tr}^{\partial\Omega_k} \text{E}_{\Omega} \text{Tr}^{\partial\Omega} u}_{\text{id}_{H^1(\mathbb{R}^n)}} - \text{Tr}^{\partial\Omega_k} u_k \right\|_{\mathcal{B}(\partial\Omega_k)} \leq \|u - u_k\|_{H^1(\mathbb{R}^n)},$$

and Lemma C.10 yields the result. \square

Using the results proved above, we establish the convergence of the Dirichlet harmonic extension operators.

Theorem C.17. *Let $(\Omega_k)_{k \in \mathbb{N}}$ be a sequence of two-sided extension domains of \mathbb{R}^n , and Ω be another two-sided extension domain such that $\Omega_k \xrightarrow{1,K} \Omega$ and satisfying the boundary norm convergence (C.4). Then,*

(i) *if the H^1 -extension operators $\text{E}_{\Omega_k} : H^1(\Omega_k) \rightarrow H^1(\mathbb{R}^n)$, $k \in \mathbb{N}$, are uniformly bounded, then:*

$$\tilde{\mathcal{D}}_{\Omega_k} \in \mathcal{L}(\mathcal{B}(\partial\Omega_k), H^1(\Omega_k)) \xrightarrow[k \rightarrow \infty]{ksm} \tilde{\mathcal{D}}_{\Omega} \in \mathcal{L}(\mathcal{B}(\partial\Omega), H^1(\Omega)),$$

in the sense of Definition C.13;

(ii) *if the H^1 -extension operators $\text{E}_{\Omega_k^c} : H^1(\Omega_k^c) \rightarrow H^1(\mathbb{R}^n)$, $k \in \mathbb{N}$, are uniformly bounded, then:*

$$\tilde{\mathcal{D}}_{\Omega_k^c} \in \mathcal{L}(\mathcal{B}(\partial\Omega_k), H^1(\Omega_k^c)) \xrightarrow[k \rightarrow \infty]{ksm} \tilde{\mathcal{D}}_{\Omega^c} \in \mathcal{L}(\mathcal{B}(\partial\Omega), H^1(\Omega^c)),$$

where the exterior extensions are chosen as vanishing at infinity.

Proof. (i) Consider the quadratic forms associated to the Dirichlet harmonic extension operators:

$$\begin{aligned} Q_k : \mathcal{B}(\partial\Omega_k) &\longrightarrow \mathbb{R}_+ && \text{for } \tilde{\mathcal{D}}_{\Omega_k}, \\ f_k &\longmapsto \|\tilde{\mathcal{D}}_k f_k\|_{H^1(\Omega_k)}^2 = \|\text{H}_{\partial\Omega_k} f_k|_{\Omega_k}\|_{H^1(\Omega_k)}^2 \\ \\ Q : \mathcal{B}(\partial\Omega) &\longrightarrow \mathbb{R}_+ && \text{for } \tilde{\mathcal{D}}_{\Omega}, \\ f &\longmapsto \|\tilde{\mathcal{D}} f\|_{H^1(\Omega)}^2 = \|\text{H}_{\partial\Omega} f|_{\Omega}\|_{H^1(\Omega)}^2 \end{aligned}$$

where $\text{H}_{\partial\Omega_k}$ denotes the harmonic extension from $\partial\Omega_k$ to \mathbb{R}^n and $\text{H}_{\partial\Omega}$ from $\partial\Omega$ to \mathbb{R}^n , vanishing at infinity. Let $f \in \mathcal{B}(\partial\Omega)$ and $(f_k \in \mathcal{B}(\partial\Omega_k))_{k \in \mathbb{N}}$ are such that $f_k \xrightarrow{ks} f$. Without loss of generality, we may assume that the sequence

$(Q_k(f_k)) \in \mathbb{R}_+^{\mathbb{N}}$ has a limit: that consists in considering a subsequence which goes to the lowest subsequential limit. We may also assume that:

$$L := \lim_{k \rightarrow \infty} Q_k(f_k) < +\infty, \quad (\text{C.5})$$

otherwise condition (i) directly holds. In that case, it holds:

$$\sup_k \left\| \mathbb{H}_{\partial\Omega_k} f_k|_{\Omega_k} \right\|_{H^1(\Omega_k)} < +\infty$$

By uniform boundedness of the extension operators, we may deduce:

$$\sup_k \|\mathbb{H}_{\partial\Omega_k} f_k\|_{H^1(\mathbb{R}^n)} < +\infty. \quad (\text{C.6})$$

Consequently we may, passing to a subsequence, assume that there is some $v \in H^1(\mathbb{R}^n)$ such that

$$\mathbb{H}_{\partial\Omega_k} f_k \rightharpoonup v \quad \text{in } H^1(\mathbb{R}^n). \quad (\text{C.7})$$

From the convergence in the sense of characteristic functions, we may deduce:

$$\mathbb{1}_{\Omega_k} \mathbb{H}_{\partial\Omega_k} f_k \xrightarrow{H^1(\mathbb{R}^n)} \mathbb{1}_{\Omega} v.$$

Therefore,

$$\|v\|_{H^1(\Omega)}^2 \leq \liminf_{k \rightarrow \infty} \|\mathbb{H}_{\partial\Omega_k} f_k\|_{H^1(\Omega_k)}^2 = L. \quad (\text{C.8})$$

Moreover, since $\mathbb{H}_{\partial\Omega_k} f_k$ is harmonic on Ω_k for all $k \in \mathbb{N}$, v is harmonic on Ω and it holds by Proposition C.16:

$$\begin{aligned} \text{Tr}^{\partial\Omega_k} \mathbb{H}_{\partial\Omega_k} f_k &\xrightarrow{ks} \text{Tr}^{\partial\Omega} v \\ &= \\ f_k &\xrightarrow{ks} f \end{aligned}$$

hence $\text{Tr}_{\partial\Omega} v = f$. By energy minimization and (C.8) it follows that:

$$Q(f) = \|\mathbb{H}_{\partial\Omega} f\|_{H^1(\Omega)}^2 \leq \|v\|_{H^1(\Omega)}^2 \leq L,$$

which proves condition (i).

Suppose now that $f \in \mathcal{B}(\partial\Omega)$. For any $k \in \mathbb{N}$, let us denote $f_k := \text{Tr}_{\partial\Omega_k} \mathbb{H}_{\partial\Omega} f$. Then it holds $f_k \xrightarrow{ks} f$ and, by energy minimization:

$$Q_k(f_k) = \|\mathbb{H}_{\partial\Omega_k} f_k\|_{H^1(\Omega_k)}^2 \leq \|\mathbb{H}_{\partial\Omega} f\|_{H^1(\Omega_k)}^2.$$

From the convergence in the sense of characteristic functions, we may deduce:

$$\lim_{k \rightarrow \infty} \|\mathbb{H}_{\partial\Omega_k} f_k\|_{H^1(\Omega_k)}^2 = \|\mathbb{H}_{\partial\Omega} f\|_{H^1(\Omega)}^2 = Q(f).$$

That proves the convergence in the Kuwae-Shioya-Mosco sense.

The proof of (ii) is similar. \square

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Résumé

Que ce soit dans le domaine médical ou pour la détection d'impuretés dans les alliages métalliques, les problèmes d'imagerie trouvent de nombreux domaines d'application. D'un point de vue mathématique, ces problèmes sont modélisés par des problèmes de transmission. Dans cette thèse, on étudie ces problèmes dans le cadre de domaines dont le bord peut être irrégulier, par exemple fractal. En se plaçant sur une large classe de domaines d'extension de Sobolev, que l'on nomme domaines d'extension bilatéraux, on généralise des notions relatives aux problèmes de transmission pour des bords Lipschitziens telles que les opérateurs potentiels simple et double couches, Poincaré-Steklov et Neumann-Poincaré. En particulier, on généralise la formule de représentation des solutions du problème de transmission à deux phases pour des domaines d'extension. On souligne le lien qu'ont les problèmes de transmission sur des domaines d'extension avec le concept probabiliste de chaîne de Markov, ainsi qu'avec la notion de transformée de Hilbert que l'on généralise dans le cadre non-Lipschitzien. On montre que tout domaine peut être approximé par une suite de domaines dyadiques et l'on prouve la convergence d'extensions harmoniques depuis le bord le long d'une suite convergente de domaines.

Summary

Whether it be in the medical field or for the detection of impurities in metal alloys, imaging problems have many areas of application. From a mathematical point of view, these problems are modelled by transmission problems. In this thesis, we study those problems in the context of domains with irregular boundaries such as fractals. By focusing on a large class of Sobolev extension domains, which we call two-sided extension domains, we generalize notions regarding transmission problems for Lipschitz boundaries such as the single and double layer potential, the Poincaré-Steklov and the Neumann-Poincaré operators. In particular, we generalize the representation formula for the solutions to the two-phase transmission problem to extension domains. The connection of transmission problems on extension domains with the probabilistic concept of the Markov chain is emphasised, and so is their link with the notion of the Hilbert transform, which is generalized to the non-Lipschitz framework. We establish any domain can be approximated by a sequence of dyadic domains, and prove the convergence of harmonic extensions from the boundary along a sequence of converging domains.